Chapter 3

Functions and Their Graphs

Section summaries

Section 3.1 Functions

A function from a set X to a set Y is a rule or correspondence that associates with each element of X exactly one element of Y. (In Math 110, these sets usually consist of real numbers.) With the function notation $y = f(x)$, each x value has only one corresponding y value.

You can think of a function as being like a program. The x-values are the inputs, and the y-values are the outputs. The possible inputs form the domain of the function, and the possible outputs form its range. For the functions that we are dealing with, the numbers that we need to exclude from the domain are numbers that lead to division by zero, or the square root of a negative number. (If the function was a program, trying to divide by zero or take the square root of a negative number would give an error message.)

Given two functions, we can make a new function from their sum, difference, product, or quotient. (See pages 217–218.)

Review problems: p219 #39,41,51,59,61,75,79,89

Section 3.2 The Graph of a Function

The **vertical line test**: A set of points in the (x, y)-plane is the graph of a function precisely when every vertical line intersects the set in at most one point.

Review problems: p226 #9,25,27,29

Section 3.3 Properties of Functions

A function is called **even** if $f(-x) = f(x)$ (the graph is symmetric about the y-axis) and **odd** if $f(-x) = -f(x)$ (the graph that is symmetric about the origin).

A function is **increasing** on an interval if its values keep going up, and **decreasing** on an interval if its values keep going down. A high point on the graph is called a **local**

maximum, and this corresponds to a change from increasing to decreasing. A low point on the graph is called a **local minimum**, and corresponds to a change from decreasing to increasing. *Note: Since we are not using calculators, you won't be asked to actually compute local maximum and local minimum values in this section.*

The average rate of change of a function is found by dividing the change in y by the change in x. If you go from x to c on the x-axis, then the corresponding change in y is $f(x) - f(c)$. We get this formula for the average rate of change: $\dfrac{f(x) - f(c)}{x - c}$, where $x \neq c$.

Review problems: p 239 #33,35,39,43,53,61,63

Section 3.4 A Library of Functions

This section gives the graphs of some functions you need to be able to recognize and to graph on your own.

Straight lines: $f(x) = mx + b$

The square function: $f(x) = x^2$ The cube function: $f(x) = x^3$

The square root function: $f(x) = \sqrt{x}$ The cube root function: $f(x) = \sqrt[3]{x}$

The reciprocal function: $f(x) = \dfrac{1}{x}$ The absolute value function: $f(x) = |x|$

There is no reason that a function has to have the same formula at each point in its domain. Of course, which formula is used for which numbers has to be spelled out very carefully. These functions get their name from being defined in "pieces". Example: the absolute value function $f(x) = |x|$ takes any number and makes it non-negative. It is convenient to express this with two different formulas: if x is already positive or zero, we don't need to make any change. But if x is negative, we need to change the sign.

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Review problems: p249 #19,21,25,31,37

Section 3.5 Graphing Techniques: Transformations

The basic model for a linear function is $f(x) = x$, whose graph is a straight line through the origin that slopes up at a 45^o angle. The family of linear functions includes all functions of the form $f(x) = ax + b$. We can get all of these by multiplying the basic example by a and adding b. The numbers a and b tell us all about the new line: if a is positive it slopes up, if a is negative it slopes down; b gives the y-intercept, and tells how far the line has been moved up (if $b > 0$) or down (if $b < 0$).

In Section 3.4, besides linear functions, we studied the basic examples of several families of functions:

quadratic functions	$f(x) = x^2$	cubic functions	$f(x) = x^3$
square root functions	$f(x) = \sqrt{x}$	cube root functions	$f(x) = \sqrt[3]{x}$
reciprocal functions	$f(x) = \dfrac{1}{x}$	absolute value functions	$f(x) = \mid x \mid$

In Section 3.5 we study the graphs that we get when the basic examples in each family are shifted up or down, shifted left or right, stretched or compressed, or reflected about one of the axes. We start with a function $f(x)$, and positive numbers a, h, and k, where $a > 1$. Changing the function does the following:

$f(x) + k$	shift up by k	$f(x) - k$	shift down by k
$f(x - h)$	shift right by h	$f(x + h)$	shift left by h
$af(x)$	stretch vertically	$\dfrac{1}{a}f(x)$	compress vertically
$-f(x)$	reflect about the x-axis	$f(-x)$	reflect about the y-axis

Review problems: p261 #27,32,53,59,69

Section 3.6 Mathematical Models

Some of the models involve geometry. You should review the Pythagorean theorem on page 30, and the geometry formulas on page 31 (for the area of a rectangle, a triangle, or a circle). You need to know that the volume of a box is its length times width times height. On the exam, if you need any other formulas for volumes, the question will include the formula.

