M-INJECTIVE MODULES AND PRIME M-IDEALS

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For a left R-module M, we identify certain submodules of M that play a role analogous to that of prime ideals in the ring R. Using this definition, we investigate conditions on the module M which imply that there is a one-to-one correspondence between isomorphism classes of indecomposable M-injective modules and "prime M-ideals".

We assume throughout the paper that R is an associative ring with identity, and that ${}_RM$ is a fixed left R-module. The module ${}_RX$ is called *M-injective* if each R-homomorphism $f : K \to X$ defined on a submodule K of M can be extended to an R-homomorphism $\widehat{f} : M \to X$ with $f = \widehat{f}\,i$, where $i : K \to M$ is the natural inclusion mapping. We note that Baer's criterion for injectivity shows that any R-injective module is injective in the category R–Mod of all left R-modules.

It is well-known that if R is a commutative Noetherian ring, then there is a one-to-one correspondence between isomorphism classes of indecomposable injective R-modules and prime ideals of R. Gabriel showed in [1] that this one-to-one correspondence remains valid for any left Noetherian ring that satisfies what he called "condition H". In current terminology, a module ${}_RX$ is said to be *finitely annihilated* if there is a finite subset $x_1, \ldots, x_n$ of X with $\mathrm{Ann}_R(X) = \mathrm{Ann}_R(x_1, \ldots, x_n)$.

Then by definition the ring R satisfies condition H if and only if every cyclic left R-module is finitely annihilated. It follows immediately that R satisfies condition H if and only if every finitely generated left R-module is finitely annihilated. We note the stronger result due to Krause [2] that if R is left Noetherian, then there is a one-to-one correspondence between isomorphism classes of indecomposable injective left R-modules and prime ideals of R if and only if R is left fully bounded (see Theorem 8.12 of [3] for a proof).

In Theorem 6.7 we show that Gabriel's correspondence can be extended to M-injective modules, provided that $\mathrm{Hom}_R(M, X) \neq 0$ for every submodule X of a homomorphic image of a direct sum of copies of M. In preparation for this result, we introduce the notion of an M-ideal in Section 1, and the notion of an M-prime module in Section 2. In Section 3 we define a prime M-ideal to be the annihilator in M of an M-prime module, and show that if M is Noetherian, then associated primes exist. In Sections 4 and 5 we show that many of the familiar results for prime ideals can be extended to prime M-ideals. In particular, we show that certain simplifications occur if M satisfies a weak form of projectivity.

Just as Baer's criterion shows that R-injective modules are actually injective in the category $R\text{–}Mod$, there is a subcategory of $R\text{–}Mod$ in which M-injective modules are injective. To define this subcategory we first recall some necessary terminology. The module ${}_RX$ is said to be *M-generated* if there exists an R-epimorphism from a direct sum of copies of M onto X. Equivalently, for each nonzero R-homomorphism $f : X \to Y$ there exists an R-homomorphism $g : M \to X$ with $fg \neq 0$. The *trace* of M in X is defined to be

$$\mathrm{tr}^M(X) = \sum_{f \in \mathrm{Hom}(M,X)} f(M) ,$$

and thus X is M-generated if and only if $\mathrm{tr}^M(X) = X$.

The category $\sigma[M]$ is defined to be the full subcategory of $R\text{–}Mod$ that contains all modules ${}_RX$ such that X is isomorphic to a submodule of an M-generated module. The reader is referred to [4], [5], and [6] for results on the category $\sigma[M]$. The subcategory $\sigma[M]$ coincides with $R\text{–}Mod$ if and only if R belongs to $\sigma[M]$; this occurs if and only if M is faithful and finitely annihilated. More generally, if M is finitely annihilated, then $R/\mathrm{Ann}_R(M)$ belongs to $\sigma[M]$, and it follows that $\sigma[M]$

coincides with the class of left modules over $R/\operatorname{Ann}_R(M)$. It is shown in [5] that $\sigma[M]$ is closed under taking homomorphic images, submodules, and direct sums. Injective modules and injective envelopes exist in $\sigma[M]$, since the injective envelope of X in $\sigma[M]$ is $\operatorname{tr}^M(\mathrm{E}(X))$, where $\mathrm{E}(X)$ denotes the injective envelope of X is R–Mod.

Our first goal is to extend the analogy between $\sigma[M]$ and R–Mod by introducing the notion of an "ideal" of M, and to use this notion in the further study of $\sigma[M]$.

1 M-ideals

The two-sided ideals of the ring R correspond to the annihilators of left R-modules. Furthermore, for a left R-module X, we have

$$\operatorname{Ann}_R(X) = \{r \in R \mid rx = 0 \text{ for all } x \in X\} = \bigcap_{f \in \operatorname{Hom}(R,X)} \ker(f) \ .$$

More generally, the annihilator of any class of modules is a two-sided ideal of R, and this motivates the following definition.

Definition 1.1 *Let M be any left R-module, let $\mathcal{C}$ be a class of modules in R–Mod, and let Ω be the set of kernels of R-homomorphisms from M into $\mathcal{C}$. That is,*

$$\Omega = \{K \subseteq M \mid \exists W \in \mathcal{C} \text{ and } f \in \operatorname{Hom}_R(M,W) \text{ with } K = \ker(f)\} \ .$$

We define the annihilator *of $\mathcal{C}$ in M to be*

$$\operatorname{Ann}_M(\mathcal{C}) = \bigcap_{K \in \Omega} K \ .$$

In [7], what we have called the annihilator of $\mathcal{C}$ in M is called the reject of $\mathcal{C}$ in M, and is denoted by $\operatorname{Rej}_M(\mathcal{C})$. For our purposes, the term annihilator is more congenial, and is closer to the terminology of [4]. It allows us to draw a clear analogy with the definition of a two-sided ideal in the ring R.

Definition 1.2 *The submodule N of M is called an M-ideal if there is a class $\mathcal{C}$ of modules in $\sigma[M]$ such that $N = \operatorname{Ann}_M(\mathcal{C})$.*

Note that although the definition of an M-ideal is given relative to the subcategory $\sigma[M]$, it is easy to check that N is an M-ideal if and only if $N = \mathrm{Ann}_M(\mathcal{C})$ for some class $\mathcal{C}$ in R–Mod. This also follows immediately from the proof of Proposition 1.3.

A subfunctor ρ of the identity on R–Mod is called a *radical* if $\rho(X/\rho(X)) = (0)$, for all modules ${}_RX$. We say that ${}_RX$ is ρ-*torsionfree* if $\rho(X) = (0)$, and ρ-*torsion* if $\rho(X) = X$. If ρ is a radical, then since $X/\rho(X)$ is ρ-torsionfree, it follows that $\rho(X) = \mathrm{Ann}_X(\mathcal{C})$, where $\mathcal{C}$ is the class of ρ-torsionfree modules. In addition, $\rho(X)$ is the intersection of all submodules X' of X such that X/X' is ρ-torsionfree.

For a class $\mathcal{C}$ of R-modules, the radical of R–Mod *cogenerated* by $\mathcal{C}$ is defined by setting $\mathrm{rad}_{\mathcal{C}}(X) = \mathrm{Ann}_X(\mathcal{C})$, for all modules ${}_RX$. If the class $\mathcal{C}$ consists of a single module ${}_RW$, we use the notation rad_W, and note that rad_W is the largest radical for which W is torsionfree. If ${}_RV$ cogenerates W, then $\mathrm{rad}_V(W) = (0)$, so $\mathrm{rad}_V \leq \mathrm{rad}_W$, and therefore $\mathrm{Ann}_M(V) \subseteq \mathrm{Ann}_M(W)$.

If I is any left ideal of R, it is possible to define a radical ρ of R–Mod by setting $\rho(X) = IX$, for each module ${}_RX$. It is evident that ρ is a subfunctor of the identity, and ρ is a radical since $\rho(X/\rho(X)) = I \cdot (X/IX) = (0)$. It follows that $IX = \mathrm{Ann}_X(\mathcal{C})$ for the class $\mathcal{C}$ of all modules ${}_RW$ with $IW = (0)$. Thus for any left ideal I of R, the submodule IM provides an example of an M-ideal.

The following conditions characterizing M-ideals were obtained in Proposition 3.3 of [8]. A proof is included here for the convenience of the reader.

Proposition 1.3 ([8]) *The following conditions are equivalent for a submodule $N \subseteq M$.*

(1) N is an M-ideal;

(2) there exists a radical ρ of R–Mod such that $N = \rho(M)$;

(3) $g(N) = (0)$ for all $g \in \mathrm{Hom}_R(M, (M/N))$;

(4) $N = \mathrm{Ann}_M(M/N)$.

Proof. (1) $\Rightarrow$ (2): Assume that $N = \mathrm{Ann}_M(\mathcal{C})$ for the class $\mathcal{C}$ of modules in $\sigma[M]$. Then $N = \mathrm{rad}_{\mathcal{C}}(M)$ for the radical $\rho = \mathrm{rad}_{\mathcal{C}}$ cogenerated by the class $\mathcal{C}$.

(2) $\Rightarrow$ (3): Assume that ρ is a radical of R–Mod, with $N = \rho(M)$. Since ρ is a radical, we have $\rho(M/N) = \rho(M/\rho(M)) = (0)$. If $g \in \mathrm{Hom}_R(M, (M/N))$, then $g(N) = g(\rho(M)) \subseteq \rho(M/N) = (0)$.

(3) $\Rightarrow$ (4): It is always true that $\mathrm{Ann}_M(M/N) \subseteq N$. Condition (3) implies that $N \subseteq \bigcap_{f:M\to M/N} \ker(f)$, so we have equality in this case.

