5.3.2. (15 pts) Let $I, J$ be ideals of the commutative ring $R$. Show that if $\sqrt{I} + \sqrt{J} = R$, then $I + J = R$.

**Solution:** Let $a \in \sqrt{I}$ and $b \in \sqrt{J}$ with $a + b = 1$. There exist $m, n$ with $a^m \in I$ and $b^n \in J$, and then $(a + b)^m+n-1 = 1$. Expanding this expression we have a sum of terms of the form $a^ib^j$, with $i + j = m + n - 1$ and the appropriate binomial coefficient. All of the first terms in which $i \geq m$ belong to $I$, and we denote their sum by $x$. The remaining terms, with $i \leq m - 1$, and hence $j \geq n$, belong to $J$, and we denote their sum by $y$. Thus we have $x + y = 1$, for $x \in I$ and $y \in J$, and so $I + J = R$.

5.3.3. (15 pts) Prove that if $I, Q$ are ideals of the commutative ring $R$ with $I \subseteq Q$, then $Q$ is a primary ideal of $R$ if and only if $Q/I$ is a primary ideal of $R/I$.

**Solution:** The proof can be modeled on the corresponding one for prime ideals. In that case, $Q$ is prime if and only if $R/Q$ is an integral domain, and so we only need to observe that $(R/I)/(Q/I)$ is isomorphic to $R/Q$. Thus we need a criterion for being primary that can be applied to the factor ring $R/Q$.

The ideal $Q$ is a primary ideal of $R$ if it satisfies the following condition for all $a, c \in R$: $ac \in Q$ implies $a \in Q$ or $c^n \in Q$, for some $n \in \mathbb{Z}^+$. This translates into the following condition in $R/Q$. The ideal $Q$ is a primary ideal of $R$ if and only if in the factor ring $R/Q$ we have the following condition for all elements $\overline{a}, \overline{c} \in R/Q$: $\overline{ac} = 0$ implies $\overline{a} = 0$ or $\overline{c^n} = 0$, for some $n \in \mathbb{Z}^+$. It then follows as above that $Q$ is a primary ideal of $R$ if and only if $Q/I$ is a primary ideal of $R/I$.

5.3.5. (20 pts) Let $F$ be a field, and consider the ideal $I = (x^2, xy)$ of $F[x, y]$.

(a) Show that $I$ is not a primary ideal. (b) Show that $I = (x) \cap (x^2, y)$.

(c) Show that $(x^2, ax + y)$ is a primary ideal, for any $a \in F$, and show that $I = (x) \cap (x^2, ax + y)$, so that $I$ can be represented in infinitely many ways as an intersection of primary ideals.

**Solution:** (a) The ideal $I$ contains the product $xy$, but does not contain $x$ and does not contain any power of $y$. Thus $I$ not a primary ideal.

(b) and (c) Since $xy = x(ax + y) - ax^2$, we have $I \subseteq (x) \cap (x^2, ax + y)$ because the generators $x^2$ and $xy$ belong to both ideals. On the other hand, if $f(x, y) \in (x^2, ax + y)$, then $f(x, y) = g(x, y)x^2 + h(x, y)(ax + y)$, or $f(x, y) = g(x, y)x^2 + ah(x, y)x + h(x, y)y$. If $f(x, y) \in (x)$, then $x$ must be a factor of $h(x, y)$, showing that $f(x, y)$ has the form $f(x, y) = (g(x, y) + ak(x, y))x^2 + k(x, y)xy$, and thus $f(x, y) \in (x^2, xy)$. The special case $a = 0$ proves (b).

