Math 521 Midterm Exam 3/7/2005

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Answer 2 questions from part A and 2 questions from part B.

Part A

1. Let F be a field. In a brief paragraph write out the sequence of propositions and theorems necessary to prove that $F[x_1, \ldots, x_n]$ is a unique factorization domain. (No proofs are necessary.)

2. (a) State the definition of a *prime ideal* of a commutative ring.
(b) Let R and S be commutative rings. Show that any prime ideal of the direct sum $R \oplus S$ must have the form $P \oplus S$ for a prime ideal P of R or $R \oplus P$ for a prime ideal P of S.

3. Let D be an integral domain with quotient field $Q(D)$, and let P be a prime ideal of D.
(a) State the definition of the *localization* D_P of D at P.
(b) Prove that if J is any proper ideal of D_P, then there exists an ideal I of D with $I \subseteq P$ such that $J = I_P$.

Part B

4. Let R be a ring and let M be a left R-module.
(a) State the definition of $\operatorname{Ann}(M)$, and show that it is an ideal of R.
(b) State the definition of a *faithful* module.
(c) Prove that M can be thought of as a faithful $R/\operatorname{Ann}(M)$-module.

5. (a) State the definition of a *simple* module.
(b) Show that ${}_RM$ is a simple module if and only if $Rm = M$, for each $0 \neq m \in M$.
(c) Prove Schur's lemma, which states that if ${}_RM$ is a simple module that $\operatorname{End}_R(M)$ is a division ring.

6. (a) State the definition of an *idempotent element*. What does it mean to say that a finite set of idempotent elements is an *orthogonal* set?
(b) Show that R is a direct sum of left ideals $A_1, \ldots, A_n$ if and only if there exists a set $e_1, \ldots, e_n$ of orthogonal idempotent elements of R such that $A_i = Re_i$ for $1 \leq j \leq n$ and $e_1 + \cdots e_n = 1$.