There are also some basic models related to economics. You need to remember that when items are sold the revenue (money taken in) is found by multiplying the number of items sold by the price per unit. That's just common sense, and you shouldn't have to make a big deal about remembering the formula $R(x) = px$, where x is the number sold and p is the price per unit.

Some of the problems in the text ask you to find the model and then find a maximum or minimum value, using a calculator. Obviously, though this is an important idea, we will not test you on it. But if it happens that the model gives you a quadratic function, then it *is* a fair question to ask you to find the maximum or minimum value on the exam, because you can use the techniques from Section 4.3.

Review problems: p280 #5,8,15,23

Sample Questions

3.1 A. For the function $f(x) = x^3 + x$, find $f(-2)$.

(a) 6
(b) 10
(c) -6
(d) -10
(e) None of these

3.1 Example 6. For the function $f(x) = 2x^2 - 3x$, find $f(3x)$.

(a) $36x^2 - 9x$
(b) $36x^2 - 3x$
(c) $18x^2 - 9x$
(d) $18x^2 - 3x$
(e) None of these

3.1 B. For the function $f(x) = x^2 - 2$, find $f(y + 2)$.

(a) $x^2y + 2x^2 - 2y - 4$
(b) $y^2 + 4y + 2$
(c) y^2
(d) $y^2 + 2$
(e) None of these

3.1 #51. What is the domain of the function $f(x) = \dfrac{x}{x^2 - 16}$?

(a) All real numbers
(b) All real numbers except 0
(c) All real numbers except $4, -4$
(d) All real numbers except 16
(e) None of these

3.1 C. What is the domain of the function $f(x) = \dfrac{x+1}{x-1}$?

(a) All real numbers except -1
(b) All real numbers except $1, -1$
(c) All real numbers except 1
(d) All real numbers
(e) None of these.

3.1 D. What is the domain of the function $g(x) = \dfrac{x^2+1}{x^3 - 4x}$?

(a) All real numbers except -4
(b) All real numbers except $2, -2$
(c) All real numbers except $0, 2, -2$
(d) $\{0, 2, -2\}$
(e) $\{0, 2, -2, -4\}$

3.1 #57. Find the domain of the function $f(x) = \dfrac{4}{\sqrt{x-9}}$?

(a) $(9, \infty)$
(b) $[9, \infty)$
(c) $(4/9, \infty)$
(d) $(-\infty, 9)$
(e) $(-\infty, 9]$

3.1 E. What is the domain of the function $f(x) = \dfrac{x+2}{\sqrt{5+3x}}$?

(a) $(-\infty, 0]$
(b) $(-\infty, 0)$
(c) $(-2, \infty)$
(d) $(-5/3, \infty)$
(e) $(-\infty, -5/3)$

3.1 F. Let $f(x) = 2x^2 - 4$. Find $f(x-3)$.

(a) $2x^2 - 12x + 18$
(b) $2x^2 - 12x + 14$
(c) $2x^2 - 22$
(d) $2x^2 + 14$
(e) None of these

3.1 G. If $f(x) = \dfrac{2x+1}{3x-5}$, then what is $f(3x+1)$?

(a) $f(3x+1) = 3 \cdot \left(\dfrac{2x+1}{3x-5}\right) + 1$
(b) $f(3x+1) = \dfrac{6x+2}{9x-4}$
(c) $f(3x+1) = \dfrac{6x+3}{9x-2}$
(d) $f(3x+1) = \dfrac{(2x+1)(3x+1)}{(3x-5)}$
(e) None of these

3.1 #75. For $f(x) = x^2 - x + 4$, find and simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$, where $h \neq 0$.