(4) $\Rightarrow$ (1): This follows immediately from the definition of an M-ideal. □

It is easy to show that the intersection of any collection of M-ideals is again an M-ideal. This follows from condition (2) of Proposition 1.3, since the intersection of any collection of radicals defines a radical. This result can also be shown using condition (3) of Proposition 1.3, since if $N = \bigcap_{\alpha\in I} N_\alpha$, where each submodule N_α is an M-ideal, then M/N can be embedded in the direct product $\prod_{\alpha\in I} M/N_\alpha$, and it follows easily that $f(N) = (0)$ for any homomorphism $f : M \to M/N$.

Let N be a submodule of M. Applying the functor $\mathrm{Hom}_R(- , M/N)$ to the exact sequence $0 \to N \to M \to M/N \to 0$ yields the exact sequence

$$0 \to \mathrm{Hom}_R(M/N, M/N) \to \mathrm{Hom}_R(M, M/N) \to \mathrm{Hom}_R(N, M/N)\ .$$

It follows that condition (3) of Proposition 1.3 is equivalent to the condition that the natural mapping $\mathrm{Hom}_R(M, M/N) \to \mathrm{Hom}_R(N, M/N)$ is zero. Therefore N is an M-ideal if and only if the natural mapping $\mathrm{Hom}_R(M/N, M/N) \to \mathrm{Hom}_R(M, M/N)$ is an isomorphism.

Proposition 1.4 *Let K be an M-ideal, and let $N \subseteq K$ be a submodule of M.*

(a) The factor module K/N is an (M/N)-ideal (in M/N).

(b) If N is a K-ideal (in K), then N is an M-ideal.

Proof. (a) If $f \in \mathrm{Hom}_R(M/N, M/K)$, then consider $f\pi$, where π is the natural projection from M onto M/N. Since K is an M-ideal, we have $f\pi(K) = (0)$. Thus $f(K/N) = (0)$, showing that K/N is an (M/N)-ideal.

(b) If $f \in \mathrm{Hom}_R(M, M/N)$, consider the composition πf, where π is the projection of M/N onto M/K. Since K is an M-ideal, we have $\pi f(K) = (0)$, and so $f(K) \subseteq K/N$. Thus it is possible to define $g : K \to K/N$ by setting $g(x) = f(x)$, for all $x \in K$. Then $g(N) = (0)$ since N is an K-ideal, and it follows that $f(N) = (0)$, showing that N is an M-ideal. $\square$

The next step is to define the product of two M-ideals. We give the definition more generally, constructing a product $N \cdot X$ for any submodule N of M and any module ${}_RX$. As a model we use the fact that if I is a left ideal of R, then for any module ${}_RX$ we have $IX = \text{Ann}_X(\mathcal{C})$, where $\mathcal{C}$ is class of modules ${}_RW$ such that $IW = (0)$. The condition $IW = (0)$ can be expressed by stating that $f(I) = (0)$ for all $f \in \text{Hom}_R(R, W)$, and it is this statement that can be generalized.

Definition 1.5 *Let N be a submodule of M. For each module ${}_RX$ we define*

$$N \cdot X = \text{Ann}_X(\mathcal{C}) \ ,$$

where $\mathcal{C}$ is the class of modules ${}_RW$ such that $f(N) = (0)$ for all $f \in \text{Hom}_R(M, W)$.

It follows immediately from the definition that $N \cdot X = (0)$ if and only if $f(N) = (0)$ for all $f \in \text{Hom}_R(M, X)$. It is evident that the class $\mathcal{C}$ in Definition 1.5 is closed under formation of submodules and direct products, and so $N \cdot X$ is the smallest submodule $Y \subseteq X$ such that $N \cdot (X/Y) = (0)$. We also note that

$$\textstyle\sum_{f \in \text{Hom}(M,X)} f(N) \subseteq N \cdot X \ ,$$

since for any $f \in \text{Hom}_R(M, X)$ we must have $\pi f(N) = (0)$ for the natural projection $\pi : X \to X/(N \cdot X)$.

Since the functor $N \cdot (-)$ defines a radical, it follows that if X_0 is a submodule of X, then $N \cdot X_0 \subseteq N \cdot X$, and if $f : X \to Y$ is any R-homomorphism, then $f(N \cdot X) \subseteq N \cdot Y$.

Proposition 1.6 *Let N be a submodule of M. Then for any module ${}_RX$ we have $N \cdot X = (0)$ if and only if $N \subseteq \text{Ann}_M(X)$.*

Proof. We have $N \cdot X = (0)$ if and only if $f(N) = (0)$ for all $f \in \text{Hom}_R(M, X)$, and this occurs if and only if $N \subseteq \text{Ann}_M(X)$. $\square$

The next corollary is an immediate consequence of condition (4) of Proposition 1.3.

Corollary 1.7 *If N is a submodule of M, then N is an M-ideal if and only if $N \cdot (M/N) = (0)$.*

A class $\mathcal{T}$ of modules in R–Mod is a *torsion class* if it is closed under homomorphic images, direct sums, and extensions. A class $\mathcal{F}$ of modules is called a *torsionfree class* if it is closed under submodules, direct products, and extensions. (See Propositions 2.1 and 2.2 in Chapter VI of [9].) The smallest torsion class containing M is constructed by taking $\mathcal{F}$ to be all modules ${}_RW$ such that $\mathrm{Hom}_R(M, W) = 0$, and then letting the torsion class consist of all modules ${}_RX$ such that $\mathrm{Hom}_R(X, W) = 0$ for all W in $\mathcal{F}$. It follows directly from Definition 1.5 that the smallest torsion class that contains M is the class of all modules ${}_RX$ such that $M \cdot X = X$.

If ρ is a radical of R–Mod, then $\rho(R)X \subseteq \rho(X)$, for all modules ${}_RX$. This result extends to $\sigma[M]$, since Proposition 3.3 of [8] shows that if N is an M-ideal, and ρ is a radical of R–Mod such that $N = \rho(M)$, then $N \cdot X \subseteq \rho(X)$, for all modules ${}_RX$. We give a somewhat more general result, which implies that for any submodule $N \subseteq M$, the radical $N \cdot (-)$ is the smallest radical ρ of R–Mod for which $N \subseteq \rho(M)$.

Lemma 1.8 *Let N be a submodule of M. If ρ is a radical of R–Mod such that $N \subseteq \rho(M)$, then $N \cdot X \subseteq \rho(X)$, for all modules ${}_RX$.*

Proof. If ${}_RW$ is any module with $\rho(W) = (0)$, then $f(N) = f(\rho(M)) \subseteq (0)$ for all $f \in \mathrm{Hom}_R(M, W)$. Thus $N \cdot W = (0)$ for all ρ-torsionfree modules W, and therefore $N \cdot X \subseteq \rho(X)$ for all modules ${}_RX$. $\square$

We next show that some standard results for two-sided ideals of the ring R can be extended to M-ideals. In particular, the product of two M-ideals is an M-ideal that is contained in their intersection.

Proposition 1.9 *Let N and K be a submodules of M.*

(a) If $N \subseteq K$, then $N \cdot X \subseteq K \cdot X$, for all modules ${}_RX$.

(b) If K is an M-ideal, then so is $N \cdot K$.

(c) The submodule $N \cdot M$ is the smallest M-ideal that contains N.

(d) If N is an M-ideal, then $N \cdot K \subseteq N \cap K$.

Proof. (a) If $N \subseteq K$, then $K \cdot W = (0)$ implies $N \cdot W = (0)$, for any module ${}_RW$, and therefore $N \cdot X \subseteq K \cdot X$, for all modules ${}_RX$.

(b) By definition, $N \cdot K$ is a K-ideal (in K). Since K is assumed to be an M-ideal, it follows from Proposition 1.4 (b) that $N \cdot K$ is an M-ideal.

(c) If K is an M-ideal and $N \subseteq K$, then $K = \rho(M)$, for some radical ρ. By Lemma 1.8 we have $N \cdot M \subseteq \rho(M) = K$.

(d) We certainly have $N \cdot K \subseteq K$. On the other hand, $N \cdot K \subseteq N \cdot M = N$ by part (c), since N is assumed to be an M-ideal. $\square$

Some results for ideals cannot be extended to this situation. For example, if A, B, C are ideals of R, with $C \subseteq B$, then $AB \subseteq C$ if and only if $A(B/C) = (0)$. The corresponding result, that $N \cdot K \subseteq P$ implies $N \cdot (K/P) = (0)$ for M-ideals N, K, P with $P \subseteq K$, holds if M is projective in $\sigma[M]$, as will be shown in Proposition 5.5.

We can now extend some standard results on annihilators.

Proposition 1.10 *Let ${}_RX$ be a module.*

(a) $\mathrm{Ann}_M(X) = \mathrm{Ann}_M(\mathrm{tr}^M(X)) = \mathrm{Ann}_M(M \cdot X)$.

(b) If N is any submodule of M, let $T = \sum_{f \in \mathrm{Hom}_R(M,X)} f(N)$, and let $A =$ $\mathrm{Ann}_M(X)$. *Then* $\mathrm{Ann}_R(T) = \mathrm{Ann}_R((N + A)/A)$.

(c) $\mathrm{Ann}_R(\mathrm{tr}^M(X)) = \mathrm{Ann}_R(M/\,\mathrm{Ann}_M(X))$.

