On flatness and the Ore condition

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In the standard theory of localization of a commutative Noetherian ring R at a prime ideal P, it is well-known that the localization R_P is a flat R-module. In the case of a prime ideal of a noncommutative Noetherian ring, it is not always possible to obtain a similar ring of fractions. An exposition of the standard theory in this more general situation can be found in [5]. The largest set in $R \backslash P$ that we can hope to invert is

$$\mathcal{C}(P) = \{c \in R \backslash P \mid cr \in P \text{ or } rc \in P \text{ implies } r \in P\} .$$

It is well-known that there exists a ring of left fractions R_P in which each element of $\mathcal{C}(P)$ is invertible if and only if $\mathcal{C}(P)$ satisfies the left Ore condition; that is, if and only if for each $a \in R$ and $c \in \mathcal{C}(P)$ there exist $b \in R$ and $d \in \mathcal{C}(P)$ with $da = bc$. In this case R_P is flat as a right R-module, as shown in Proposition II.3.5 of [5].

Even if $\mathcal{C}(P)$ does not satisfy the left Ore condition, it was shown by Cohn in [4] that it is still possible to obtain a localization at P, by inverting matrices rather than elements. Let $\Gamma(P)$ be the set of all square matrices C such that C is not a divisor of zero in the matrix ring $\mathrm{Mat}_n(R/P)$ (where C is an $n \times n$ matrix). The universal localization $R_{\Gamma(P)}$ of R at $\Gamma(P)$ is defined to be the universal $\Gamma(P)$-inverting ring. As shown in [4], the universal localization always exists, and has the desirable property that the canonical $\Gamma(P)$-inverting homomorphism $\theta : R \to R_{\Gamma(P)}$ is an epimorphism of rings. Furthermore, if $J(R_{\Gamma(P)})$ denotes the Jacobson radical of $R_{\Gamma(P)}$ and $Q_{cl}(R/P)$

denotes the classical ring of left quotients of R/P, then $R_{\Gamma(P)}/J(R_{\Gamma(P)})$ is naturally isomorphic to $Q_{cl}(R/P)$, which is a simple Artinian ring by Goldie's theorem.

It was shown by the author in [1] that $R_{\Gamma(P)}$ is flat as a right R-module only when the Ore condition is satisfied, in which case $R_{\Gamma(P)}$ coincides with R_P, the Ore ring of left fractions with denominators in $\mathcal{C}(P)$. There are similar results due to Braun [3] and Teichner [6] (see Corollaries 3 and 4, respectively). The goal of this paper is to find a general setting in which it is possible to give a common proof.

We will use the characterization of flat modules given in Proposition 10.7 of Chapter I of [5], which can be written in vector notation in the following way. The module M_R is flat $\Longleftrightarrow$ if $\mathbf{m} \cdot \mathbf{r}^t = 0$ for $\mathbf{m} = (m_1, \ldots, m_n) \in M^n$ and $\mathbf{r} = (r_1, \ldots, r_n) \in R^n$, then there exist $A = (a_{ij}) \in \mathrm{Mat}_{k,n}(R)$ and $\mathbf{x} = (x_1, \ldots, x_k) \in M^k$ with $A\mathbf{r}^t = \mathbf{0}$ and $\mathbf{x}A = \mathbf{m}$. As a consequence, if $\theta : R \to T$ is a ring homomorphism, then θ induces on T the structure of a flat right R-module $\Longleftrightarrow$ if $\mathbf{t} \cdot \theta(\mathbf{r})^t = 0$ for $\mathbf{t} = (t_1, \ldots, t_n) \in T^n$ and $\mathbf{r} = (r_1, \ldots, r_n) \in R^n$, then there exist $A = (a_{ij}) \in \mathrm{Mat}_{k,n}(R)$ and $\mathbf{u} = (u_1, \ldots, u_k) \in T^k$ with $A\mathbf{r}^t = \mathbf{0}$ and $\mathbf{u}\theta(A) = \mathbf{t}$.

This brief discussion brings us to the main theorem. Note that the statement of the theorem is independent of any chain conditions on the ring R.

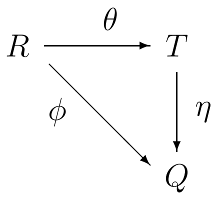
Theorem 1 *Let $\phi : R \to Q$ be a ring homomorphism such that for all $q_1, \ldots, q_n \in Q$ there exists a unit $u \in Q$ with $uq_i \in \phi(R)$, for $1 \leq i \leq n$. Let $S \subseteq R$ be the set of elements inverted by ϕ. If there exists an S-inverting ring homomorphism $\theta : R \to T$ such that*

(i) there exists a ring homomorphism $\eta : T \to Q$ with $\eta\theta = \phi$, and

(ii) T is flat as a right R-module,

then S satisfies the left Ore condition.