(a) $h + 1$
(b) $h - 1$
(c) $2x + h + 1$
(d) $2x + h - 1$
(e) None of these

3.1 #79. For $f(x) = x^3 - 2$, find and simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$, where $h \neq 0$

(a) h^2
(b) $x^2 + xh + h^2$
(c) $3x^2 + 3xh + h^2$
(d) $3x^2 + 3xh^2 + h^3$
(e) None of these

3.2 #25c. For the function $f(x) = \dfrac{x+2}{x-6}$, if $f(x) = 2$, then $x =$

(a) -1
(b) 8
(c) 14
(d) There is no answer
(e) None of these

3.2 #25e. Find the x-intercept(s) of the graph of the function $f(x) = \dfrac{x+2}{x-6}$.

(a) 2
(b) -2
(c) $2, -6$
(d) $-1/3$
(e) None of these

3.2 #25f. Find the y-intercept(s) of the graph of the function $f(x) = \dfrac{x+2}{x-6}$.

(a) 2
(b) -2
(c) $2, -6$
(d) $-1/3$
(e) None of these

3.2 #27c. For the function $f(x) = \dfrac{2x^2}{x^4+1}$, if $f(x) = 1$, then $x =$

(a) 1
(b) -1
(c) $1, -1$
(d) There is no answer
(e) None of these

3.2 #27e. Find the x-intercept(s) of the graph of the function $f(x) = \dfrac{2x^2}{x^4+1}$.

(a) 0
(b) 1
(c) 2
(d) There is no x-intercept
(e) None of these

3.2 #27f. Find the y-intercept(s) of the graph of the function $f(x) = \dfrac{2x^2}{x^4+1}$.

(a) 0
(b) 1
(c) 2
(d) There is no y-intercept
(e) None of these

3.3 #39. The function $f(x) = x + |x|$ is

(a) Odd
(b) Even
(c) Neither odd nor even
(d) Both odd and even
(e) None of these

3.3 #41. The function $f(x) = \dfrac{1}{x^2}$ is

(a) Odd
(b) Even
(c) Neither odd nor even
(d) Both odd and even
(e) None of these

3.3 #43. The function $f(x) = \dfrac{-x^3}{3x^2 - 9}$ is

(a) Odd
(b) Even
(c) Neither odd nor even
(d) Both odd and even
(e) None of these

3.3 A. If f is an odd function and (a, b) lies on the graph of f, what other point(s) must also lie on the graph of f?

(a) $(-a, b)$
(b) $(-a, -b)$
(c) $(a, -b)$
(d) $(a, -b)$ and $(-a, b)$
(e) $(a, -b)$ and $(-a, -b)$

3.3 #55. For the function $f(x) = x^3 - 2x + 1$, find the average rate of change from $x = -1$ to $x = 1$.

(a) 0
(b) 1
(c) -1
(d) 2
(e) None of these

3.3 B. For the function $f(x) = \dfrac{4}{x^2}$, the equation of the secant line joining $(1, f(1))$ and $(2, f(2))$ is:

(a) $y = 3x + 1$
(b) $y = 3x - 5$
(c) $y = -3x + 7$
(d) $y = -3x + 13$
(e) None of these

3.3 C. For $f(x) = \sqrt{x}$, the equation of the line joining $(1, f(1))$ and $(4, f(4))$ is

(a) $y = \frac{1}{3}x + \frac{2}{3}$
(b) $y = \frac{1}{3}x + \frac{10}{3}$
(c) $y = 3x$
(d) $y = 3x - 2$
(e) None of these

3.4 A. Find the x-intercept(s) and the y-intercept of the function

$$f(x) = \begin{cases} 3 + x & \text{if } -3 \le x < 0 \\ 2 & \text{if } x = 0 \\ \sqrt{x} - 1 & \text{if } 0 < x \end{cases}$$

(a) $x = -3$ and $y = 2$
(b) $x = -3$ and $y = 3$
(c) $x = 3$ and $y = 2$
(d) $x = 3$ and $y = 3$
(e) None of these

3.4 #25. If $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 2 & \text{if } x = 0 \\ 2x + 1 & \text{if } 0 < x \end{cases}$, find $f(x + 3)$ when $x > 0$.

(a) $x^2 + 6x + 9$
(b) $x^2 + 6x + 10$
(c) $2x + 6$
(d) $2x + 7$
(e) None of these

3.4 #35. If $f(x) = \begin{cases} 1 + x & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x \end{cases}$, find the range of $f(x)$.

