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A NOTE ON COHN'S UNIVERSAL LOCALIZATION AT A SEMIPRIME IDEAL

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The universal localisation $R_{\Gamma(S)}$ at a semiprime ideal S of a left Noetherian ring R was defined and studied by P. M. Cohn. In this note we investigate the interaction between the universal localisation $R_{\Gamma(S)}$, the Ore localisation at S, and the torsion-theoretic localisation at the injective envelope $E(R/S)$ of the module ${}_R(R/S)$.

Throughout this note, R will denote a left Noetherian ring (with identity), and S will denote a semiprime ideal of R. The study of the universal localisation $R_{\Gamma(S)}$ was initiated by P. M. Cohn, and continued by the present author (in [**3, 4, 5, 6**]) and others. It was shown in [**5**] that Goldie's localisation, defined in [**11**], is also related to the universal localisation (it is a homomorphic image of the latter).

Recently there has been renewed interest in the construction of the universal localisation because of its connection with some questions in topology (see [**10, 12**]). One of the questions of interest has been to find conditions under which $R_{\Gamma(S)}$ is flat as a left R-module. [**4**, Theorem 3.1] shows that this is equivalent to the statement that $C(S)$ is left localizable, in which case $R_{\Gamma(S)}$ is naturally isomorphic to the Ore localisation R_S. In this note we give two localizability conditions phrased in terms of the connection between the universal localisation at S and the torsion theoretic localisation at the injective envelope $E(R/S)$. We provide a counterexample to the statement of [**8**, Theorem 4.2], in which it is asserted that there always exists a ring homomorphism from the torsion theoretic localisation into the universal localisation, and investigate conditions under which such a homomorphism does exist.

For any ideal I of R, the set of elements $c \in R$ that are regular modulo I will be denoted by $C(I)$. We need to extend this definition relative to S, as follows. For any positive integer n, let $\Gamma_n(S)$ denote the set of all matrices C such that C belongs to the $n \times n$ matrix ring $M_n(R)$ and the image of C in $M_n(R/S)$ is a regular element. This will be abbreviated by saying that C is regular modulo S. Note that $C \in \Gamma_n(S)$ if and only if the image of C is invertible under the canonical mapping from $M_n(R)$ into the left classical quotient ring $Q_{cl}(M_n(R/S)) \cong M_n(Q_{cl}(R/S))$. The union over all $n > 0$ of $\Gamma_n(S)$ will be denoted by $\Gamma(S)$.

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The universal localisation $R_{\Gamma(S)}$ of R at $\Gamma(S)$ is defined as the universal $\Gamma(S)$-inverting ring. It can be constructed as follows (see [**8, 9**] for details). For each n and each $n \times n$ matrix $[c_{ij}]$ in $\Gamma(S)$, take a set of n^2 symbols $[d_{ij}]$, and take a ring presentation of $R_{\Gamma(S)}$ consisting of all of the elements of R, as well as all of the elements d_{ij} as generators; as defining relations take all of the relations holding in R, together with all of the relations $[c_{ij}][d_{ij}] = I$ and $[d_{ij}][c_{ij}] = I$ which define all of the inverses of the matrices in $\Gamma(S)$. Any left $R_{\Gamma(S)}$-module is a left R-module, using the canonical ring homomorphism $\lambda : R \to R_{\Gamma(S)}$, and as such it has no torsion relative to $C(S)$.

The torsion-theoretic localisation of R at S is defined as follows. The torsion radical $\gamma = \mathrm{rad}_{E(R/S)}$ is defined for any module ${}_RM$ by letting $\mathrm{rad}_{E(R/S)}(M)$ be the intersection of all kernels of R-homomorphisms from M into $E(R/S)$. A module ${}_RM$ is called γ-torsion if $\gamma(M) = M$ and γ-torsionfree if $\gamma(M) = 0$; a submodule M' is γ-dense if M/M' is γ-torsion and γ-closed if M/M' is γ-torsionfree. The γ-closure of M' in M is defined as the intersection of all γ-closed submodules of M which contain M'. For the torsion radical γ, a left ideal $A \subseteq R$ is γ-closed if and only if A is the left annihilator of a subset of $E(R/S)$. In particular, the ideal $\gamma(R)$ is the left annihilator of W. A module ${}_RM$ is called γ-injective if each homomorphism $f : N' \to M$ such that N' is a γ-dense submodule of ${}_RN$ can be extended to N.