Proof. (a) If $f \in \mathrm{Hom}_R(M, X)$, then $f(M) \subseteq \mathrm{tr}^M(X) \subseteq M \cdot X$. It follows that $\mathrm{Hom}_R(M, X) = \mathrm{Hom}_R(M, \mathrm{tr}^M(X)) = \mathrm{Hom}_R(M, M \cdot X)$, and so

$$\bigcap_{f \in \mathrm{Hom}(M,X)} \ker(f) = \bigcap_{f \in \mathrm{Hom}(M,\mathrm{tr}^M(X))} \ker(f) = \bigcap_{f \in \mathrm{Hom}(M,M \cdot X)} \ker(f) \ .$$

(b) On the one hand, $M/\,\mathrm{Ann}_M(X) = M/\,\mathrm{Ann}_M(\mathrm{tr}^M(X))$ can be embedded in a direct product of copies $\mathrm{tr}^M(X)$. It follows that $(N + A)/A$ can be embedded in a direct product of copies of T, and so

$$\mathrm{Ann}_R(T) \subseteq \mathrm{Ann}_R((N + A)/A) \ .$$

On the other hand, T is a homomorphic image of a direct sum of copies of N. Since each homomorphism in $\mathrm{Hom}_R(M, \mathrm{tr}^M(X))$ factors through

$$M/\,\mathrm{Ann}_M(X) = M/\,\mathrm{Ann}_M(\mathrm{tr}^M(X)) \ ,$$

it is possible to write T as a homomorphic image of a direct sum of copies of $(N + A)/A$, and so it follows that

$$\text{Ann}_R((N + A)/A) \subseteq \text{Ann}_R(T) \ .$$

This completes the proof.

(c) This follows immediately from part (b), by taking $N = M$. $\Box$

2 M-prime modules

In this section we give the definition of an M-prime module, and then in the next section we will define an M-ideal to be prime if it is the annihilator in M of an M-prime module. Recall that the module ${}_RX$ is said to be a prime module if X is nonzero and $\text{Ann}_R(Y) = \text{Ann}_R(X)$ for all nonzero submodules $Y \subseteq X$. The definition of an M-prime module is complicated by the fact that it is possible to have $\text{Hom}_R(M, X) = 0$ even though X is a nonzero member of $\sigma[M]$.

Definition 2.1 *The module ${}_RX$ is said to be M-prime if* $\text{Hom}_R(M, X) \neq 0$*, and* $\text{Ann}_M(Y) = \text{Ann}_M(X)$ *for all submodules* $Y \subseteq X$ *such that* $\text{Hom}_R(M, Y) \neq 0$.

In case $M = R$, we have $\text{Hom}_R(M, X) = \text{Hom}_R(R, X) \neq 0$ for all nonzero modules ${}_RX$, and so the notion of an R-prime module reduces to the familiar definition of a prime module in R–Mod. For the sake of clarity, for the remainder of the paper we will use the term "R-prime module" rather than "prime R-module".

The next proposition gives several characterizations of M-prime modules, extending the following conditions, which are equivalent for any nonzero module ${}_RX$:

(1) X is an R-prime module;

(2) $IY = (0)$ implies $IX = (0)$, for any left ideal I of R and any nonzero submodule $Y \subseteq X$;

(3) for each $a \in R \setminus \text{Ann}_R(X)$ and each $0 \neq x \in X$ there exists $r \in R$ with $arx \neq 0$;

(4) $IY = (0)$ implies $IX = (0)$, for any two-sided ideal I of R and any nonzero submodule $Y \subseteq X$.

Proposition 2.2 *The following conditions are equivalent for any left R-module X such that* $\mathrm{Hom}_R(M, X) \neq 0$.

(1) X is an M-prime module;

(2) $N \cdot Y = (0)$ implies $N \cdot X = (0)$, for any submodule $N \subseteq M$ and any submodule $Y \subseteq X$ with $M \cdot Y \neq (0)$;

(3) for each $m \in M \setminus \mathrm{Ann}_M(X)$ and each $0 \neq f \in \mathrm{Hom}_R(M, X)$, there exists $g \in \mathrm{Hom}_R(M, f(M))$ such that $g(m) \neq 0$;

(4) $N \cdot Y = (0)$ implies $N \cdot X = (0)$, for any M-ideal $N \subseteq M$ and any nonzero M-generated submodule $Y \subseteq X$.

Proof. (1) $\Rightarrow$ (2): Let N be any submodule of M, and let Y be any submodule of X with $M \cdot Y \neq (0)$. Then $\mathrm{Hom}_R(M, Y) \neq 0$, and so $\mathrm{Ann}_M(Y) \subseteq \mathrm{Ann}_M(X)$ since X is assumed to be M-prime. It follows from Proposition 1.6 that if $N \cdot Y = (0)$, then $N \cdot X = (0)$.

(2) $\Rightarrow$ (3): Let $m \in M \setminus \mathrm{Ann}_M(X)$ and $0 \neq f \in \mathrm{Hom}_R(M, X)$. Then by assumption we must have $N \cdot f(M) \neq (0)$ for $N = Rm$, so Rm is not contained in $\mathrm{Ann}_M(f(M))$, and therefore there exists $g \in \mathrm{Hom}_R(M, f(M))$ with $g(m) \neq 0$.

(3) $\Rightarrow$ (4): Let Y be a nonzero M-generated submodule of X, and let N be any M-ideal with $N \cdot Y = (0)$. If $N \cdot X \neq (0)$, then there exists $m \in N \setminus \mathrm{Ann}_M(X)$. Since Y is M-generated, there exists $0 \neq f \in \mathrm{Hom}_R(M, Y)$, and so by assumption there exists $g \in \mathrm{Hom}_R(M, Y)$ with $g(m) \neq 0$. Thus $Rm \cdot Y \neq (0)$, a contradiction.

(4) $\Rightarrow$ (1): Assume Y is a nonzero submodule of X with $\mathrm{Hom}_R(M, Y) \neq 0$, and let $N = \mathrm{Ann}_M(Y)$. By assumption there exists $0 \neq f \in \mathrm{Hom}_R(M, Y)$, and then $N \cdot f(M) = (0)$. Since $f(M)$ is certainly M-generated, by assumption we have $N \cdot X = (0)$, and thus $\mathrm{Ann}_M(Y) = \mathrm{Ann}_M(X)$, showing that X is an M-prime module. □

Corollary 2.3 *Let ${}_RX$ be an M-prime module. A submodule Y of X is M-prime if and only if* $\mathrm{Hom}_R(M, Y) \neq 0$.

Proof. Condition (2) of Proposition 2.2 is inherited by any submodule. Thus a nonzero submodule Y is M-prime if and only if $\mathrm{Hom}_R(M, Y) \neq 0$. □

Corollary 2.4 *The following conditions are equivalent for a module $_RX$ such that* $\text{Hom}_R(M, X) \neq 0$.

(1) X is an M-prime module;

(2) $M \cdot X$ is an M-prime module;

(3) $\text{tr}^M(X)$ is an M-prime module.

Proof. Since $\text{tr}^M(X) \subseteq M \cdot X \subseteq X$, it follows from Corollary 2.3 that (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3). The implication (3) $\Rightarrow$ (1) follows from condition (4) of Proposition 2.2, since every M-generated submodule of X is contained in $\text{tr}^M(X)$. $\square$

We note that, in particular, the module $_RX$ is M-prime if it has no proper nontrivial M-generated submodules. If R is a simple ring, then every nonzero left R-module is R-prime. Similarly, if M has no proper nontrivial M-ideals, then every module $_RX$ with $\text{Hom}_R(M, X) \neq 0$ is M-prime.

In the following example, we show that an M-prime module need not be R-prime. We also show that an R-prime module need not be M-prime.

Example 2.1

Let $M = \mathbf{Z}_{p^\infty}$, in the category of $\mathbf{Z}$-modules. Every proper factor of M is isomorphic to M itself, and so no proper nontrivial submodule can be an M-ideal. It also follows that no proper nontrivial submodule of M is M-generated. Thus M is M-prime, but each proper submodule has nonzero annihilator in $\mathbf{Z}$, and so M is not a $\mathbf{Z}$-prime module.

On the other hand, $\mathbf{Z}$ itself is certainly a $\mathbf{Z}$-prime module, but it is not M-prime since $\text{Hom}_Z(\mathbf{Z}_{p^\infty}, \mathbf{Z}) = 0$. In fact, $\text{Hom}_Z(\mathbf{Z}_{p^\infty}, X) = 0$ for all $\mathbf{Z}$-prime modules.

Proposition 2.5 *Let $_RX$ be an M-prime module. If* $\text{Hom}_R(M, Y) \neq 0$ *for all nonzero submodules $Y \subseteq X$, then* $\text{tr}^M(X)$ *is an R-prime module.*

Proof. Without loss of generality we can assume that $X = \mathrm{tr}^M(X)$. Then X is M-generated, and so it follows from Proposition 1.10 (c) that

$$\mathrm{Ann}_R(X) = \mathrm{Ann}_R(M/\mathrm{Ann}_M(X)) \ .$$

By assumption, every nonzero submodule X' of X contains a nonzero M-generated submodule Y. Since X is M-prime we have $\mathrm{Ann}_M(Y) = \mathrm{Ann}_M(X)$, and therefore

$$\mathrm{Ann}_R(X') \subseteq \mathrm{Ann}_R(Y) = \mathrm{Ann}_R(M/\mathrm{Ann}_M(Y)) = \mathrm{Ann}_R(M/\mathrm{Ann}_M(X)) = \mathrm{Ann}_R(X).$$

It follows that X is an R-prime module. □

In [10], a nonzero module ${}_RM$ is called prime if $\mathrm{rad}_N = \mathrm{rad}_M$ for all nonzero submodules N of M. It is shown in Proposition 2.3 of [10] that a nonzero module M satisfies this definition if and only if it is cogenerated by each of its nonzero submodules. To relate this notion to our definition of an M-prime module, we have the following proposition, which shows that such modules are "universally" M-prime.

Proposition 2.6 *The following conditions are equivalent for a nonzero module ${}_RX$.*

(1) Each nonzero submodule of X cogenerates X;

(2) X is M-prime for each module ${}_RM$ such that $\sigma[M] = \sigma[X]$ and X is M-generated;

(3) X is M-prime for each module ${}_RM$ such that $\mathrm{Hom}_R(M, X) \neq 0$.

Proof. (1) ⇒ (3): If Y is a nonzero submodule of X, then for any module M we have $\mathrm{Ann}_M(Y) \subseteq \mathrm{Ann}_M(X)$ since Y cogenerates X. Thus X is an M-prime module if and only if $\mathrm{Hom}_R(M, X) \neq 0$.