(a) $(-\infty, 1)$
(b) $(0, \infty)$
(c) $[0, \infty)$
(d) $(-\infty, \infty)$
(e) None of these

3.5 A. Using the function $f(x) = x^3$, find the equation of the corresponding function whose graph is shifted 2 units to the right then shifted up 5 units.

(a) $f(x) = (x + 2)^3 + 5$
(b) $f(x) = (x - 2)^3 + 5$
(c) $f(x) = (x + 5)^3 + 2$
(d) $f(x) = (x - 5)^3 + 2$
(e) None of these

3.5 B. Using the function $f(x) = x^3$, find the equation of the corresponding function whose graph is shifted left 1 unit, reflected about the x-axis, and shifted up 2 units.

(a) $f(x) = -(x + 2)^3 + 1$
(b) $f(x) = -(x - 2)^3 + 1$
(c) $f(x) = -(x + 1)^3 + 2$
(d) $f(x) = -(x - 1)^3 + 2$
(e) None of these

3.5 C. Using the function $f(x) = x^3$, find the equation of the corresponding function whose graph is reflected about the x-axis and then shifted down 4 units.

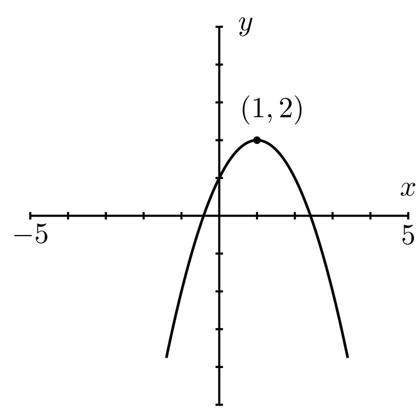
(a) $f(x) = -(x - 4)^3$
(b) $f(x) = -(x + 4)^3$
(c) $f(x) = -x^3 + 4$
(d) $f(x) = -x^3 - 4$
(e) None of these

3.5 D. If the graph of $y = 3x^2 + 4x - 5$ is reflected about the y-axis, the new equation is

(a) $y = -3x^2 - 4x + 5$
(b) $y = -3x^2 + 4x + 5$
(c) $y = 3x^2 - 4x + 5$
(d) $y = 3x^2 - 4x - 5$
(e) None of these

3.5 E. Which function represents the graph at the right?

(a) $f(x) = -(x - 1)^2 + 2$
(b) $f(x) = -(x - 2)^2 + 1$
(c) $f(x) = -(x - 1)^2 - 2$
(d) $f(x) = -(x - 2)^2 - 1$
(e) None of these



3.6 #7. A rectangle has one corner on the graph of $y = 16 - x^2$, another at the origin, a third on the positive y-axis, and the fourth on the positive x-axis. Express the area as a function of x.

(a) $A(x) = -x^2 + x + 16$
(b) $A(x) = -x^3 + 16x$
(c) $A(x) = -x^3 + 8x$
(d) $A(x) = -x^2 + 16$
(e) None of these

3.6 #13. A wire of length x is bent into the shape of a circle. Express the area $A(x)$ as a function of x.

(a) $A(x) = x^2$

(b) $A(x) = \pi x^2$

(c) $A(x) = \dfrac{x^2}{4}$

(d) $A(x) = \dfrac{x^2}{4\pi}$

(e) None of these

3.6 A. Alex has 400 feet of fencing to enclose a rectangular garden. One side of the garden lies along the barn, so only three sides require fencing. Express the area $A(x)$ of the rectangle as a function of x, where x is the length of the side perpendicular to the side of the barn.

(a) $A(x) = -x^2 + 200x$

(b) $A(x) = -x^2 + 400x$

(c) $A(x) = -2x^2 + 200x$

(d) $A(x) = -2x^2 + 400x$

(e) None of these

3.6 B. Germaine has 40 feet of fencing to enclose a rectangular pool. One side of the pool lies along the house, so only three sides require fencing. Express the area $A(x)$ of the rectangle as a function of x, where x is the length of the side perpendicular to the side of the house.