To prove that $(x^2, ax + y)$ is primary it is helpful to first prove the following result:

*If $I$ is an ideal of $R$ such that $\sqrt{I}$ is a maximal ideal of $R$, then $I$ is a primary ideal of $R$.*

**Proof:** Let $a, b \in R$ with $ab \in I$. If $b^n \in I$ for some $n$, we are done. If not, then $b \notin \sqrt{I}$, and so $b$ is invertible modulo $\sqrt{I}$ since $R/\sqrt{I}$ is a field. Since $\sqrt{I}/I$ is the Jacobson radical of $R/I$, it follows that $b$ is invertible modulo $I$, and therefore $ab \in I$ implies $a \in I$.

We have shown that $xy \in (x^2, ax + y)$, and thus $y^2 = y(ax + y) - axy \in (x^2, ax + y)$. Therefore $(x, y)$ is contained in the radical of $(x^2, ax + y)$, and so $(x, y) = \sqrt{(x^2, ax + y)}$ since $(x, y)$ is maximal. It follows from the above result that $(x^2, ax + y)$ is primary.

An alternate proof can be given by making use of the automorphism of $F[x, y]$ that is defined by substituting $ax + y$ for $y$. This is a ring homomorphism, and substituting $y - ax$ defines an inverse. Under this automorphism, the ideal $(x^2, y)$ corresponds to $(x^2, ax + y)$, and we already know that $(x^2, y)$ is primary since modulo $(y)$ it is a power of a maximal ideal.
In class

1. (25 pts) State 3 of the following definitions: (a) projective module; injective module; (b) the tensor product of modules $M_R$ and $N_R$; (c) for any ring) prime ideal; primitive ideal; maximal ideal; (d) the biendomorphism ring of the module $R^M$.

2. (25 pts) State 3 of the following theorems, including any necessary definitions not given in #1:
   (a) the Hilbert basis theorem; (b) the Jordan–Hölder theorem; (c) the Krull–Schmidt theorem; (d) the fundamental theorem for finitely generated modules over a PID; (e) the Artin–Wedderburn theorem; (f) Hopkins’s theorem; (g) the Jacobson density theorem.

3. (100 points) Prove 4 of the following:
   (a) Prove that if $R^M$ is a module, and $N$ is a submodule of $M$, then $M$ is Noetherian if and only if both $N$ and $M/N$ are Noetherian. Solution: This is Proposition 2.4.5 (a).

   (b) Let $M$ be a left module, and let $M^n$ denote the direct sum of $n$ copies of $M$, with elements written as column vectors. (i) Show that $M^n$ is a module over the matrix ring $S = M_n(R)$. (ii) Show that there is a one-to-one correspondence between $S$-submodules of $M^n$ and $R$-submodules of $M$.
   Solution: (i) Multiplication of a matrix with entries from $R$ and a column vector with entries from $M$ is a well-defined operation. The associativity of matrix multiplication can be extended to this situation—I did not expect a complete proof. I just expected you to mention the necessary equations: $(AB)X = A(BX)$, $A(X + Y) = AX + AY$, $(A + B)X = AX + BX$, $IX = X$. (ii) It is clear that if $N$ is a submodule of $M$, then $N^n$, the set of all column vectors whose entries come from $N$, is a submodule of $M^n$. We want to know that the converse is more interesting to prove, and requires more work. If $K$ is any submodule of $M^n$, let $N$ be the set of all elements of $M$ that occur as the first entry of an element of $K$ that has zero’s in all of the remaining entries. It is easy to check that $N$ is a submodule of $M$, since it is clearly closed under addition and scalar multiplication. Finally, if $x$ is any element of $K$, it is possible to show that each entry of $x$ belongs to $N$, since we can multiply $x$ by a matrix unit which shifts the entry $x_i$ to the first position and annihilates every other entry.

