Chapter 5

Polynomial and Rational Functions

Section summaries

Section 5.1 Polynomial Functions

The **general form of a polynomial function** is $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. The **degree** of $f(x)$ is the largest exponent in the formula. Linear functions $f(x) = mx + b$ and quadratic functions $f(x) = ax^2 + bx + c$ are the simplest cases. If $|x|$ is large, then the term $a_n x^n$ is much larger than the others, so the "big picture" of $f(x)$ is that its graph follows the pattern of x^n, flipped over if a_n is negative.

The number of different x-intercepts of a polynomial of degree n is at most n, because a polynomial equation of degree n has at most n roots. The same is true of any horizontal line–the graph of a polynomial of degree n can cross the line at most n times. This means that the graph has at most $n - 1$ "turning points" (see the discussion on the top of page 321), and this helps you in graphing.

Finally, a polynomial has two types of behavior at an x-intercept. It may cross the x-axis, like $y = x^3$, or it may just touch the x-axis, like $y = x^2$. If you can factor the function completely, you can tell whether it crosses or touches by looking at the exponent of the factor that corresponds to the root you are interested in. If the exponent is odd, the graph will cross the axis because the y-values will change sign, but if the exponent is even, the graph will just touch the axis and stay on the same side.

You should review the summaries on pages 336 and 338 very carefully.

Review problems: p340 #31,37,45,55,63,71,75

Section 5.2 Properties of Rational Functions

We have been building up to more and more complicated functions. This section deals with some basic properties of functions of the form $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials. These are called **rational** functions. The functions $f(x) = \dfrac{1}{x}$ and $f(x) = \dfrac{1}{x^2}$ are two familiar examples.

Just as $f(x) = \frac{1}{x}$ has a graph that is asymptotic to the axes, a general rational function can have horizontal and vertical asymptotes. It may or may not cross the x-axis.

x-intercepts: The only way $f(x)$ can be zero is if the numerator is zero, so you can find the x-intercepts by setting the numerator equal to zero, and solving the equation $p(x) = 0$.

vertical asymptotes: These can be found by looking at the values of x at which $f(x)$ is not defined (because of division by zero). You just need to set the denominator equal to zero, and solve the equation $q(x) = 0$. Note: you must first make sure that the numerator and denominator do not have any common factors.

horizontal asymptotes: if $|x|$ is large, the function

$$f(x) = \frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + a_1 x + a_0}$$

behaves like $y = \frac{a_m x^m}{b_n x^n}$, just like we learned for polynomials. If the numerator and denominator have the same degree, this reduces to a constant, and gives the equation of the asymptote. If the denominator has larger degree than the numerator, then $y = 0$ is a horizontal asymptote. If the numerator has larger degree than the denominator, then there is no horizontal asymptote (you will not be tested on oblique asymptotes). To just find the horizontal asymptotes you do *not* need to use long division to write $f(x)$ as a “mixed fraction.”

Review problems: p352 #15,21,27,35,39,41,45,47

Section 5.4 Polynomial and Rational Inequalities

The method of solution is given on page 370. In terms of the graph of a polynomial or rational function, we need to determine when the graph is above or below the x-axis. We first decide where the graph can change from positive to negative, and from negative to positive. Then if we know that we have found an interval on which the graph cannot change sign, it is enough to test one point in the interval. Just be very careful in working these problems.

Review problems: p373 #5,13,25,29,33,51

Sample Questions

5.1 #38. Form a polynomial function of degree 3 with zeros -2, 2, 3.

(a) $f(x) = (x^2 - 4)(x - 3)^2$

(b) $f(x) = (x^2 - 4)(x + 3)^2$

(c) $f(x) = (x^2 - 4)(x - 3)$

(d) $f(x) = (x^2 - 4)(x + 3)$

(e) None of these

5.1 A. The polynomial function $f(x)$ has a zero at $x = 2$ with multiplicity 3. We know that

(a) since 3 is an odd number, the graph touches but does not cross the x-axis.