(3) ⇒ (2): This is clear.

(2) ⇒ (1): Given any nonzero submodule Y of X, consider the module $M = X \oplus Y$. Then X is M-generated, and $\sigma[M] = \sigma[X]$ since Y belongs to $\sigma[X]$. By assumption X is M-prime, and so $\mathrm{Ann}_M(Y) = \mathrm{Ann}_M(X) = (0)$. This implies that $\mathrm{Ann}_X(Y) = (0)$, and therefore Y cogenerates X. □

If ${}_RX$ is a module for which $\mathrm{Hom}_R(M, X) \neq 0$, then to show that X is an M-prime module it is only necessary to show that every nonzero M-generated submodule of X cogenerates X.

A nonzero module ${}_RX$ is said to be *semi-compressible* if for each nonzero submodule $Y \subseteq X$ there exists a monomorphism $f : X \to Y^n$, for a finite direct sum Y^n of copies of Y. It is clear that any nonzero semi-compressible module is an R-prime module. Proposition 9.3.3 of [11] implies that any uniform left ideal of a semiprime left Goldie ring is a semi-compressible module.

Corollary 2.7 *If ${}_RX$ is semi-compressible or $X \cong R/P$ for a prime ideal P of R, then X is an M-prime module if and only if* $\mathrm{Hom}_R(M, X) \neq 0$.

Proof. The conditions of Proposition 2.6 are satisfied in either case. □

We now turn our attention to the case in which M itself is M-prime.

Proposition 2.8 *The module M is an M-prime module if and only if $f(M)$ cogenerates M, for each nonzero endomorphism $f \in \mathrm{End}_R(M)$.*

Proof. Assume that M is an M-prime module. Then for any nonzero submodule N with $\mathrm{Hom}_R(M, N) \neq 0$ we have $\mathrm{Ann}_M(N) = \mathrm{Ann}_M(M) = (0)$. It follows that N cogenerates M.

To prove the converse we can use condition (3) of Proposition 2.2. Let $0 \neq m \in M$, and let $0 \neq f \in \mathrm{End}_R(M)$. By hypothesis, $f(M)$ cogenerates M, and so there exists $g \in \mathrm{Hom}_R(M, f(M))$ with $g(m) \neq 0$. Thus condition (3) of Proposition 2.2 is satisfied, and M is an M-prime module. □

A module ${}_RX$ is called *monoform* if each nonzero homomorphism $f : Y \to X$ defined on a submodule Y of X must be a monomorphism. It is clear from the definition that any monoform module satisfies the hypothesis of Proposition 2.8.

Corollary 2.9 *If M is a monoform module, then M is an M-prime module.*

We note that any monoform module must be uniform, and the converse holds for nonsingular modules. A module X is called α-critical, for the ordinal α, if it has Krull dimension α, but every proper factor module X/Y has Krull dimension less than α. Lemma 6.2.13 of [11] shows that any α-critical module is monoform.

The next theorem shows that under certain conditions the properties under consideration can be translated to properties of the endomorphism ring.

Theorem 2.10 *If M is M-prime and quasi-projective, then $\text{End}_R(M)$ is a prime ring. Conversely, if $\text{End}_R(M)$ is a prime ring and $\text{Hom}_R(M, K) \neq 0$ for all nonzero submodules $K \subseteq M$, then M is an M-prime module.*

Proof. Assume that M is M-prime and quasi-projective, and let f, g be nonzero elements of $\text{End}_R(M)$. By condition (3) of Proposition 2.2 there exists a homomorphism $h \in \text{Hom}_R(M, f(M))$ with $hg \neq 0$. Since M is quasi-projective, the homomorphism h can be lifted to $\widehat{h} \in \text{End}_R(M)$, with $f\widehat{h} = h$. It follows that $f\widehat{h}g \neq 0$, and therefore $\text{End}_R(M)$ is a prime ring.

Conversely, suppose that $\text{End}_R(M)$ is a prime ring and $\text{Hom}_R(M, K) \neq 0$ for all nonzero submodules $K \subseteq M$. Let $f \in \text{End}_R(M)$ be a nonzero endomorphism. Given any nonzero submodule $K \subseteq M$ there exists a nonzero element $g \in \text{Hom}_R(M, K)$, which we can view as an element of $\text{End}_R(M)$. Since $\text{End}_R(M)$ is a prime ring, there exists $h \in \text{End}_R(M)$ with $fhg \neq 0$. It follows that $f(M)$ cogenerates M, since $fh : M \to f(M)$ and $fh(K) \neq (0)$. $\square$

Proposition 2.11 *If P is an M-ideal, then M/P is an M-prime module if and only if M/P is an M/P-prime module.*

Proof. The factor module M/P is M-prime if and only if $\text{Ann}_M(Y) = P$ or $\text{Ann}_M(Y) = M$, for all nonzero submodules $Y \subseteq M/P$. It is M/P-prime if and only if $\text{Ann}_{M/P}(Y) = (0)$ or $\text{Ann}_{M/P}(Y) = M/P$, for all nonzero submodules $Y \subseteq M/P$. Since P is an M-ideal, we have $f(P) = (0)$ for all $f \in \text{Hom}_R(M, M/P)$, and so $\text{Ann}_M(Y) = P$ if and only if $\text{Ann}_{M/P}(Y) = (0)$, and $\text{Ann}_M(Y) = M$ if and only if $\text{Ann}_{M/P}(Y) = M/P$. Thus M/P is M-prime if and only if it is M/P-prime. $\square$

3 Prime M-ideals

We will proceed by analogy with the situation for ideals of the ring R. Our next step is to define the notion of a prime M-ideal. The existence of associated primes, when M is Noetherian, provides the motivation for the definition of an M-prime module, and for that of a prime M-ideal. Requiring that the usual proof remains valid in this more general setting means that the following definition of a prime M-ideal is weaker than one would hope to have.

Definition 3.1 *The M-ideal P is said to be a* prime M-ideal *if there exists an M-prime module ${}_RX$ such that $P = \mathrm{Ann}_M(X)$.*

Although the above definition is given in R–Mod, the next proposition shows that it is actually a definition in $\sigma[M]$.

Proposition 3.2 *If P is a prime M-ideal, then $P = \mathrm{Ann}_M(X)$ for an M-generated M-prime module ${}_RX$.*

Proof. Let P be a prime M-ideal, and let ${}_RY$ be an M-prime module with $P = \mathrm{Ann}_M(Y)$. If we let $X = \mathrm{tr}^M(Y)$, then X is M-generated. It follows from part (c) of Corollary 2.4 that X is an M-prime module. Finally, it follows from Proposition 1.10 (a) that $P = \mathrm{Ann}_M(X)$. □

We next consider prime M-ideals associated to a module.

Definition 3.3 *Let P be a prime M-ideal. We say that P is a prime M-ideal associated to the module X if $P = \mathrm{Ann}_M(Y)$ for an M-prime submodule Y of X.*

Proposition 3.4 *Assume that M is a Noetherian module, and let X be any left R-module. If $\mathrm{Hom}_R(M, X) \neq 0$, then X has an associated prime M-ideal.*

Proof. Consider the set of all proper M-ideals that are annihilators of submodules of X. Since $\mathrm{Hom}_R(M,X) \neq 0$, this set contains $\mathrm{Ann}_M(X)$, and so it is nonempty. Since M is Noetherian, there exists a maximal element P in the set, with $P = \mathrm{Ann}_M(Y)$. If Y' is any submodule of Y, then $\mathrm{Ann}_M(Y) \subseteq \mathrm{Ann}_M(Y')$, and so by definition Y is M-prime. □

The Jacobson radical of the ring R is generally defined to be the intersection of maximal left ideals of R, and then this intersection is shown to be the same as the intersection of all primitive ideals of R. The definition is extended to modules, by defining the Jacobson radical $\mathrm{J}(X)$ of a module ${}_RX$ to be the intersection of all maximal submodules of X (or just X, if X has no maximal submodules). Equivalently, $\mathrm{J}(X) = \mathrm{Ann}_X(\mathcal{C})$, where $\mathcal{C}$ is the class of simple left R-modules. Using the notion of an M-ideal, we can define the notion of a primitive M-ideal, from which it follows that $\mathrm{J}(M)$ is equal to the intersection of all primitive M-ideals.

Definition 3.5 *An M-ideal P is said to be a* primitive *M-ideal if $P = \mathrm{Ann}_M(S)$ for a simple module ${}_RS$.*

The notion of a maximal ideal of R can also be generalized immediately to M-ideals, and the next proposition shows that the expected results hold.

Proposition 3.6 *Let P be a proper M-ideal.*

(a) If P is a maximal M-ideal, then P is a prime M-ideal.

(b) If P is a primitive M-ideal, then P is a prime M-ideal.

Proof. (a) Suppose that P is a maximal M-ideal. If K/P is any nonzero submodule of M/P with $\mathrm{Hom}_R(M,K/P) \neq 0$, then $P = \mathrm{Ann}_M(M/P) \subseteq \mathrm{Ann}_M(K/P)$ and $\mathrm{Ann}_M(K/P) \neq M$, so the maximality of P forces $\mathrm{Ann}_M(K/P) = \mathrm{Ann}_M(M/P)$. Thus M/P is an M-prime module with $\mathrm{Ann}_M(M/P) = P$.

(b) If P is a primitive M-ideal, then $P = \mathrm{Ann}_M(S)$ for a simple left R-module S. It follows immediately from Corollary 2.9 that S is an M-prime module. □

The next proposition generalizes the well-known result that a simple Artinian ring is a finite direct sum of minimal left ideals.