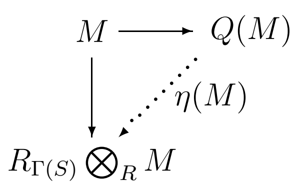
The set $\Gamma(S)$ defines an idempotent radical as follows. For each module ${}_RM$, let $\mathrm{rad}_{\Gamma(S)}(M)$ be the set of elements $m \in M$ such that m is a component of a vector $\mathbf{u}$ such that $C\mathbf{u} = \mathbf{0}$ for some matrix $C \in \Gamma(S)$ of the appropriate size. [**4**, Lemma 2.1] shows that γ is the largest torsion radical for which $\gamma \le \mathrm{rad}_{\Gamma(S)}$.

For the torsion radical γ, the full subcategory determined by all modules ${}_RM$ such that $E(M)$ and $E(M)/M$ are γ-torsionfree is called the quotient category determined by γ, and will be denoted by R–Modγ. Note that R–Modγ can also be described as the full subcategory of all γ-torsionfree, γ-injective modules. The inclusion functor from R–Mod$\gamma \to R$–Mod has a left adjoint, denoted by Q_γ, and defined by letting $Q_\gamma(M)$ be the γ-closure of $M/\gamma(M)$ in $E(M)/\gamma(M)$.

For any module $M \in R$–Modγ, and any element $m \in M$, the homomorphism $[r \mapsto rm] : R \to M$ defined by multiplication can be extended uniquely to $\rho_m : Q_\gamma(R) \to M$. For any element $q \in Q_\gamma(R)$, the homomorphism ρ_q can be used to define right multiplication by q, and this induces a ring structure on $Q_\gamma(R)$. Furthermore, any module $M \in R$–Modγ becomes a left $Q_\gamma(R)$-module by defining $qm = \rho_m(q)$, for all $q \in Q_\gamma(R)$ and $m \in M$. The ring $Q_\gamma(R)$ is called the ring of quotients determined by γ. The quotient category R–Modγ is also a quotient category of $Q_\gamma(R)$–Mod, and the functor Q_γ can be viewed as a functor from R–Mod to $Q_\gamma(R)$–Mod, although as such it may be only left exact. We can define an associated quotient functor $Q : R$–Mod $\to R$–Mod by letting Q be the composition of the functor Q_γ and the inclusion from R–Modγ into R–Mod.

[**8**, Theorem 4.2] asserts that in the above situation there is a natural homomorphism from $Q_\gamma(R)$ into $R_{\Gamma(S)}$ and that this is an isomorphism if and only if $C(S)$ is a left denominator set in R. The proof depends upon the existence of a natural transformation from the functor Q_γ to the functor $R_{\Gamma(S)} \bigotimes_R (-)$, which is constructed under the assumption that for any left $R_{\Gamma(S)}$-module X, the factor module $E(X)/X$ is torsionfree relative to the idempotent radical $\mathrm{rad}_{\Gamma(S)}$ defined by $\Gamma(S)$. Example 1 will show that this basic assumption may not hold, and in this example the natural transformation and ring homomorphism discussed above do not exist. The first proposition clarifies the situation with regard to the existence of a natural transformation from Q_γ to $R_{\Gamma(S)} \bigotimes_R (-)$.

PROPOSITION 1. *Let S be a semiprime ideal of R, let γ be the torsion radical of R–*Mod *determined by $E(R/S)$, and let Q : R–*Mod $\to$ R–Mod *be the associated quotient functor. Let $R_{\Gamma(S)}$ denote the universal localisation at S. Then every $R_{\Gamma(S)}$-module is γ-injective if and only if there exists a natural transformation η from the functor Q :* R–Mod $\to$ R–Mod *to the functor* $R_{\Gamma(S)} \bigotimes_R (-)$: R–Mod $\to$ R–Mod *such that the following diagram commutes for all modules $_RM$.*



PROOF: First suppose that we have a natural transformation as described in the statement of the theorem. Let $_RM$ be an $R_{\Gamma(S)}$-module. Then the natural R-homomorphism from M to $R_{\Gamma(S)} \bigotimes_R M$ is an isomorphism, and so it follows that the homomorphism from $Q(M)$ into $R_{\Gamma(S)} \bigotimes_R M$ is onto. On the other hand, since M is an $R_{\Gamma(S)}$-module, by construction it must be γ-torsionfree, and so the canonical R-homomorphism from M into $Q(M)$ is one-to-one. Since M is an essential submodule of $Q(M)$, it follows that the R-homomorphism from $Q(M)$ into $R_{\Gamma(S)} \bigotimes_R M$ is also one-to-one. Therefore $Q(M)$ is isomorphic to $R_{\Gamma(S)} \bigotimes_R M$, which is isomorphic, in turn, to M, so M is γ-injective since $Q(M)$ is γ-injective.