(b) since 3 is an odd number, the graph crosses the x-axis.

(c) since 2 is an even number, the graph touches but does not cross the x-axis.

(d) since 2 is an even number, the graph crosses the x-axis.

(e) none of these occurs.

5.1 B. Find all the zeros and their multiplicities for the polynomial $p(x) = 11x(x-1)^5(x+6)$.

(a) -1 is a zero of multiplicity 5; 6 is a zero of multiplicity 1; 0 is a zero of multiplicity 1

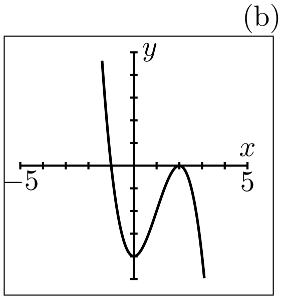
(b) 1 is a zero of multiplicity 5; -6 is a zero of multiplicity 1; 0 is a zero of multiplicity 1

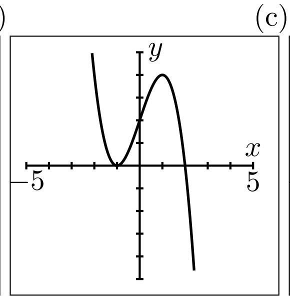
(c) -1 is a zero of multiplicity 5 and 6 is a zero of multiplicity 1

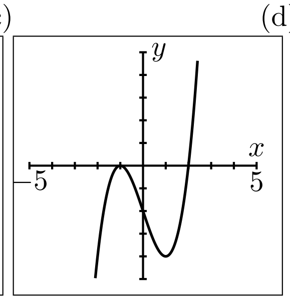
(d) 1 is a zero of multiplicity 5; -6 is a zero of multiplicity 1.

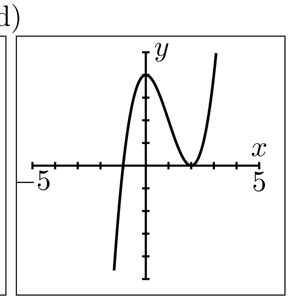
(e) None of these

5.1 C. Which of the following is the graph of $f(x) = (x + 1)^2(x - 2)$?

(a)


(b)


(c)


(d)


5.1 D. The function $f(x) = x^2(x - 2)(x + 3)^2$ has

(a) one zero of multiplicity 1 and one zero of multiplicity 2.

(b) one zero of multiplicity 1 and two zeros of multiplicity 2.

(c) one zero of multiplicity 1 and three zeros of multiplicity 2.

(d) three zeros of multiplicity 1 and one zero of multiplicity 2.