Proposition 3.7 *Assume that M is a finitely generated, Artinian module. If* (0) *is a maximal M-ideal, then M is a homogeneous semisimple module.*

Proof. Since M is finitely generated, it has a maximal submodule M_0, and we can consider the radical rad_S defined by the simple module $S = M/M_0$. Since $\text{rad}_S(M) \neq M$, by assumption we must have $\text{rad}_S(M) = (0)$, and so (0) is an intersection of maximal submodules. Since M is Artinian, (0) must be a finite intersection of maximal submodules, and therefore M is isomorphic to a finite direct sum of copies of S. □

4 Modules with finite length

Using the notion of an M-ideal, we can obtain some information on modules with finite length. The definition of the Jacobson radical of M also provides an illustration of the use of the product of M-ideals.

Let $\mathcal{S}$ denote the class of simple modules in $\sigma[M]$, and let J denote the radical cogenerated by $\mathcal{S}$, so that for a module ${}_RX$ in $\sigma[M]$, the submodule $\text{J}(X)$ is the intersection of all maximal submodules of X. Our first result is a version of Nakayama's lemma.

Proposition 4.1 *If the module ${}_RX$ is finitely generated and belongs to $\sigma[M]$, then* $\text{J}(M) \cdot X = X$ *implies* $X = (0)$.

Proof. Assume that X is a nonzero finitely generated module in $\sigma[M]$. Then X has at least one maximal submodule X_0, and X/X_0 is a simple module in $\sigma[M]$, so $\text{J}(X) \neq X$. If X is nonzero, then it follows from Lemma 1.8 that $\text{J}(M) \cdot X \subseteq \text{J}(X) \neq X$. □

We recall that a submodule Y of the module ${}_RX$ is said to be a small (or superfluous) submodule if $Y + X' = X$ implies $X' = X$, for any submodule X' of X.

Corollary 4.2 *Let N be an M-ideal.*

(a) If $N \subseteq \mathrm{J}(M)$, then $N \cdot X$ is a small submodule of X, for all finitely generated generated modules ${}_RX$ in $\sigma[M]$.

(b) Conversely, if M is finitely generated and $N \cdot X$ is small in X for all finitely generated modules ${}_RX$ in $\sigma[M]$, then $N \subseteq \mathrm{J}(M)$.

Proof. (a) Suppose that Y is a submodule of X such that $N \cdot X + Y = X$. Then for the natural projection $\pi : X \to X/Y$ we have $\pi(N \cdot X) = X/Y$, and so $N \cdot (X/Y) = X/Y$. Thus $\mathrm{J}(M) \cdot (X/Y) = X/Y$, and since X/Y is finitely generated, it follows from Proposition 4.1 that $X/Y = (0)$. Thus $Y = X$, and it follows that $N \cdot Y$ is a small submodule of X.

(b) Since M is finitely generated and N is an M-ideal, by assumption $N = N \cdot M$ is a small submodule of M. Thus $N \subseteq \mathrm{J}(M)$, since the intersection of all maximal submodules of M is equal to the sum of all small submodules of M. □

Definition 4.3 *If N is an M-ideal, then successive powers of N are defined as follows. First, $N^2 = N \cdot N$. Then by induction, for any integer $k > 2$, we define $N^k = N \cdot N^{k-1}$.*

With the above convention, we obtain the following result.

Proposition 4.4 *If M is a module with finite length, and $\mathrm{J}(M)$ is the Jacobson radical of M, then $\mathrm{J}(M)$ is nilpotent.*

Proof. Since M is Noetherian, every submodule of M is finitely generated, so it follows from Proposition 4.1 that if $\mathrm{J}(M)^n \neq (0)$, then $\mathrm{J}(M) \cdot \mathrm{J}(M)^n$ is properly contained in $\mathrm{J}(M)^n$. The descending chain $M \supset \mathrm{J}(M) \supset \cdots \supset \mathrm{J}(M)^n \supset \cdots$ must terminate at (0), since M is Artinian. □

It is well-known that the following conditions are equivalent for a nonzero module ${}_RX$ (see Proposition 21.3 of [5]): (i) X contains an essential Artinian submodule; (ii) if a collection of submodules of X has zero intersection, then so does a finite subset of the collection; (iii) X has a finitely generated essential socle. The module X is called *essentially Artinian* (or finitely cogenerated) if the above conditions hold. It follows that X is Artinian if and only if every factor module is essentially Artinian. Proposition 5 of [12] shows that the ring R is left Artinian if and only if for each ideal I of R the ring R/I is essentially left Artinian. This result can be extended to Noetherian modules (we state a somewhat stronger result).

Proposition 4.5 *Assume that each M-ideal is a finitely generated submodule of M. Then M has finite length if and only if for each M-ideal N the factor module M/N is essentially Artinian.*

Proof. If M has finite length, then every factor module is Artinian.

To prove the converse, let $N = \bigcap_{k=1}^{\infty} \mathrm{J}(M)^k$. Then N is an M-ideal since the intersection of any collection of M-ideals is again an M-ideal. By assumption, M/N is essentially Artinian, so N is the intersection of finitely many terms in the sequence $\left\{\mathrm{J}(M)^k\right\}_{k=1}^{\infty}$. It follows that there exists a positive integer n such that $\mathrm{J}(M)^k = \mathrm{J}(M)^n$ for all integers $k > n$. Since $\mathrm{J}(M)^n$ is assumed to be finitely generated, this contradicts Proposition 4.1, unless $\mathrm{J}(M)^n = (0)$.

In each factor module $\mathrm{J}(M)^{k-1}/\mathrm{J}(M)^k$, the (0) submodule is an intersection of maximal submodules, and this intersection reduces to a finite intersection. It follows that $\mathrm{J}(M)^{k-1}/\mathrm{J}(M)^k$ can be embedded in a finitely generated semisimple module, so it has a composition series. Thus M itself has a composition series. □

Proposition 4.6 *Assume that M has finite length, and that* $\mathrm{Hom}_R(M, N) \neq 0$ *for every nonzero M-ideal $N \subseteq M$. If M is an M-prime module, then M is semisimple and homogeneous.*

Proof. It follows from Proposition 4.4 that $\mathrm{J}(M)$ is nilpotent of some degree, say n. Then $\mathrm{J}(M) \cdot \mathrm{J}(M)^{n-1} = \mathrm{J}(M)^n = (0)$, but $\mathrm{J}(M)^{n-1} \neq (0)$. By assumption $\mathrm{Hom}_R(M, \mathrm{J}(M)^{n-1}) \neq 0$, and since M is an M-prime module, it follows that $\mathrm{J}(M) = (0)$, and thus $n = 1$. Since M has finite length, this implies that M can be embedded in a finite direct sum of simple modules, and hence M is a semisimple module.

Suppose that M contains two non-isomorphic simple modules. Let S and T be the corresponding homogeneous components of M. Then $\mathrm{Hom}_R(M, T) \neq 0$, and so

Proposition 2.8 implies that T cogenerates S, since M is M-prime. This contradicts the assumption that S and T are generated by nonisomorphic simple modules, and so it follows that M is homogeneous. $\square$

Example 4.1

This example exhibits a module M of finite length such that M is M-prime but not semisimple. By construction, M is a finitely generated, projective module. In addition, the ring R can be embedded in $M \oplus M$, and so $\sigma[M] = R\text{–}Mod$.

Let F be a field, and let $R = \begin{bmatrix} F & 0 \\ F & F \end{bmatrix}$ be the ring of lower triangular 2×2 matrices over F. Let M be the left ideal $\begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix}$, and let N be the subset $\begin{bmatrix} 0 & 0 \\ F & 0 \end{bmatrix}$. The lattice of submodules of M is just $(0) \subset N \subset M$, so M is not semisimple. It is easy to check that the simple modules M/N and N are not isomorphic. Furthermore, $\text{Hom}_R(M/N, N) = 0$, which implies that N is not an M-generated submodule. Thus M has no proper nontrivial M-generated submodules, and so it must be M-prime. We note, in addition, that $\text{Hom}_R(N, M/N) = 0$ implies that N is an M-ideal.

5 Projective modules

In this section we investigate the simplifications obtained in case M is projective in the subcategory $\sigma[M]$. For certain results, the weaker assumption that M is quasi-projective is all that is necessary.

Proposition 5.1 *If M is quasi-projective, then a submodule N of M is an M-ideal if and only if it is a fully invariant submodule of M.*

Proof. Suppose that M is quasi-projective, and N is a fully invariant submodule of M. If $f \in \text{Hom}_R(M, M/N)$, and $\pi : M \to M/N$ is the canonical projection, then f lifts to $\widehat{f} \in \text{End}_R(M)$, with $f = \pi\widehat{f}$, since M is quasi-projective. Then

$\widehat{f}(N) \subseteq N$ since N is fully invariant, and so $f(N) = \pi\widehat{f}(N) = (0)$, showing that N is an M-ideal. $\square$

Example 5.1

Let $M = \mathbf{Z}_{p^\infty}$, in the category of $\mathbf{Z}$-modules. Every proper factor module of M is isomorphic to M itself, and so no proper nontrivial submodule can be an M-ideal. On the other hand, every submodule of M is fully invariant in M. Thus a fully invariant submodule of M need not be an M-ideal.

Example 5.2

If $M = R/I$, where I is an ideal of R, then $\sigma[M]$ coincides with R/I-*Mod*. By Proposition 5.1, the M-ideals coincide with the ideals of R/I, since they are precisely the fully-invariant submodules of R/I.

Proposition 5.2 *Assume that M is quasi-projective, and let K and N be submodules of M.*

(a) Assume that $N \subseteq K$ and N is an M-ideal. If K/N is an (M/N)-ideal (in M/N), then K is an M-ideal.

(b) If N and K are M-ideals, then $N + K$ is an M-ideal.

