MATH 521 TEST I Professor J. Beachy, 3/31/93

PART I. (40 points) Choose either problem 1 or problem 2.

1. (general ring theory)

(a) Let I and J be ideals of the ring R such that $I + J = R$. Prove that the rings $R/I \oplus R/J$ and $R/(I \cap J)$ are isomorphic.

Hint: You may assume that $\phi : R \to R/I \oplus R/J$ defined by $\phi(r) = (r + I, r + J)$ for all $r \in R$ is a ring homomorphism.

(b) In a noncommutative ring R, an ideal P of R is called *prime* if $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ for all ideals A, B of R which contain P. Find all prime ideals of the ring R of all 2×2 lower triangular matrices over $\mathbf{Z}$.

2. (unique factorization)

Let D be a Noetherian integral domain. Prove that D is a unique factorization domain if and only if any two elements of D have a greatest common divisor.

Hint: Model your proof of the "if" part on the proof that any principal ideal domain is a unique factorization domain.

PART II. (60 points) Choose either problem 3 or problem 4.

3. (chain conditions)

(a) Let M be a left R-module with a submodule N such that M and M/N are Artinian. Prove that M is Artinian.

(b) Let M be a left R-module which is both Artinian and Noetherian. Prove that M has a composition series.

(c) Let D be a principal ideal domain, and let I be any nonzero ideal of D. Prove that D/I has a composition series as a left D-module.

4. (direct summands)

(a) Let M and N be left R-modules. Let $f \in \text{Hom}_R(M, N)$ and $g \in \text{Hom}_R(N, M)$ with $fg = 1_N$. Prove that $M = \ker(f) \oplus \text{Im}(g)$.

(b) Use part (a) to prove that if M is a left R-module and $f \in \text{End}_R(M)$ with $f^2 = f$, then $M = \ker(f) \oplus \text{Im}(f)$.

(c) Prove that any direct summand of an injective left R-module is injective.

MATH 521 FINAL EXAM Professor J. Beachy, 5/4/93

1. (40 pts) Choose two of the following four proofs from the class notes.

   (a) State and prove Maschke's Theorem.

   (b) Outline the proof of the existence of the tensor product of two modules, and give a complete proof of the uniqueness of the construction.

   (c) Let D be a principal ideal domain, let $0 \neq a \in D$, and let $R = D/aD$. Prove that R is injective as a left module over itself.

   (d) Let R be a commutative Noetherian ring, and suppose that (0) is an irreducible ideal of R. Prove that (0) is a primary ideal of R.

2. (20 pts) Let M, M' be right R-modules, let N, N' be left R-modules, and let $f : M \to M'$ and $g : N \to N'$ be *onto* R-homomorphisms. Let K be the subgroup of $M \otimes_R N$ generated by all elements of the form $x \otimes y$ such that either $x \in \ker(f)$ or $y \in \ker(g)$. Prove that $K = \ker(f \otimes g)$, and hence that $M' \otimes_R N' \cong (M \otimes_R N)/K$.

3. (20 pts) Let I be the ideal of the ring $\mathbf{Z}[\mathbf{x}]$ of all polynomials with integer coefficients defined by
$$\{a_0 + a_1x + a_2x^2 + \ldots a_nx^n \in \mathbf{Z}[\mathbf{x}] \mid \mathbf{a_0 = 0} \text{ and } \mathbf{2|a_1}\} .$$
   Find $\sqrt{I}$, and find the primary decomposition of I.

4. (20 pts) Choose one of the following problems.

   (a) Let R be a left Artinian ring in which $I^2 = (0)$ implies $I = (0)$, for all two-sided ideals I of R. Prove that R is semisimple Artinian (i.e. the module ${}_RR$ is semisimple).

   (b) Let R be the subring of the field $\mathbf{Q}$ of rational numbers consisting of all fractions with odd denominator. Prove that R is a principal ideal domain.

   (c) Let ${}_RQ$ be an injective module such that $\text{Hom}_R(S, Q) \neq (0)$ for all simple left R-modules S. Prove that if ${}_RM$ is any left R-module, and m is any nonzero element of M, then there exists $f \in \text{Hom}_R(M, Q)$ with $f(m) \neq 0$.

   (d) The following conditions are equivalent for a ring R. (1) for each $a \in R$ there exists $b \in R$ with $aba = a$; (2) each principal left ideal of R is generated by an idempotent element; (3) each finitely generated left ideal of R is generated by an idempotent element.

   (i) Prove (1) $\Leftrightarrow$ (2) in the above result.

   (ii) Let R be a left Noetherian ring which satisfies the given conditions. Prove that R is semisimple Artinian (i.e. the module ${}_RR$ is semisimple).