Directions: Choose two questions from part A, one from part B, and one from part C. The questions are worth 25 points each. I suggest 1, 3, 5, and 8.

Part A (choose two)

about central idempotents and direct sums of rings:
1. (a) Let \( R \) be a ring, let \( e_1 \) be an idempotent \((e_1^2 = e_1)\) element in the center of \( R \), and let \( e_2 = 1 - e_1 \). Show that \( Re_1 \) and \( Re_2 \) are subrings of \( R \), and that each element of \( R \) can be represented uniquely in the form \( ae_1 + be_2 \), for some \( a, b \in R \).

(b) Let \( R \) be a ring isomorphic to the direct sum \( S \oplus T \) of rings \( S \) and \( T \). Prove that \( M_n(R) \) is isomorphic to the direct sum of the rings \( M_n(S) \oplus M_n(T) \).

Hint: Find central idempotent elements \( e_1 \) and \( e_2 \) of \( R \) such that \( e_1 + e_2 = 1 \), where there are obvious ring isomorphisms between \( Re_1 \) and the ring \( M_n(S) \), and between \( Re_2 \) and the ring \( M_n(T) \).

about group rings:
2. Let \( F \) be a field, and let \( G \) be a finite group.

(a) Give the definition of the group ring \( FG \). That is, describe the elements of \( FG \), and give the addition and multiplication in \( FG \).

(b) For a ring \( R \), give the definition of the opposite ring \( R^{\text{op}} \).

(c) Show that \( FG \) is isomorphic to \((FG)^{\text{op}} \). You may assume the fact that any group homomorphism from \( G \) into the group of units of \((FG)^{\text{op}} \) will extend uniquely to a ring homomorphism from \( FG \) into \((FG)^{\text{op}} \).

about localization at a prime ideal:
3. Let \( D \) be an integral domain, and let \( P \) be a prime ideal of \( P \).

(a) Describe the construction of the localization \( D_P \) of \( D \) at \( P \). Describe the connection between the ideals of \( D \) and those of \( D_P \).

(b) Prove that if every ideal of \( D \) is finitely generated, then the same holds for every ideal of the localization \( D_P \).

about polynomial rings and Gauss’s lemma:
4. (a) Show that if \( P \) is a prime ideal of the commutative ring \( R \), then \( P[x] \) is a prime ideal of the polynomial ring \( R[x] \). (You may assume that \( P[x] \), the set of all polynomials with coefficients in \( P \), is an ideal of \( R[x] \).)

(b) Use part (a) to prove Gauss’s lemma. That is, prove that if \( D \) is a unique factorization domain, then the product of two primitive polynomials in \( D[x] \) is again a primitive polynomial. (Recall that a nonconstant polynomial \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \) in \( D[x] \) is called primitive if there is no irreducible element \( p \in D \) such that \( p | a_i \) for all \( i \).)
Part B (choose one)

about simple modules:
5. Let $M$ be a left $R$-module.
   (a) Prove that $M$ is a simple module if and only if $M = Rm$, for all nonzero $m \in M$.
   (b) Prove that $M$ is simple if and only if $M \cong R/A$ for a maximal left ideal $A \subseteq R$.
   (c) (Schur’s lemma) Prove that if $M$ is simple, then $\text{End}_R(M)$ is a division ring.

about the homework problem on splitting, and a generalization:
6. Let $M$ be a left $R$-module, and let $f$ be an $R$-endomorphism of $M$.
   (a) Prove that if $f^2 = f$, then $M = \ker(f) \oplus \text{Im}(f)$.
   (b) Prove that if there exists an integer $n$ such that $\text{Im}(f^n) = \text{Im}(f^{n+1}) = \ldots = \text{Im}(f^{2n})$ and $\ker(f^n) = \ker(f^{n+1}) = \ldots = \ker(f^{2n})$, then $M = \ker(f^n) \oplus \text{Im}(f^n)$.

about a new definition related to the socle:
7. Let $M$ be a left $R$-module. A submodule $K \subseteq M$ is said to be essential if $K \cap N \neq (0)$ for all nonzero submodules $N \subseteq M$.
   (a) Show that $\mathbb{Z}$ is an essential submodule of $\mathbb{Q}$.
   (b) Give the definition of the socle of $M$, and prove that it is equal to the intersection of all essential submodules of $M$.