Proof. These results clearly hold for fully invariant submodules. Since M is assumed to be quasi-projective and N is a fully invariant submodule, Proposition 18.2 (4) of [5] shows that M/N is also quasi-projective. It then follows from Proposition 5.1 that the desired results hold for M-ideals. $\square$

The modules defined in Example 4.1 show that in the next proposition the condition that $\mathrm{Hom}_R(M, N) \neq 0$ for all fully invariant submodules $N \subseteq M$ is a necessary condition. In fact, in Example 4.1 the module M is quasi-projective and M-prime, but it is not R-prime, since for the nonzero submodule $N \subseteq M$ we have $\mathrm{Ann}_R(N) \neq \mathrm{Ann}_R(M)$. Note that the submodule N is fully invariant, with $\mathrm{Hom}_R(M, N) = 0$.

Proposition 5.3 *Let M be an M-prime module. If M is quasi-projective and* $\text{Hom}_R(M, N) \neq 0$ *for every nonzero fully invariant submodule $N \subseteq M$, then M is an R-prime module.*

Proof. Let I be an ideal of R. If $N = \{m \in M \mid Im = 0\}$ is nonzero, then by assumption $\text{Hom}_R(M, N) \neq 0$, since N is a fully invariant submodule. It follows from Proposition 2.8 that N must cogenerate M, so we may conclude that $N = M$, showing that M is an R-prime module. □

The converse of Proposition 5.3 does not hold. The next example exhibits a module M that is not M-prime, although it is R-prime and quasi-projective, and also satisfies the condition that $\text{Hom}_R(M, N) \neq 0$ for all submodules $N \subseteq M$.

Example 5.3

> Let R be a simple Noetherian ring that has two nonisomorphic simple modules S_1 and S_2. In this case let $M = S_1 \oplus S_2$. Then M is an R-prime module, since every nonzero R-module has zero annihilator. On the other hand, it follows from Proposition 2.8 that M is not an M-prime module. It is clear that $\text{Hom}_R(M, N) \neq 0$ for the two proper nonzero submodules of M.

The next results appear to require the stronger hypothesis that M is projective as an object of the category $\sigma[M]$.

Lemma 5.4 *Assume that M is projective in $\sigma[M]$, and let N be any submodule of M. If $p : X \to Y$ is an epimorphism, for any module ${}_RX$ in $\sigma[M]$, then p maps $\sum_{f \in \text{Hom}(M,X)} f(N)$ onto $\sum_{f \in \text{Hom}(M,Y)} f(N)$.*

Proof. Let $y \in Y$, with $y = \sum_{i=1}^{n} f_i(m_i)$, where $f_i \in \text{Hom}_R(M, Y)$ and $m_i \in N \subseteq M$ for $i = 1, \ldots, n$. Since M is projective in $\sigma[M]$, each homomorphism f_i can be lifted to $\widehat{f_i} : M \to X$ with $p\widehat{f_i} = f_i$. Thus $y = \sum_{i=1}^{n} p\widehat{f_i}(m_i) = p\left(\sum_{i=1}^{n} \widehat{f_i}(m_i)\right)$, showing that p maps $\sum_{f \in \text{Hom}(M,X)} f(N)$ onto $\sum_{f \in \text{Hom}(M,Y)} f(N)$. □

Proposition 5.5 *Assume that M is projective in $\sigma[M]$, and let N be any submodule of M. The following conditions hold for any module ${}_RX$ in $\sigma[M]$, and any submodule $Y \subseteq X$.*

(a) $N \cdot X = \sum_{f \in \mathrm{Hom}(M,X)} f(N)$

(b) $N \cdot (X/Y) = (0)$ *if and only if* $N \cdot X \subseteq Y$

(c) If $N = \mathrm{Ann}_M(X/Y)$, *then* $\mathrm{Ann}_M(X/(N \cdot X)) = N$.

Proof. (a) It is always true that $\sum_{f \in \mathrm{Hom}(M,X)} f(N) \subseteq N \cdot X$. To show the reverse inclusion, let $Y = \sum_{f \in \mathrm{Hom}(M,X)} f(N)$. By Lemma 5.4, $f(N) = (0)$ for all homomorphisms $f \in \mathrm{Hom}_R(M, X/Y)$, and so by definition $N \cdot X \subseteq Y$.

(b) By definition, if $N \cdot (X/Y) = (0)$, then $N \cdot X \subseteq Y$. On the other hand, if $N \cdot X \subseteq Y$, then $\sum_{f \in \mathrm{Hom}(M,X)} f(N) \subseteq Y$, by part (a). It follows from Lemma 5.4 that $\sum_{f \in \mathrm{Hom}(M,X/Y)} f(N) = 0$, and so $N \cdot (X/Y) = (0)$.

(c) Now suppose that $N = \mathrm{Ann}_M(X/Y)$, and let $\mathrm{Ann}_M(X/(N \cdot X)) = A$. Since M is projective in $\sigma[M]$, it follows from part (b) that $N \cdot (X/(N \cdot X)) = (0)$, and so $N \subseteq A$. On the other hand, since $A \cdot (X/(N \cdot X)) = (0)$, it follows that $A \cdot X \subseteq N \cdot X \subseteq Y$. Then since M is projective in $\sigma[M]$, it follows from part (b) that $A \cdot (X/Y) = (0)$, and so $A \subseteq \mathrm{Ann}_M(X/Y) = N$. □

As an immediate consequence of Proposition 5.5, we note that if M is projective in $\sigma[M]$ and $N \cdot X = (0)$, then $N \cdot f(X) = (0)$ for any homomorphic image $f(X)$ of X. If N is an M-ideal, and X is a direct sum of copies of M/N, then $N \cdot X = (0)$. It follows immediately that $N \cdot X = (0)$ for any module X in $\sigma[M/N]$.

Under the assumption that M is projective in $\sigma[M]$, it follows from Proposition 5.5 that if K is any submodule of M, then $\mathrm{Ann}_M(M/K)$ is the largest M-ideal contained in K. This provides the first indication that in this situation prime M-ideals behave more like ordinary ideals. We first show that the multiplication we have introduced is associative. (The argument used is similar to that in Lemma 2.1 of [10].) We then show that an M-ideal P is prime if and only if M/P is an M-prime module.

Proposition 5.6 *Assume that M is projective in $\sigma[M]$, and let K and N be submodules of M. Then $(K \cdot N) \cdot X = K \cdot (N \cdot X)$ for any module ${}_RX$ in $\sigma[M]$.*

Proof. We use the description of the product given in part (a) of Proposition 5.5.

To show that $(K \cdot N) \cdot X \subseteq K \cdot (N \cdot X)$, let $x \in (K \cdot N) \cdot X$. Then for some positive integer n we have $x = \sum_{i=1}^{n} f_i(m_i)$, for homomorphisms $f_i \in \mathrm{Hom}_R(M, X)$ and elements $m_i \in K \cdot N$. But each element m_i has the form $m_i = \sum_{j=1}^{k(j)} g_{ij}(x_{ij})$, where $g_{ij} \in \mathrm{Hom}_R(M, N)$ and $x_{ij} \in K$. Thus $x = \sum_{i=1}^{n} \sum_{j=1}^{k(j)} f_i g_{ij}(x_{ij})$, with $x_{ij} \in M$. Furthermore, since $x_{ij} \in K$ and $g_{ij}(M) \subseteq N$, for each homomorphism g_{ij}, we actually have $f_i g_{ij} \in \mathrm{Hom}_R(M, N \cdot X)$. This implies that $x \in K \cdot (N \cdot X)$.

To show that $(K \cdot N) \cdot X \supseteq K \cdot (N \cdot X)$, let $x \in K \cdot (N \cdot X)$. Then for some positive integer n we have $x = \sum_{i=1}^{n} f_i(m_i)$, for homomorphisms $f_i \in \mathrm{Hom}_R(M,\, N \cdot X)$ and elements $m_i \in K$.

We first consider the case of a single element $m \in K$ and a single homomorphism $f \in \mathrm{Hom}_R(M,\, N \cdot X)$. Let $M^{(I)}$ denote the direct sum of I copies of M, taken over the index set $I = \mathrm{Hom}_R(M, X)$, and let $h : M^{(I)} \to X$ be the homomorphism whose component indexed by $f \in I$ is f itself. If g is the restriction of h to $N^{(I)}$, then by Proposition 5.5 (a) the image of g is $N \cdot X$. Since M is projective in $\sigma[M]$, it is possible to lift f to $\widehat{f} : M \to N^{(I)}$, with $g\widehat{f} = f$. We have $\widehat{f}(m) = \sum_{j=1}^{n} \lambda_j \pi_j(\widehat{f}(m))$, for some finite subset of projections $\pi_j : M^{(I)} \to M$ and inclusions $\lambda_j : M \to M^{(I)}$. We note that $\pi_j \widehat{f} : M \to M$, for each index j, with $\pi_j \widehat{f}(M) \subseteq N$. Since $m \in K$, we can consider $\pi_j \widehat{f}(m)$ to be an element of $K \cdot N$. Then $h\lambda_j : M \to X$, and so $h\lambda_j \pi_j \widehat{f}(m) \in h\lambda_j(K \cdot N)$, which implies that $h\lambda_j \pi_j \widehat{f}(m) \in (K \cdot N) \cdot X$. Finally, $f(m) = h\widehat{f}(m) = \sum_{j=1}^{n} h\lambda_j \pi_j \widehat{f}(m)$, and therefore $f(m) \in (K \cdot N) \cdot X$.

Returning to the element $x \in K \cdot (N \cdot X)$, with $x = \sum_{i=1}^{n} f_i(m_i)$, for homomorphisms $f_i : M \to N \cdot X$, the preceding paragraph shows that $x \in (K \cdot N) \cdot X$, completing the proof. □

Theorem 5.7 *Let $P \subseteq M$ be an M-ideal, and assume that M is projective in $\sigma[M]$. The following conditions are equivalent.*

(1) P is a prime M-ideal;

(2) $N \cdot K \subseteq P$ implies $N \subseteq P$ or $K \subseteq P$, for all M-ideals N and K such that K is M-generated;

(3) M/P is an M-prime module.