(e) None of these

5.1 E. Which one of these functions might have the given graph?

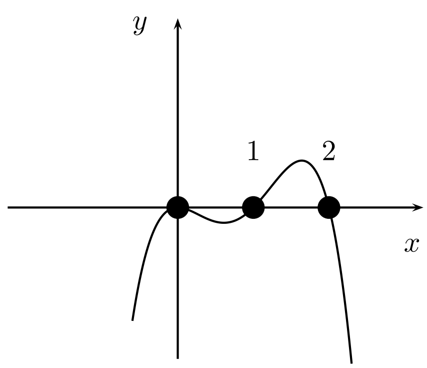
(a) $f(x) = x(x-1)(x-2)^2$

(b) $f(x) = -x(x-1)(x-2)$

(c) $f(x) = x(x-1)(x-2)$

(d) $f(x) = x^2(x-1)(x-2)$

(e) $f(x) = -x^2(x-1)(x-2)$



5.2 A. Find the domain of the function $f(x) = \dfrac{x-2}{x+1}$.

(a) All real numbers except -1

(b) All real numbers except 1

(c) All real numbers except -2

(d) All real numbers except 2

(e) None of these

5.2 B. (c) What is the domain of the function G defined by $G(x) = \dfrac{x+4}{x^3-4x}$?

(a) all reals except -4

(b) all reals except $2, -2$

(c) all reals except $0, 2, -2$

(d) $\{0, 2, -2\}$

(e) $\{0, 2, -2, -4\}$

5.2 C. The graph of $y = \dfrac{1}{(x-4)^2}$ looks like that of $y = \dfrac{1}{x^2}$ but is shifted

(a) left 4 units

(b) right 4 units

(c) down 4 units

(d) up 4 units

(e) None of these

5.2 D. Find the vertical asymptotes of the graph of $f(x) = \dfrac{x^2 - 3x}{x^2 - 2x - 8}$.

(a) $x = -4$ and $x = 2$

(b) $x = 4$ and $x = -2$

(c) $x = 0$ and $x = 3$

(d) $x = 4$, $x = -2$, $x = 0$ and $x = 3$

(e) None of these

5.2 E. The line $x = 4$ is a vertical asymptote of the graph of which of the following functions?

(a) $f(x) = x - 4$

(b) $f(x) = \dfrac{2x - 8}{x - 4}$

(c) $f(x) = \dfrac{1}{x^2 - 16}$

(d) $f(x) = \dfrac{x - 4}{x + 3}$

(e) $f(x) = \dfrac{4x + 1}{x + 2}$

5.2 F. Find the x-intercepts and vertical asymptotes of the graph of $f(x) = \dfrac{x^2 - 3x}{x^2 - 2x - 8}$

(a) x-intercepts $(4, 0), (3, 0), (-2, 0), (0, 0)$, vertical asymptotes $x = 4, x = 3, x = -2, x = 0$

(b) x-intercepts $(-4, 0), (2, 0)$, vertical asymptotes $x = 0, x = 3$

(c) x-intercepts $(4, 0), (-2, 0)$, vertical asymptotes $x = 0, x = 3$

(d) x-intercepts $(0, 0), (3, 0)$, vertical asymptotes $x = 4, x = -2$

(e) x-intercepts $(0, 0), (3, 0)$, vertical asymptotes $x = -4, x = 2$

5.2 G. Find the vertical asymptotes of $f(x) = \dfrac{(x - 1)(x + 2)(x - 3)}{x(x - 4)^2}$.

(a) $x = 1, -2, 3$

(b) $x = 0, 1, -2, 3, 4$

(c) $x = 0, 4$

(d) $x = 4$

(e) None of these

5.2 H. Find the horizontal asymptote of the graph of $f(x) = \dfrac{x - 3}{5x + 2}$.

(a) $y = \frac{1}{5}$

(b) $y = -\frac{3}{2}$

(c) $x = -\frac{2}{5}$

(d) $x = -\frac{1}{3}$

(e) None of these

5.2 K. Find the horizontal asymptote for the graph of $f(x) = \dfrac{5x-1}{2x+3}$.

(a) $x = -\frac{3}{2}$
(b) $x = \frac{5}{2}$
(c) $y = -\frac{3}{2}$
(d) $y = \frac{5}{2}$
(e) None of these

5.2 #48. Find the horizontal asymptote (if any) of $f(x) = \dfrac{-2x^2+1}{2x^3+4x^2}$.

(a) $y = -2$
(b) $y = -1$
(c) $y = 1$
(d) $y = 0$
(e) None of these

5.2 L. Which one of these functions does **not** have a horizontal asymptote?

(a) $f(x) = \dfrac{2}{3x-5}$
(b) $f(x) = \dfrac{2x^2+1}{3x-5}$
(c) $f(x) = \dfrac{2x^2+1}{3x^2-5}$
(d) $f(x) = \dfrac{2x}{3x^2-5}$
(e) $f(x) = 2 + \dfrac{6}{3x^2-5}$