To prove the converse, suppose that every $R_{\Gamma(S)}$-module is γ-injective as an R-module. Then the canonical R-homomorphism from M into $R_{\Gamma(S)} \bigotimes_R M$ extends to an R-homomorphism $\eta(M) : Q(M) \to R_{\Gamma(S)} \bigotimes_R M$, and this homomorphism is unique since $R_{\Gamma(S)} \bigotimes_R M$ is γ-torsionfree. The uniqueness of the extension implies immediately that we have defined a natural transformation from the functor Q to the functor $R_{\Gamma(S)} \bigotimes_R (-)$. The commutativity of the above diagram also follows directly from the uniqueness in definition of the natural transformation η. $\square$

EXAMPLE 1. Let R be the ring $\begin{bmatrix} F & 0 \\ F & F \end{bmatrix}$ of 2×2 lower triangular matrices over a field

F, let P be the prime ideal $\begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix}$, and let $I = \begin{bmatrix} 0 & 0 \\ F & 0 \end{bmatrix}$. Since P is an idempotent ideal, it follows from [**4**, Lemma 1.5] that P is the kernel of the mapping from R into $R_{\Gamma(S)}$. Since R/P is a field, the universal localisation of R at P is simply R/P. As a left R-module, R/P is isomorphic to I, and P is the injective envelope of I. Since P/I is not isomorphic to R/P (as left R-modules), we see that P/I is torsion relative to the torsion radical γ determined by $E(R/P)$, and therefore the conditions of Proposition 1 are not satisfied.

Since R is γ-torsionfree, and R is γ-dense in its injective envelope $\begin{bmatrix} F & F \\ F & F \end{bmatrix}$, it follows that $Q_\gamma(R)$ is the full ring of 2×2 matrices over F. It is then easy to see that there is no ring homomorphism (over R) from $Q_\gamma(R)$ into $R_{\Gamma(S)}$, contradicting the assertion in [**8**, Theorem 4.2]. □

[**4**, Theorem 3.1] states that in the situation under discussion (assuming that S is a semiprime ideal of a left Noetherian ring R) the existence of any ring homomorphism from $R_{\Gamma(S)}$ into $Q_\gamma(R)$ (over R) is equivalent to the condition that $C(S)$ is a left denominator set. On the other hand, it will be shown in Example 2 that it *is* possible to have a ring homomorphism from $Q_\gamma(R)$ into $R_{\Gamma(S)}$ that is not an isomorphism. The next proposition provides a condition sufficient to guarantee that this occurs.

PROPOSITION 2. *If* $R_{\Gamma(S)}$ *is* $\mathrm{rad}_{\Gamma(S)}$*-closed in its* R*-injective envelope, and* $\mathrm{rad}_{\Gamma(S)}(R) = \gamma(R)$, *then there is a natural ring homomorphism from* $Q_\gamma(R)$ *into* $R_{\Gamma(S)}$.

PROOF. Without loss of generality we can assume that $\mathrm{rad}_{\Gamma(S)}(R) = \gamma(R) = (0)$. Let $\lambda : R \to R_{\Gamma(S)}$ be the canonical embedding, and assume that $R_{\Gamma(S)}$ is $\mathrm{rad}_{\Gamma(S)}$-closed in its R-injective envelope $E(R_{\Gamma(S)})$. (That is, assume that $E(R_{\Gamma(S)})/R_{\Gamma(S)}$ is $\mathrm{rad}_{\Gamma(S)}$-torsionfree.) Any R-homomorphism $f : R \to R_{\Gamma(S)}$ extends to $\overline{f} : R_{\Gamma(S)} \to E(R_{\Gamma(S)})$; the image of $\overline{f}$ must be in $R_{\Gamma(S)}$ since $R_{\Gamma(S)}/\lambda(R)$ is torsion relative to $\mathrm{rad}_{\Gamma(S)}$, whereas $E(R_{\Gamma(S)})/R_{\Gamma(S)}$ is by assumption torsionfree relative to $\mathrm{rad}_{\Gamma(S)}$. Furthermore, the extension is unique since $R_{\Gamma(S)}/\lambda(R)$ is torsion relative to $\mathrm{rad}_{\Gamma(S)}$ and $E(R_{\Gamma(S)})$ is torsionfree relative to $\mathrm{rad}_{\Gamma(S)}$. It follows that if $q \in R_{\Gamma(S)}$, then the R-endomorphism $\rho_q : R_{\Gamma(S)} \to R_{\Gamma(S)}$ defined by $\rho_q = [x \mapsto xq]$ is the unique extension of the R-homomorphism $[x \mapsto xq] : R \to R_{\Gamma(S)}$.