Proof. (1) $\Rightarrow$ (2): Suppose that P is a prime M-ideal, and that $N \cdot K \subseteq P$, for M-ideals N and K such that $K \not\subseteq P$ and K is M-generated. By assumption there is an M-prime module X with $P = \mathrm{Ann}_M(X)$, and so there exists $f \in \mathrm{Hom}_R(M/P, X)$ with $f((K+P)/P) \neq (0)$. Since $N \cdot K \subseteq P$, we have $N \cdot K \subseteq P \cap K$, and so Proposition 5.5 (b) implies that $N \cdot f((K+P)/P) = (0)$. Then $f((K+P)/P)$ is an M-generated submodule of X, since $(K+P)/P \cong K/(P \cap K)$ and K is M-generated. It follows that $N \cdot X = (0)$, since X is M-prime, and thus $N \subseteq P$ since $P = \mathrm{Ann}_M(X)$.

(2) $\Rightarrow$ (3): Let $P \subseteq K \subseteq M$, and assume that K/P is a nonzero M-generated submodule of M/P such that $N \cdot (K/P) = (0)$ for an M-ideal N. Since K/P is M-generated and M is projective in $\sigma[M]$, there exists an M-generated submodule L of M such that $(L+P)/P = K/P$. (The argument is similar to the one used in the proof of Lemma 5.4.) Since K/P is nonzero, we have $L \not\subseteq P$. Since $N \cdot (K/P) = (0)$, we have $N \cdot ((L+P)/P) = (0)$, which implies that $N \cdot L \subseteq N \cdot (L+P) \subseteq P$.

Now consider the submodule $L \cdot M$, which is the smallest M-ideal that contains L. Since M is projective in $\sigma[M]$, and L is M-generated, it follows from Proposition 5.5 (a) that $L \cdot M$ is M-generated. Furthermore, Proposition 5.6 implies that $N \cdot (L \cdot M) = (N \cdot L) \cdot M$. Since $N \cdot L \subseteq P$, and $P \cdot M = M$ since P is an M-ideal, we have

$$N \cdot (L \cdot M) = (N \cdot L) \cdot M \subseteq P \cdot M = P\ .$$

Thus $L \cdot M$ is an M-generated M-ideal with $N \cdot (L \cdot M) \subseteq P$, but $L \cdot M \not\subseteq P$. The hypothesis implies that $N \subseteq P$, and therefore $N \cdot (M/P) = (0)$. It follows from condition (4) of Proposition 2.2 that M/P is an M-prime module.

(3) $\Rightarrow$ (1): If M/P is an M-prime module, then P is a prime M-ideal since $P = \mathrm{Ann}_M(M/P)$. □

Proposition 5.8 *Let N be a submodule of M, and let I be a left ideal of R.*

(a) The left ideal $N \cdot R$ is a two-sided ideal of R. If N is an M-ideal, then $(N \cdot R)M \subseteq N$.

(b) The submodule IM is an M-ideal. If I is a two-sided ideal of R, then $(IM) \cdot R \subseteq I$.

(c) If M is projective in $\sigma[M]$, then $(IN) \cdot X = I(N \cdot X)$ for any X in $\sigma[M]$.

Proof. (a) Since $N\cdot(-)$ defines a radical, it follows that $N\cdot R$ is a two-sided ideal of R. If N is an M-ideal, then it follows from Lemma 1.8 that $(N\cdot R)M \subseteq N\cdot M = N$.

(b) We have already noted that IM is an M-ideal. If I is a two-sided ideal of R, let $f \in \mathrm{Hom}_R(M,(R/I))$. Then for any element $m = \sum_{i=1}^n a_i m_i \in IM$, with $a_i \in I$, we have $f(m) = \sum_{i=1}^n a_i f(m_i) = 0$. Thus $IM\cdot(R/I) = (0)$, and hence $(IM)\cdot R \subseteq I$.

(c) Assume that M is projective in $\sigma[M]$. By Proposition 5.5 (a), for any module ${}_RX$ in $\sigma[M]$ we have

$$(IN)\cdot X = \sum_{f\in \mathrm{Hom}(M,X)} f(IN) = I\left(\sum_{f\in \mathrm{Hom}(M,X)} f(N)\right) = I\left(N\cdot X\right) \ .$$

This completes the proof. □

6 Noetherian modules

Before investigating the correspondence between indecomposable M-injective modules and prime M-ideals, we show that several familiar results for Noetherian rings can be extended to modules.

Proposition 6.1 *Let M be a Noetherian module, and assume that the module ${}_RX$ is a homomorphic image of a finite direct sum of copies of M. Then there exists a chain of submodules*

$$(0) = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X$$

such that for $1 \le i \le n$, each factor module X_i/X_{i-1} is an M-prime module.

Proof. Since X is M-generated, we certainly have $\mathrm{Hom}_R(M,X) \neq 0$, and so it follows from Proposition 3.4 that X has an M-prime submodule X_1. The hypotheses of Proposition 3.4 are also satisfied by X/X_1, and so it contains an M-prime submodule which we denote by X_2/X_1. We can continue this process inductively. Since X is a homomorphic image of a finite direct sum of copies of M, it is Noetherian, and so the ascending chain $X_1 \subset X_2 \subset \cdots$ must terminate at X after finitely many steps. □

Definition 6.2 *The prime radical of the module M, denoted by* $\mathrm{P}(M)$*, is defined to be the intersection of all prime M-ideals.*

We note that there is another notion of the prime radical of a module (see [13], for example), where the prime radical of the module M is defined to be the intersection of all R-prime submodules of M. The two notions differ, as can be seen in Example 4.1.

Before proceeding, we make the following useful observation. Since each prime M-ideal is the annihilator in M of an M-prime module, it follows that $\mathrm{P}(M) = \mathrm{rad}_{\mathcal{C}}(M)$, where $\mathcal{C}$ is the class of all M-prime left R-modules. If ${}_RX$ is any module with a submodule Y such that X/Y is M-prime, then $\mathrm{rad}_{\mathcal{C}}(X) \subseteq Y$. In this case it follows from Lemma 1.8 that $\mathrm{P}(M) \cdot X \subseteq Y$.

Proposition 6.3 *If M is a Noetherian module, then the prime radical* $\mathrm{P}(M)$ *is a nilpotent M-ideal.*

Proof. Using Proposition 6.1, it is possible to construct a sequence of submodules

$$(0) = M_n \subset M_{n-1} \subset \cdots \subset M_1 \subset M_0 = M$$

such that each factor is an M-prime module. Then $\mathrm{P}(M) \subseteq M_1$ since M/M_1 is an M-prime module, and in general

$$\mathrm{P}(M)^{i+1} = \mathrm{P}(M) \cdot \mathrm{P}(M)^i \subseteq \mathrm{P}(M) \cdot M_i \subseteq M_{i+1}$$

for $1 \leq i \leq n-1$. Using this inductive definition of the powers of $\mathrm{P}(M)$ (see Definition 4.3) we have $\mathrm{P}(M)^n = (0)$. □

Definition 6.4 *The module ${}_RX$ in $\sigma[M]$ is said to be finitely M-generated if there exists an epimorphism $f : M^n \to X$, for some positive integer n. It is said to be finitely M-annihilated if there exists a monomorphism $g : M/\operatorname{Ann}_M(X) \to X^m$, for some positive integer m.*

Definition 6.5 *The module ${}_RM$ is said to satisfy condition H if every finitely M-generated module is finitely M-annihilated.*

Note that if $M = R$ and R is a fully bounded Noetherian ring, then M satisfies condition H. The same is true if M is an Artinian module, since then M/K has the finite intersection property.

In the following results on Gabriel's correspondence, in addition to condition H we assume that $\text{Hom}_R(M, X) \neq 0$ for all modules ${}_R X$ in $\sigma[M]$. The next example shows that this assumption is weaker than the assumption that M is a generator in $\sigma[M]$.

Example 6.1

Let F be a field, let R be the ring of lower triangular 3×3 matrices over F, and let J denote the Jacobson radical of R. We use the following notation.

$$e_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad e_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let M be the module $M = Re_1 \oplus (Re_2/Je_2) \oplus Re_3$. Since Re_2 is isomorphic to a submodule of Re_1, it follows that R can be embedded in M^2, so M is cofaithful, and therefore $\sigma[M] = R\text{–}Mod$. Each simple left R-module is a homomorphic image of M, and hence $\text{Hom}_R(M, X) \neq 0$ for each left R-module X. On the other hand, M is not a generator for $R\text{–}Mod$, since Re_2 is not M-generated. This follows from the fact that $\text{Hom}_R(Re_1, Re_2) = 0$, while $(Re_2/Je_2) \oplus Re_3$ is a semisimple module and therefore cannot generate Re_2.

Lemma 6.6 *Assume that M is a Noetherian module such that* $\text{Hom}_R(M, X) \neq 0$ *for all nonzero modules X in $\sigma[M]$. Then each nonzero uniform module in $\sigma[M]$ has a unique prime M-ideal associated to it.*

Proof. Let ${}_R X$ be a nonzero uniform module in $\sigma[M]$. Then X contains an M-prime submodule since by assumption $\text{Hom}_R(M, X) \neq 0$. Suppose that P and Q are prime M-ideals associated to X. Then $P = \text{Ann}_M(Y)$ and $Q = \text{Ann}_M(W)$, where

Y and W are M-prime submodules of X. Since X is uniform, we have $Y \cap W \neq (0)$, and so by assumption $\mathrm{Hom}_R(M, Y \cap W) \neq 0$. Therefore

$$P = \mathrm{Ann}_M(Y) = \mathrm{Ann}_M(Y \cap W) = \mathrm{Ann}_M(W) = Q \ ,$$

which completes the proof. $\square$

Theorem 6.7 *Let M be a Noetherian module. If M satisfies condition H and* $\mathrm{Hom}_R(M, X) \neq 0$ *for all modules X in $\sigma[M]$, then there is a one-to-one correspondence between isomorphism classes of indecomposable M-injective modules in $\sigma[M]$ and prime M-ideals.*

Proof. Let U be an indecomposable M-injective module in $\sigma[M]$. Then U is uniform, so Lemma 6.6 implies that U has a unique M-prime ideal P associated to it. Thus the function that assigns to U its associated prime M-ideal is well-defined.