5.2 M. Find the asymptotes of the following function. $f(x) = \dfrac{3x^2-3x}{x^2+x-12}$

(a) The horizontal asymptote is $y = 0$; the vertical asymptotes are $x = -4$ and $x = 3$.
(b) The horizontal asymptote is $y = 3$; the vertical asymptotes are $x = 4$ and $x = -3$.
(c) The horizontal asymptote is $y = 3$; the vertical asymptotes are $x = -4$ and $x = 3$.
(d) The horizontal asymptote is $y = -4$; the vertical asymptote is $x = 3$.
(e) None of these

5.4 #5. Solve the inequality $x^3 - 4x^2 > 0$.

(a) $(4, \infty)$
(b) $(0, \infty)$
(c) $(-\infty, 4)$
(d) $(-\infty, 0)$
(e) None of these

5.4 #13. Solve the inequality $(x-1)(x-2)(x-3) \leq 0$.

(a) $(-\infty, 1] \cup [2, \infty]$

(b) $(-\infty, 2] \cup [3, \infty]$

(c) $(-\infty, 1] \cup [2, 3]$

(d) $[1, 2] \cup [3, \infty)$

(e) None of these

Answer Key

5.1 #38. (c)

5.1 A. (b)

5.1 B. (b)

5.1 C. (c)

5.1 D. (b)

5.1 E. (e)

5.2 A. (a)

5.2 B. (c)

5.2 C. (b)

5.2 D. (b)

5.2 E. (c)

5.2 F. (d)

5.2 G. (c)

5.2 H. (a)

5.2 K. (c)

5.2 #48. (d)

5.2 L. (b)

5.2 M. (c)

5.4 #5. (a)

5.4 #13. (c)

Solutions

5.1 #38. Form a polynomial function of degree 3 with zeros -2, 2, 3.

Solution: (c) The polynomial should include a linear factor for each zero, so we should use $f(x) = (x+2)(x-2)(x-3) = (x^2-4)(x-3)$.

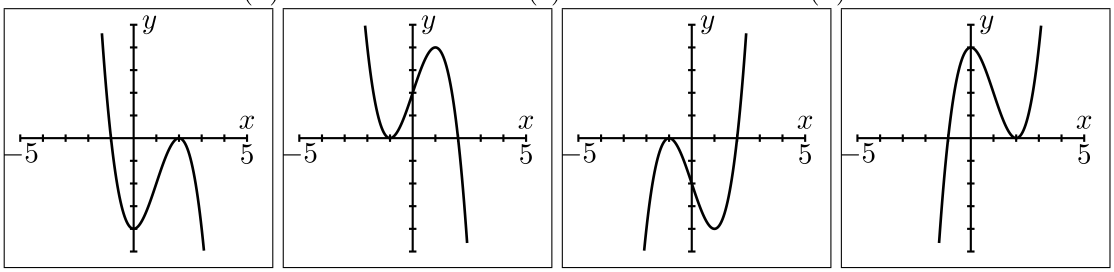
5.1 A. The polynomial function $f(x)$ has a zero at $x = 2$ with multiplicity 3. Answer: (b) Since the multiplicity of the zero is 3, which is an odd number, the graph crosses the x-axis.

5.1 B. Find all the zeros and their multiplicities for the polynomial $p(x) = 11x(x-1)^5(x+6)$.

Solution: (b) Setting the factors x, $x-1$, and $x+6$ each equal zero shows that the zeros are 0, 1, and -6. The next step is to look at the exponent of each factor: x has exponent 1, so the multiplicity of the root 0 is 1; $x-1$ has exponent 5, so the multiplicity of the root 1 is 5; $x+6$ has exponent 1, so the multiplicity of the root -6 is 1.