If we let $\epsilon : R \to Q_\gamma(R)$ be the canonical embedding, then the assumption on $R_{\Gamma(S)}$ implies that $\lambda : R \to R_{\Gamma(S)}$ has a unique extension $\phi : Q_\gamma(R) \to R_{\Gamma(S)}$ such that $\phi\epsilon = \lambda$. Since ϕ is an R-homomorphism, we only need to show that ϕ preserves multiplication. Given $a, b \in Q_\gamma(R)$, we can interpret $\phi(a)\phi(b)$ as $[\rho_{\phi(b)}](\phi(a)) = [\rho_{\phi(b)}\phi](a)$. Similarly, we can interpret $\phi(ab)$ as $\phi(\tau_b(a)) = [\phi\tau_b](a)$, where τ_b is the R-endomorphism $[x \mapsto xb]$ of Q_γ that defines right multiplication by b. For any $r \in R$, we have $[\rho_{\phi(b)}\phi](\epsilon(r)) = \rho_{\phi(b)}\phi\epsilon(r) = \rho_{\phi(b)}\lambda(r) = \lambda(r) \cdot \phi(b)$ and $[\phi\tau_b](\epsilon(r)) = \phi(\epsilon(r) \cdot b)$. But since ϕ is an R-homomorphism, we must have $\phi(\epsilon(r) \cdot b) = \lambda(r) \cdot \phi(b)$, and so $\rho_{\phi(b)}\phi\epsilon = \phi\tau_b\epsilon$. Uniqueness of the extensions implies that $\rho_{\phi(b)}\phi = \phi\tau_b$, showing that ϕ preserves multiplication. $\square$

EXAMPLE 2. Let R be the ring $\begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$ of 2×2 lower triangular matrices over the ring of integers $\mathbb{Z}$, and let S be the semiprime ideal $\begin{bmatrix} 0 & 0 \\ \mathbb{Z} & p\mathbb{Z} \end{bmatrix}$, where p is a prime. In [**4**], as a special case of a result on triangular matrix rings, the universal localisation $R_{\Gamma(S)}$ was shown to be $\begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Z}_{(p)} \end{bmatrix}$.

On the other hand, the torsion theoretic localisation $Q_\gamma(R)$ is the ring $T = \begin{bmatrix} \mathbb{Z}_{(p)} & 0 \\ \mathbb{Z}_{(p)} & \mathbb{Z}_{(p)} \end{bmatrix}$. This can be seen by noting first that T/R is $C(S)$-torsion, and then showing that T is γ-closed in its R-injective envelope $\begin{bmatrix} \mathbb{Q} & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Q} \end{bmatrix}$. It is then clear that in this case $Q_\gamma(R)$ can be embedded as a subring of $R_{\Gamma(S)}$. $\square$

The localisation $Q_\gamma(S)$ of the ideal S is always a left ideal of $Q_\gamma(R)$, but it is of interest to know when it is a two-sided ideal. The next lemma gives an equivalent condition. Note that in Example 1 we have $Q_\gamma(S) = \begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix}$, and so it is not an ideal of $Q_\gamma(R) = \begin{bmatrix} F & F \\ F & F \end{bmatrix}$. On the other hand, in Example 2 it can be shown that $Q_\gamma(S) = \begin{bmatrix} 0 & 0 \\ \mathbb{Z}_{(p)} & p\mathbb{Z}_{(p)} \end{bmatrix}$, and this *is* an ideal of $Q_\gamma(R) = \begin{bmatrix} \mathbb{Z}_{(p)} & 0 \\ \mathbb{Z}_{(p)} & \mathbb{Z}_{(p)} \end{bmatrix}$.