By Corollary 2.3 we can assume without loss of generality that $P = \mathrm{Ann}_M(U_1)$, where U_1 is an M-prime submodule that is a homomorphic image of M. Since M satisfies condition H and U_1 is finitely M-generated, it follows that U_1 is finitely M-annihilated. Thus there exists an embedding

$$0 \to M/P \to U_1^n \to U^n$$

of M/P into a finite direct sum of copies of U. Considering this embedding in R–Mod gives an embedding

$$0 \to E(M/P) \to E(U)^n$$

of the respective injective envelopes. Since U is uniform, the injective module $E(U)$ is indecomposable, and it follows from the Krull-Schmidt-Azumaya theorem that $E(M/P)$ is isomorphic to a direct sum of copies of $E(U)$. If X is any indecomposable M-injective module in $\sigma[M]$ with P as its associated prime M-ideal, we obtain a similar sequence

$$0 \to E(M/P) \to E(X)^m$$

for some integer m. Since X is uniform, the Krull-Schmidt-Azumaya theorem implies that that $E(M/P)$ is isomorphic to a direct sum of copies of $E(X)$, and hence $E(X) \cong E(U)$. It follows that

$$X = \mathrm{tr}^M(E(X)) \cong \mathrm{tr}^M(E(U)) = U \ ,$$

and so the correspondence between indecomposable M-injective modules and prime M-ideals is a one-to-one correspondence. $\square$

The next proposition shows that under the assumptions of Theorem 6.7 the prime M-ideals can be described more fully. In the proposition the second hypothesis on M is necessary. This is shown by the module M defined in Example 4.1, which satisfies condition H since it is Artinian, but although (0) is a prime M-ideal, M itself is not semi-compressible. Note that in the example we have $\mathrm{Hom}_R(M, N) = (0)$.

Proposition 6.8 *Let M be a Noetherian module such that M satisfies condition H and $\mathrm{Hom}_R(M, X) \neq 0$ for all modules X in $\sigma[M]$. If P is a prime M-ideal, then M/P is a semi-compressible module.*

Proof. Assume that P is a prime M-ideal, with $P = \mathrm{Ann}_M(X)$ for an M-prime module ${}_RX$ in $\sigma[M]$. By hypothesis, there is a nonzero element $h \in \mathrm{Hom}_R(M, X)$, and $h(M)$ is M-prime by Corollary 2.3. Since $\mathrm{Ann}_M(h(M)) = P$, we can assume without loss of generality that X is finitely M-generated. Then X is finitely M-annihilated since M satisfies condition H, so there is an embedding $f : M/P \to X^n$, for some positive integer n.

Now let L be any nonzero submodule of M/P. Since M is Noetherian, L contains a uniform submodule K, and since by assumption $\mathrm{Hom}_R(M, K) \neq 0$, we can assume without loss of generality that K is a homomorphic image of M. We must have $\cap_{i=1}^n \ker(f_i) = (0)$ for the components $\{f_i\}_{i=1}^n$ of the embedding $f : M/P \to X^n$, and thus $\ker(f_j) = (0)$ for some j, since K is uniform. It follows that K is isomorphic to a submodule of X, and therefore $\mathrm{Ann}_M(K) = P$, since X is an M-prime module and $\mathrm{Hom}_R(M, K) \neq 0$.

Finally, K is finitely M-annihilated since it is finitely M-generated and M satisfies condition H, so there exists an embedding $g : M/P \to K^m$ for some positive integer m. Thus M/P can be embedded in a finite direct sum of copies of L, which shows that M/P is semi-compressible. $\square$

The next propositions generalize properties of fully bounded Noetherian rings to Noetherian quasi-projective modules that satisfy condition H.

Lemma 6.9 *Let M be a Noetherian quasi-projective module that satisfies condition H. If P is a maximal fully invariant submodule of M, then M/P is a homogeneous semisimple module.*

Proof. Let $P \subset M$ be a maximal fully invariant submodule. Since M is Noetherian, there exists a maximal submodule K of M with $P \subseteq K \subset M$. Let $f : M \to M/K$ be any R-homomorphism, and let $\pi_1 : M \to M/P$ and $\pi_2 : M/P \to M/K$ be the natural projections. Since M is assumed to be quasi-projective, f can be lifted to $\widehat{f} \in \mathrm{End}_R(M)$ with $\pi_2\pi_1\widehat{f} = f$. Then $\pi_1\widehat{f}(P) = (0)$ since P is an M-ideal, and therefore $f(P) = \pi_2\pi_1\widehat{f}(P) = (0)$, showing that $P \subseteq \mathrm{Ann}_M(M/K)$. But then $P = \mathrm{Ann}_M(M/K)$, because P is a maximal M-ideal. Since M/K is finitely M-generated, and M satisfies condition H, it follows that M/K is finitely M-annihilated. Therefore M/P can be embedded in a finite direct sum of copies of the simple module M/K, and so M/P is semisimple and homogeneous. □

Proposition 6.10 *Let M be a Noetherian quasi-projective module that satisfies condition H. If every prime M-ideal is a maximal fully invariant submodule, then M has finite length.*

Proof. By Proposition 3.4, M contains an M-prime submodule M_1. By Corollary 2.3, we can assume without loss of generality that M_1 is a homomorphic image of M, and hence a homomorphic image of M/P, for $P = \mathrm{Ann}_M(M_1)$. It follows from Lemma 6.9 that M/P is a semisimple module, and so M_1 has finite length since it is Noetherian and semisimple.

As in the proof of Proposition 6.1, it is possible to construct a sequence of submodules

$$(0) = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M$$

such that each factor is an M-prime module with finite length, and therefore M itself has finite length. □

Proposition 6.11 *Let M be any quasi-projective module that satisfies condition H. If $N \subseteq M$ is a fully invariant submodule, then M/N satisfies condition H.*

Proof. Let ${}_RX$ be finitely M/N-generated, with an epimorphism $p : (M/N)^k \to X$. Then X is finitely M-generated, so it is finitely M-annihilated since M satisfies condition H. It suffices to show that $N \subseteq \operatorname{Ann}_M(X)$, since then $(M/N)/\operatorname{Ann}_{M/N}(X)$ is isomorphic to $M/\operatorname{Ann}_M(X)$, and this yields an embedding of $(M/N)/\operatorname{Ann}_{M/N}(X)$ into X^n for some positive integer n.

Since M is assumed to be quasi-projective, Proposition 18.2 (2) of [5] implies that it is M^k-projective. Therefore any R-homomorphism $f : M \to X$ lifts to $\widehat{f} : M \to M^k$, with $p\pi\widehat{f} = f$, where $\pi : M^k \to (M/N)^k$ is the natural projection. It follows that $\pi\widehat{f}(N) = (0)$ since N is an M-ideal, and thus $N \subseteq \ker(f)$, which implies that $N \subseteq \operatorname{Ann}_M(X)$. □

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References

[1] Gabriel, P. Des catégories abéliennes, Bull. Soc. Math. France **1964**, *90*, 323–448.

[2] Krause, G. On fully left bounded left noetherian rings, J. Algebra **1972**, *23*, 88–99.

[3] Goodearl, K. R.; Warfield, R. B. Jr. *An Introduction to Noncommutative Noetherian Rings*, London Mathematical Society Student Texts, Vol. 16, Cambridge University Press: Cambridge, 1989.

[4] Dung, N. V.; Huynh, D. V.; Smith, P. F. ; Wisbauer, R. *Extending modules*, Longman Group Limited: Essex, 1994.

[5] Wisbauer, R. *Foundations of Module and Ring Theory*, Gordon and Breach: Philadelphia, 1991.

[6] Wisbauer, R. *Modules and algebras: bimodule structure and group actions on algebras*, Addison Wesley Longman, Ltd.: Harlow, 1996.

[7] Anderson, F. W.; Fuller, K. R. *Rings and Categories of Modules*, Graduate Texts in Mathematics, Vol. 13, 2^{nd} Ed., Springer-Verlag: Berlin-Heidelberg-New York, 1992.

[8] Bican, L.; Jambor, P.; Kepka, T.; Němec, P. Generation of preradicals, Czechoslovak Math. J. **1977**, *27 (102)*, 155–166.

[9] Stenström, B. *Rings of Quotients*, Springer-Verlag: Berlin-Heidelberg-New York, 1975.

[10] Bican, L.; Jambor, P.; Kepka, T.; Němec, P. Prime and coprime modules, Fund. Math. **107** (1980), 33–45.

[11] McConnell, J. C.; Robson, J. C. *Noncommutative Noetherian Rings*, John Wiley & Sons: New York, 1987.

[12] Beachy, J. A. On Quasi-Artinian Rings, J. London Math. Soc. (2) **1971**, *3*, 449–452.

[13] McCasland, R. L.; Smith, P. F. Prime Submodules of Noetherian Modules, Rocky Mountain J. Math. **1993**, *23*, 1041–1062.

[14] Azumaya, G.; Mbuntum, F.; Varadarajan, K. On M-projective and M-injective modules, Pacific J. Math. **1975**, *59*, 9–16.

[15] Beachy, J. A. A Characterization of Prime Ideals, J. Indian Math. Soc. **1973**, *37*, 343–345.

[16] Beachy, J. A. On maximal torsion radicals, *Canad. J. Math.* **1973**, *25*, 712-726.