5.1 C. Which of the following is the graph of $f(x) = (x+1)^2(x-2)$?

(a) (b) (c) (d)



Solution: (c) The function has roots -1 and 2, so these must be x-intercepts on the graph. All four graphs pass this test, so we have to look at it more deeply. The multiplicity of -1 is 2, so the graph should only touch the axis at $x = -1$. Since the multiplicity of 2 is just 1, the graph should cross the axis at $x = 2$. This eliminates answers (a) and (d). We could plot one more point: since $f(0) = -2$, the y=intercept must be -2, and this eliminates answer (b).

5.1 D. The function $f(x) = x^2(x-2)(x+3)^2$ has

Answer: (b) one zero of multiplicity one and two zeros of multiplicity two.

Solution: $x = 0$ has multiplicity 2; $x = 2$ has multiplicity 1; $x = -3$ has multiplicity 2.

5.1 E. Which one of these functions might have the given graph?

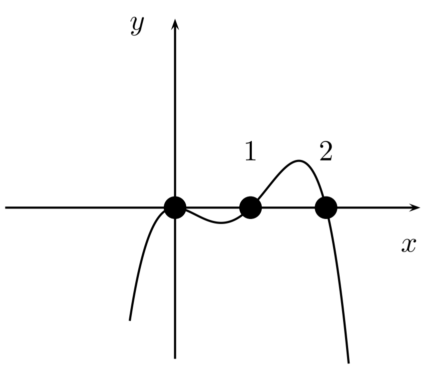
(a) $f(x) = x(x-1)(x-2)^2$

(b) $f(x) = -x(x-1)(x-2)$

(c) $f(x) = x(x-1)(x-2)$

(d) $f(x) = x^2(x-1)(x-2)$

(e) $f(x) = -x^2(x-1)(x-2)$



Solution: (e) The roots must be $x = 0$, $x = 1$, and $x = 2$, and this is true for every formula. Since that graph touches the x-axis at $x = 0$ but does not cross, the root $x = 0$ must have even multiplicity. The graph crosses the x-axis at $x = 1$ and $x = 2$, so these roots must have odd multiplicity. This eliminates the formulas in (a), (b), and (c).

Finally, to tell the difference between (d) and (e), check the value of the function at $x = -2$. In (d) we get $f(-2) = (-2)^2(-2-1)(-2-2) = 48$, so this doesn't agree with the graph, so the answer must be (e).

Alternatively, you can see that for large values of x the formula in (d) behaves like $f(x) = x^4$, while the formula in (e) behaves like $f(x) = -x^4$. The behavior of the graph is like $f(x) = -x^4$, so the answer must be (e).

5.2 A. Find the domain of the function $f(x) = \frac{x-2}{x+1}$.

Solution: (a) Set the denominator equal to 0. Exclude $x = -1$ to avoid division by 0.

5.2 B. What is the domain of the function G defined by $G(x) = \frac{x+4}{x^3-4x}$?

Solution: (c) Set $x^3 - 4x = 0$. This gives $x(x^2 - 4) = 0$, or $x(x-2)(x+2) = 0$. The numbers 0, 2, and -2 must be excluded. Answer: all real numbers except $0, 2, -2$.

5.2 C. The graph of $y = \frac{1}{(x-4)^2}$ looks like that of $y = \frac{1}{x^2}$ but is shifted

Solution: (b) right 4 units. One way to remember which way it is shifted is to observe that the graph of $y = \frac{1}{x^2}$ has a vertical asymptote at $x = 0$, while the graph of $y = \frac{1}{(x-4)^2}$ has a vertical asymptote at $x = 4$.

5.2 D. Find the vertical asymptotes of the graph of $f(x) = \frac{x^2-3x}{x^2-2x-8}$.

Solution: (b) Set the denominator equal to 0. $x^2 - 2x - 8 = 0$ $(x-4)(x+2) = 0$ The vertical asymptotes occur at $x = 4$ and $x = -2$.