Part C (no choice)

about the modular law, and an application:
8. Let $M$ be a left $R$-module, with a submodule $N$.
   (a) Prove the modular law. That is, prove that if $K \subseteq L$ are any submodules of $M$, then $L \cap (K + N) = K + (L \cap N)$.
   (b) Prove that if $K \subseteq L$ are any submodules of $M$ such that $K \cap N = L \cap N$ and $(K + N)/N = (L + N)/N$, then $K = L$. 


The final exam is scheduled for Wednesday, May 5, at 10:00 am. The first part, worth 25%, consists of three takehome questions on primary decomposition in commutative rings. The solutions are due at the start of the exam. The remaining 75% will be in-class questions from Sections 2.4–2.7 and 3.1–3.3.

5.3.2. (15 pts) Let $I, J$ be ideals of the commutative ring $R$. Show that if $\sqrt{I} + \sqrt{J} = R$, then $I + J = R$.

5.3.3. (15 pts) Prove that if $I, Q$ are ideals of the commutative ring $R$ with $I \subseteq Q$, then $Q$ is a primary ideal of $R$ if and only if $Q/I$ is a primary ideal of $R/I$.

5.3.5. (20 pts) Let $F$ be a field, and consider the ideal $I = (x^2, xy)$ of $F[x, y]$.
   (a) Show that $I$ is not a primary ideal.
   (b) Show that $I = (x) \cap (x^2, y)$.
   (c) Show that $(x^2, ax + y)$ is a primary ideal, for any $a \in F$, and show that $I = (x) \cap (x^2, ax + y)$, so that $I$ can be represented in infinitely many ways as an intersection of primary ideals.
1. (25 pts) State 3 of the following definitions:
   (a) projective module; injective module;
   (b) the tensor product of modules $M_R$ and $RN$;
   (c) (for any ring) prime ideal; primitive ideal; maximal ideal;
   (d) the biendomorphism ring of the module $R_M$.

2. (25 pts) State 3 of the following theorems, including any necessary definitions not given in question #1:
   (a) the Hilbert basis theorem;
   (b) the Jordan–Hölder theorem;
   (c) the Krull–Schmidt theorem;
   (d) the fundamental theorem for finitely generated modules over a PID;
   (e) the Artin–Wedderburn theorem;
   (f) Hopkins’s theorem;
   (g) the Jacobson density theorem.

3. (100 points) Prove 4 of the following:
   (a) Prove that if $R_M$ is a module, and $N$ is a submodule of $M$, then $M$ is Noetherian if and only if both $N$ and $M/N$ are Noetherian.
   (b) Let $M$ be a left module, and let $M^n$ denote the direct sum of $n$ copies of $M$, with elements written as column vectors.
      (i) Show that $M^n$ is a module over the matrix ring $S = M_n(R)$.
      (ii) Show that there is a one-to-one correspondence between $S$-submodules of $M^n$ and $R$-submodules of $M$.
   (c) If $R$ is any ring, and $I$ is an ideal of $R$, prove that $(R/I) \otimes_R (R/I)$ is isomorphic to $R/I$ (as abelian groups).
   (d) Let $R$ be a ring, with Jacobson radical $J$.
      (i) Prove that $J$ cannot contain a nonzero idempotent element of $R$.
      (ii) Prove that if $R$ is left Noetherian, then $J$ cannot contain a nonzero idempotent ideal of $R$ (i.e. $I^2 = I$ implies $I = (0)$, for any ideal $I \subseteq J$).
   (e) Let $R_M$ be a module, and let $f \in \text{End}_R(M)$.
      (i) Show that $\ker(f^2) = \ker(f)$, then $\ker(f) \cap \text{Im}(f) = (0)$.
      (ii) Show that if $\text{Im}(f^2) = \text{Im}(f)$, then $\ker(f) + \text{Im}(f) = M$.
   (f) Prove that the ring $R$ is a prime ring if and only if $M_n(R)$ is a prime ring.
   (g) Prove that a left Artinian ring with no nonzero divisors of zero is a division ring.