Part A

1. Let $F$ be a field. In a brief paragraph write out the sequence of propositions and theorems necessary to prove that $F[x_1, \ldots, x_n]$ is a unique factorization domain. (No proofs are necessary.)

2. (a) State the definition of a prime ideal of a commutative ring.
   (b) Let $R$ and $S$ be commutative rings. Show that any prime ideal of the direct sum $R \oplus S$ must have the form $P \oplus S$ for a prime ideal $P$ of $R$ or $R \oplus P$ for a prime ideal $P$ of $S$.

3. Let $D$ be an integral domain with quotient field $Q(D)$, and let $P$ be a prime ideal of $D$.
   (a) State the definition of the localization $D_P$ of $D$ at $P$.
   (b) Prove that if $J$ is any proper ideal of $D_P$, then there exists an ideal $I$ of $D$ with $I \subseteq P$ such that $J = IP$.

Part B

4. Let $R$ be a ring and let $M$ be a left $R$-module.
   (a) State the definition of Ann($M$), and show that it is an ideal of $R$.
   (b) State the definition of a faithful module.
   (c) Prove that $M$ can be thought of as a faithful $R/\text{Ann}(M)$-module.

5. (a) State the definition of a simple module.
   (b) Show that $RM$ is a simple module if and only if $Rm = M$, for each $0 \neq m \in M$.
   (c) Prove Schur's lemma, which states that if $RM$ is a simple module that $\text{End}_R(M)$ is a division ring.

6. (a) State the definition of an idempotent element. What does it mean to say that a finite set of idempotent elements is an orthogonal set?
   (b) Show that $R$ is a direct sum of left ideals $A_1, \ldots, A_n$ if and only if there exists a set $e_1, \ldots, e_n$ of orthogonal idempotent elements of $R$ such that $A_i = Re_i$ for $1 \leq j \leq n$ and $e_1 + \cdots + e_n = 1$. 