   (c) If $R$ is any ring, and $I$ is an ideal of $R$, prove that $(R/I) \otimes_R (R/I)$ is isomorphic to $R/I$ (as abelian groups).
   Solution: The proof is a special case of Exercise 7 in Section 2.6, but a direct proof can be given more easily. Let $\pi : R \to R/I$ be the natural projection. Then $\pi \otimes 1 : R \otimes_R (R/I) \to (R/I) \otimes_R (R/I)$ is onto, and its kernel is generated by elements of the form $r \otimes 0$ and $a \otimes x$, where $a \in \ker(\pi)$. Since $\ker(\pi) = I$, we have $a \otimes x = 1 \otimes ax = 1 \otimes 0 = 0$. Thus $\pi \otimes 1$ is an isomorphism, and we can combine this with the known isomorphism from $R \otimes_R (R/I)$ onto $R/I$ to prove that $(R/I) \otimes_R (R/I)$ is isomorphic to $R/I$.

   (d) Let $R$ be a ring, with Jacobson radical $J$. (i) Prove that $J$ cannot contain a nonzero idempotent element of $R$. (ii) Prove that if $R$ is left Noetherian, then $J$ cannot contain a nonzero idempotent ideal of $R$
   Solution: Note the correction in the statement of part (i): $J$ contains 0, which is definitely idempotent.
   (i) If $x$ is idempotent, then $e = 1 - x$ is also idempotent, and if $x \in J$, then $e$ is invertible. Thus $e^2 = e$ implies that $e = 1$, and hence $x = 0$, a contradiction.
   (ii) If $I \subseteq J$ and $I^2 = I$, then $JI = I$, and so we can apply Nakayama’s lemma, since by assumption $I$ is finitely generated. It follows that $I = (0)$, a contradiction.
(e) Let $R M$ be a module, and let $f \in \text{End}_R(M)$. (i) Show that if $\ker(f^2) = \ker(f)$, then $\ker(f) \cap \text{Im}(f) = (0)$. (ii) Show that if $\text{Im}(f^2) = \text{Im}(f)$, then $\ker(f) + \text{Im}(f) = M$.

Solution: (i) Let $m \in \ker(f) \cap \text{Im}(f)$. Then there exists $x \in M$ with $m = f(x)$, and so $f^2(x) = f(m) = 0$, showing that $x \in \ker(f)$. By assumption, $x \in \ker(f^2)$, and so $m = f(0) = 0$.

(ii) Given $m \in M$, by assumption there exists $x \in M$ with $f(m) = f^2(x)$. Let $m = (m - f(x)) + f(x)$. Then $f(x) \in \text{Im}(g)$, and $m - f(x) \in \ker(f)$ since $f(m - f(x)) = f(m) - f^2(x) = 0$.

(f) Prove that the ring $R$ is a prime ring if and only if $M_n(R)$ is a prime ring.

Solution: My solution has two ingredients. First, $R$ is a prime ring if and only if $AB = (0)$ implies $A = (0)$ or $B = (0)$, for all ideals of $R$. Secondly, the ideals of $M_n(R)$ are in one-to-one correspondence with those of $R$, where the ideal $A$ of $R$ corresponds to the ideal $M_n(A)$ consisting of all $n \times n$ matrices with entries in $A$.

Let $A$ and $B$ be ideals of $R$. Since the product $AB$ consists of sums of products of the form $ab$, for $a \in A$ and $b \in B$, each entry of any matrix in the product $M_n(A) \cdot M_n(B)$ belongs to $AB$. Thus $AB = (0)$ if and only if $M_n(A) \cdot M_n(B) = (0)$. It follows immediately that $R$ is prime if and only if $M_n(R)$ is prime.

(g) Prove that a left Artinian ring with no nonzero divisors of zero is a division ring.

Solution: In a left Artinian ring, the Jacobson radical is nilpotent, so it consists of divisors of zero. We conclude that the radical is zero, and so $R$ is semisimple. If $e \neq 1$ is a nonzero idempotent, the $e$ is a divisor of zero since $e(1 - e) = 0$. Thus $R$ cannot be a nontrivial sum of minimal left ideals, so $RR$ is a simple module, and hence $R$ is a division ring since $(0)$ is a maximal left ideal.