5.2 E. The line $x = 4$ is a vertical asymptote of the graph of which of the following functions?
(a) $f(x) = x-4$ (b) $f(x) = \frac{2x-8}{x-4}$ (c) $f(x) = \frac{1}{x^2-16}$ (d) $f(x) = \frac{x-4}{x+3}$ (d) $f(x) = \frac{4x+1}{x+2}$

Solution: (c) To have a vertical asymptote at $x = 4$, the denominator must have a factor of $x - 4$. This seems to be true for both (b) and (c). Actually, $f(x) = \frac{2x-8}{x-4} = \frac{2(x-4)}{x-4} = 2$ for all values except $x = 4$. In the function in (b), the graph is a horizontal line at $y = 2$, with the point $(4, 2)$ missing, so it has no vertical asymptote. The correct answer is that only $f(x) = \frac{1}{x^2-16}$ has a graph with a vertical asymptote at $x = 4$.

5.2 F. Find the x-intercepts and vertical asymptotes of the graph of $f(x) = \frac{x^2-3x}{x^2-2x-8}$

Solution: (d) x-intercepts $(0, 0), (3, 0)$, vertical asymptotes $x = 4, x = -2$ Set the numerator equal to 0 to find the x-intercepts. $x^2 - 3x = 0$ $x(x-3) = 0$ $x = 0$ and $x = 3$.
Set the denominator equal to 0 to find the vertical asymptotes. $x^2 - 2x - 8 = 0$ $(x-4)(x+2) = 0$ $x = 4$ and $x = -2$

5.2 G. Find the vertical asymptotes of $f(x) = \frac{(x-1)(x+2)(x-3)}{x(x-4)^2}$.

Solution: (c) Set the denominator equal to 0 to get $x = 0$ and $x = 4$.

5.2 H. Find the horizontal asymptote of the graph of $f(x) = \frac{x-3}{5x+2}$.

Solution: (a) Multiplying both the numerator and denominator by $\frac{1}{x}$ gives $f(x) = \frac{1-\frac{3}{x}}{5+\frac{2}{x}}$, and in this form we see that as x increases the function gets closer and closer to $\frac{1}{5}$.
The shortcut is to remember that the highest powers of x dictate the behavior, so $f(x)$ will behave like $\frac{x}{5x} = \frac{1}{5}$ for large values of x.

5.2 K. Find the horizontal asymptote for the graph of $f(x) = \frac{5x-1}{2x+3}$.

Solution: (d) For large values of x the function behaves like $f(x) = \frac{5x}{2x}$, so the horizontal asymptote is $y = \frac{5}{2}$.

5.2 #48. Find the horizontal asymptote (if any) of $f(x) = \frac{-2x^2+1}{2x^3+4x^2}$.

Solution: (d) For large values of x, the function behaves like $f(x) = \frac{-2x^2}{2x^3} = -\frac{1}{x}$, so $y = 0$ is a horizontal asymptote. Remember that if the denominator has higher degree than the numerator, then $y = 0$ will be a horizontal asymptote.

5.2 L. Which one of these functions does **not** have a horizontal asymptote?
(a) $f(x) = \frac{2}{3x-5}$ (b) $f(x) = \frac{2x^2+1}{3x-5}$ (c) $f(x) = \frac{2x^2+1}{3x^2-5}$ (d) $f(x) = \frac{2x}{3x^2-5}$ (e) $f(x) = 2 + \frac{6}{3x^2-5}$

Solution: (b) If the degree of the numerator is larger than the degree of the denominator, then there is no horizontal asymptote. This happens for (b).

5.2 M. Find the asymptotes of the following function. $f(x) = \frac{3x^2-3x}{x^2+x-12}$

Solution: (c) For large values of x, the function behaves like $f(x) = \frac{3x^2}{x^2}$, so $y = 3$ is a horizontal asymptote.

To find the vertical asymptotes, solve $x^2 + x - 12 = 0$. $(x+4)(x-3) = 0$ The vertical asymptotes are $x = -4$ and $x = 3$.