(a) $A(x) = -x^2 + 40x$

(b) $A(x) = -x^2 + 20x$

(c) $A(x) = -2x^2 + 40x$

(d) $A(x) = -2x^2 + 20x$

(e) None of these

Answer Key

3.1 A. (d)

3.1 Example 6. (c)

3.1 B. (b)

3.1 #51. (c)

3.1 C. (c)

3.1 D. (c)

3.1 #57. (a)

3.1 E. (d)

3.1 F. (b)

3.1 G. (c)

3.1 #75. (d)

3.1 #79. (c)

3.2 #25c. (c)

3.2 #25e. (b)

3.2 #25f. (d)

3.2 #27c. (c)

3.2 #27e. (a)

3.2 #27f. (a)

3.3 #39. (c)

3.3 #41. (b)

3.3 #43. (a)

3.3 A. (b)

3.3 #55. (c)

3.3 B. (c)

3.3 C. (a)

3.4 A. (e)

3.4 #25. (d)

3.4 #35. (d)

3.5 A. (b)

3.5 B. (c)

3.5 C. (d)

3.5 D. (d)

3.5 E. (a)

3.6 #7. (b)

3.6 #13. (d)

3.6 A. (d)

3.6 B. (c)

Solutions

3.1 A. For the function $f(x) = x^3 + x$, find $f(-2)$.
Solution: (d) $f(-2) = (-2)^3 + (-2) = -8 - 2 = -10$

3.1 B. For the function $f(x) = x^2 - 2$, find $f(y + 2)$.
Solution: (b) $f(y+2) = (y+2)^2 - 2 = (y+2)(y+2) - 2 = (y^2+4y+4) - 2 = y^2+4y+2$

3.1 C. What is the domain of the function $f(x) = \dfrac{x+1}{x-1}$?
Solution: (c) We only need to worry about division by 0, so we need to exclude $x = 1$ and the answer is: All real numbers except 1.

3.1 D. What is the domain of the function $g(x) = \dfrac{x^2+1}{x^3-4x}$?
Solution: (c) Set the denominator equal to 0 to see which numbers to exclude.
$x^3 - 4x = 0 \qquad x(x^2-4) = 0 \qquad x(x-2)(x+2) = 0$
Answer: All real numbers except $0, 2, -2$.

3.1 E. What is the domain of the function $f(x) = \dfrac{x+2}{\sqrt{5+3x}}$?
Solution: (d) We need to exclude numbers that lead to division by zero, or to the square root of a negative number. Solve: $0 < 5 + 3x \qquad -5 < 3x \qquad -\frac{5}{3} < x$
Answer (in interval notation): $(-5/3, \infty)$

3.1 F. Let $f(x) = 2x^2 - 4$. Find $f(x - 3)$.
Solution: (b) $f(x-3) = 2(x-3)^2 - 4 = 2(x^2 - 6x + 9) - 4 = 2x^2 - 12x + 18 - 4$
Answer: $f(x-3) = 2x^2 - 12x + 14$

3.1 G. If $f(x) = \dfrac{2x+1}{3x-5}$, then what is $f(3x + 1)$?
Solution: (c) You can think of the function this way: $f(\quad) = \dfrac{2(\quad)+1}{3(\quad)-5}$
Now substitute $3x + 1$ into the empty slots that used to be x.
$$f(3x+1) = \frac{2(3x+1)+1}{3(3x+1)-5} = \frac{(6x+2)+1}{(9x+3)-5} = \frac{6x+3}{9x-2}$$

3.3 A. If f is an odd function and (a, b) lies on the graph of f, what other point(s) must also lie on the graph of f?

Solution: (b) The point symmetric about the origin must be on the graph, and this is the point $(-a, -b)$. Another way to see this is that if you substitute in the negative value $x = -a$, then the y value must change sign too.

3.3 B. For the function $f(x) = \dfrac{4}{x^2}$, the equation of the secant line joining $(1, f(1))$ and $(2, f(2))$ is:

Solution: (c) First find the slope m of the line segment joining the two points.
$f(1) = \dfrac{4}{1^2} = 4$ and $f(2) = \dfrac{4}{2^2} = 1$ so $m = \dfrac{f(2) - f(1)}{2 - 1} = \dfrac{1-4}{1} = -3$
This reduces the choices to (c) $y = -3x + 7$ or (d) $y = -3x + 13$. Substituting $x = 1$ into equation (c) does give the correct y value of 4, so it must be the right equation.