LEMMA 3. *The closure $Q_\gamma(S)$ in $Q_\gamma(R)$ is an ideal of $Q_\gamma(R)$ if and only if the classical ring of quotients $Q_{cl}(R/S)$ is a left module over $Q_\gamma(R)$ (extending the action of R).*

PROOF. The lemma on page 400 of [**2**] states that if M is a γ-torsionfree left $Q_\gamma(R)$-module, then for any $m \in M$ and any left ideal A of R, we have $Q_\gamma(R)m = (0)$ if and only if $Am = (0)$. Applying this condition when $A = S$ and M is the injective envelope $E(R/S)$ of R/S, it follows that $Q_\gamma(S)m = (0)$ if and only if $m \in Q_{cl}(R/S)$. Thus if $Q_\gamma(S)$ is an ideal, it follows that $Q_{cl}(R/S)$ is a $Q_\gamma(R)$-submodule of $E(R/S)$. Conversely, if $Q_{cl}(R/S)$ is a $Q_\gamma(R)$-submodule of $E(R/S)$, then $Q_\gamma(S)$ is its annihilator in $Q_\gamma(R)$, and therefore is an ideal of $Q_\gamma(R)$. $\square$

LEMMA 4. *If the equivalent conditions of Proposition 1 are satisfied, or if there exists a ring homomorphism from $Q_\gamma(R)$ into $R_{\Gamma(S)}$ (over R), then $Q_\gamma(S)$ is an ideal of $Q_\gamma(R)$.*

PROOF: Since R/S is a semiprime Goldie ideal, the classical ring of left quotients $Q_{cl}(R/S)$ coincides with the maximal ring of quotients $Q_{max}(R/S)$, and so by [**1**, Proposition 3.4 (a)] we have $Q_{cl}(R/S) \subseteq Q_\gamma(R/S)$. By assumption, $Q_{cl}(R/S)$ is γ-injective since it is isomorphic to the $R_{\Gamma(S)}$-module $R_{\Gamma(S)}/J(R_{\Gamma(S)})$. Since $Q_{cl}(R/S)$ is γ-dense in $Q_\gamma(R/S)$, it splits off, and this contradicts the fact that it is also essential in $Q_\gamma(R/S)$. We conclude that $Q_{cl}(R/S) = Q_\gamma(R/S)$, and so [**1**, Proposition 3.4 (b)] implies that $Q_\gamma(S)$ is an ideal of $Q_\gamma(R)$.

If there exists a ring homomorphism from $Q_\gamma(R)$ into $R_{\Gamma(S)}$ (over R), then $Q_{cl}(R/S)$ has the structure of a left $Q_\gamma(R)$-module since it is always a left $R_{\Gamma(S)}$-module. Lemma 3 then implies that $Q_\gamma(S)$ is an ideal of $Q_\gamma(R)$. □

Recall that we have defined the quotient functor $Q : R\text{–Mod} \to R\text{–Mod}$ by letting Q be the composition of the functor Q_γ (defined by $E(R/S)$) and the inclusion from $R\text{–Mod}\gamma$ into $R\text{–Mod}$.

THEOREM 5. *Let R be a left Noetherian ring, and let S be a semiprime ideal of R. Then the following conditions are equivalent.*

(1) *The universal localisation $R_{\Gamma(S)}$ coincides with the Ore localisation R_S;*

(2) *There exists a ring homomorphism from the torsion theoretic localisation $Q_\gamma(R)$ into the universal localisation $R_{\Gamma(S)}$ (over R) and the quotient functor $Q : R\text{–Mod} \to R\text{–Mod}$ is exact;*

(3) *The quotient functor $Q : R\text{–Mod} \to R\text{–Mod}$ is exact, and there is a natural transformation η from Q to $R_{\Gamma(S)} \bigotimes_R (-)$ such that the following diagram commutes for all modules ${}_RM$.*

$$\begin{array}{ccc} M & \longrightarrow & Q(M) \\ \downarrow & \swarrow{\scriptstyle \eta(M)} & \\ R_{\Gamma(S)} \bigotimes_R M & & \end{array}$$

PROOF. [**7**, Theorem 3] implies that if R is left Noetherian, then the ideal S is left localizable if and only if $Q_\gamma(S)$ is an ideal of $Q_\gamma(R)$ and $Q : R\text{–Mod} \to R\text{–Mod}$ is an exact functor. The equivalence of conditions (1), (2), and (3) therefore follows from Lemma 4. □

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