3.3 C. For $f(x) = \sqrt{x}$, the equation of the line joining $(1, f(1))$ and $(4, f(4))$ is

Solution: (a) $m = \dfrac{f(4) - f(1)}{4 - 1} = \dfrac{\sqrt{4} - \sqrt{1}}{4 - 1} = \dfrac{1}{3}$ so the answer could be (a) $y = \frac{1}{3}x + \frac{2}{3}$ or (b) $y = \frac{1}{3}x + \frac{10}{3}$. Equation (a) is correct since $f(1) = 1$.

3.4 A. Find the x-intercept(s) and the y-intercept of the function

$$f(x) = \begin{cases} 3 + x & \text{if } -3 \le x < 0 \\ 2 & \text{if } x = 0 \\ \sqrt{x} - 1 & \text{if } 0 < x \end{cases}$$

Solution: (e) To find the y-intercept, substitute $x = 0$ into the formula. This means we need to use the middle formula, so $y = 2$.
To find the x-intercepts, we need to solve the first and last equations for $y = 0$. In the first equation we get $3 + x = 0$, so $x = -3$. This is a legal value in the domain of this part of the formula, so it does give us an x-intercept. In the last equation, setting $\sqrt{x} - 1 = 0$ gives us $x = 1$, and again this is a legal value in the domain of this part of the formula.
Answer: $x = -3$, $x = 1$, and $y = 2$, which is not one of the choices (a) – (d).

3.5 A. Using the function $f(x) = x^3$, find the equation of the corresponding function whose graph is shifted 2 units to the right then shifted up 5 units.

Solution: (b) $f(x) = (x - 2)^3 + 5$

3.5 B. Using the function $f(x) = x^3$, find the equation of the corresponding function whose graph is shifted left 1 unit, reflected about the x-axis, and shifted up 2 units.

Solution: (c) Shift left one unit: $f(x) = (x + 1)^3$ Then reflect about the x-axis: $f(x) = -(x + 1)^3$ Finally shift up 2 units: $f(x) = -(x + 1)^3 + 2$

3.5 C. Using the function $f(x) = x^3$, find the equation of the corresponding function whose graph is reflected about the x-axis and then shifted down 4 units.

Solution: (d) $f(x) = -x^3 - 4$

3.5 D. If the graph of $y = 3x^2 + 4x - 5$ is reflected about the y-axis, the new equation is
Solution: (d) Substitute $-x$ to get $y = 3(-x)^2 + 4(-x) - 5 = 3x^2 - 4x - 5$

3.5 E. Which function represents the graph at the right?

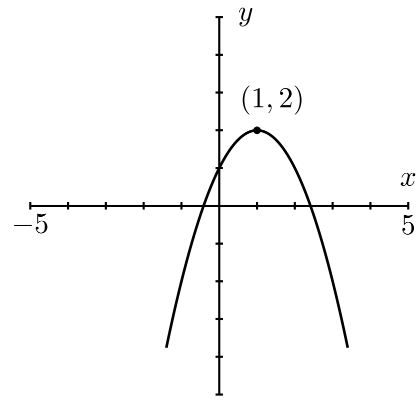
(a) $f(x) = -(x-1)^2 + 2$

(b) $f(x) = -(x-2)^2 + 1$

(c) $f(x) = -(x-1)^2 - 2$

(d) $f(x) = -(x-2)^2 - 1$

(e) None of these



Solution: (a) The graph is that of $y = x^2$ shifted 1 unit to the right, reflected about the x-axis, and then shifted up 2 units.

3.6 A. Alex has 400 feet of fencing to enclose a rectangular garden. One side of the garden lies along the barn, so only three sides require fencing. Express the area $A(x)$ of the rectangle as a function of x, where x is the length of the side perpendicular to the side of the barn.
Solution: (d) The three sides are x, x, and $400 - 2x$, so the total area is length × width, giving $A(x) = x(400 - 2x) = 400x - 2x^2 = -2x^2 + 400x$.

3.6 B. Germaine has 40 feet of fencing to enclose a rectangular pool. One side of the pool lies along the house, so only three sides require fencing. Express the area $A(x)$ of the rectangle as a function of x, where x is the length of the side perpendicular to the side of the house.
Solution: (c) The three sides are x, x, and $40 - 2x$, giving $A(x) = x(40 - 2x) = 40x - 2x^2 = -2x^2 + 40x$.