Proof. Given $a \in R$ and $c \in S$, we must find $b \in R$ and $d \in S$ with $da = bc$. To clarify the situation, we give the following commutative diagram.



Since $c \in S$ and θ is S-inverting, it follows that $\theta(c)$ is invertible in T. If we let $\mathbf{t} = (\theta(a)\theta(c)^{-1}, 1)$ and $\mathbf{r} = (c, -a)$, then $\mathbf{t} \cdot \theta(\mathbf{r})^t = \theta(a)\theta(c)^{-1}\theta(c) - \theta(a) = 0$. As in the comments preceding the theorem, by Proposition I.10.7 of [5] there exist $\mathbf{u} \in T^k$ and $A \in \mathrm{Mat}_{k,2}(R)$ such that $A\mathbf{r}^t = \mathbf{0}$ and $\mathbf{u}\theta(A) = \mathbf{t}$.

From the second component of $\mathbf{t}$ we obtain $\sum_{i=1}^k u_i\theta(a_{i2}) = 1$. By assumption there exists a unit $u \in Q$ with $u\eta(u_i) \in \phi(R)$, for $1 \le i \le k$. Thus there exist $b_1, \ldots, b_k \in R$ with $u\eta(u_i) = \phi(b_i)$, for $1 \le i \le k$. If we let $d = \sum_{i=1}^k b_i a_{i2}$, then

$$\begin{aligned} \phi(d) &= \textstyle\sum_{i=1}^k \phi(b_i)\phi(a_{i2}) \;=\; \sum_{i=1}^k u\eta(u_i)\eta\theta(a_{i2}) \\ &= u\eta(\textstyle\sum_{i=1}^k u_i\theta(a_{i2})) \;=\; u\eta(1) \;=\; u\ , \end{aligned}$$

and so $d \in S$.

From the equation $A\mathbf{r} = \mathbf{0}$ we obtain $a_{i2}a = a_{i1}c$, for $1 \le i \le k$. If we let $b = \sum_{i=1}^k b_i a_{i1}$, then

$$da = (\textstyle\sum_{i=1}^k b_i a_{i2})a = \sum_{i=1}^k b_i(a_{i2}a) = \sum_{i=1}^k b_i(a_{i1}c) = (\sum_{i=1}^k b_i a_{i1})c = bc\ .$$

Thus the left Ore condition holds in S, completing the proof. $\square$

We recall that an ideal I of the ring R is called a semiprime left Goldie ideal if the factor ring R/I is semiprime, satisfies the ascending chain condition on left annihilators, and has finite uniform dimension on the left.

Corollary 2 ([1], Corollary 3.2) *Let I be a semiprime left Goldie ideal of the ring R, and let $R_{\Gamma(I)}$ be the universal localization of R at I. If $R_{\Gamma(I)}$ is a flat right R-module, then $\mathcal{C}(I)$ is a left Ore set.*

Proof. Since I is assumed to be a semiprime left Goldie ideal, the ring R/I has a semisimple Artinian classical ring of left quotients $Q_{cl}(R/I)$. Let $Q = Q_{cl}(R/I)$, and let ϕ be the projection $R \to R/I$ followed by the canonical embedding $R/I \to Q$. An element $c \in R$ is in $\mathcal{C}(I)$ if and only if it is inverted by ϕ. It is well-known that the classical ring of left quotients has common denominators, so for $q_1, \ldots, q_k \in Q$ there exists an element $d \in R$ such that $d \in \mathcal{C}(I)$ and $\phi(d)q_i \in R/I$, for $1 \leq i \leq k$. Note that $\phi(d)$ is a unit of Q since $d \in \mathcal{C}(I)$. The canonical $\Gamma(I)$-inverting homomorphism $\theta : R \to R_{\Gamma(I)}$ inverts the subset $\mathcal{C}(I) \subseteq \Gamma(I)$, and the universality of θ guarantees the existence of $\eta : R_{\Gamma(I)} \to Q$ with $\eta\theta = \phi$, so the conditions of the theorem are satisfied. □

Corollary 3 ([3], Theorem 16) *Let I be a semiprime left Goldie ideal of the ring R, and let T be an extension ring of R such that $R \cap J(T) = I$ and $T/J(T)$ is naturally isomorphic to the left classical ring of quotients $Q_{cl}(R/I)$. If T is a flat right R-module, then $\mathcal{C}(I)$ satisfies the left Ore condition.*

Proof. Define Q and $\phi : R \to Q$ as in the proof of Corollary 2. If $\theta : R \to T$ is the inclusion mapping, and $\eta : T \to T/J(T)$ is the canonical projection, then $\eta\theta = \phi$ since we have assumed that the given isomorphism $T/J(T) \cong Q_{cl}(R/I)$ is natural. It follows that θ is a $\mathcal{C}(I)$-inverting homomorphism, since any element that is invertible modulo the Jacobson radical of a ring is invertible in the ring. Thus the conditions of the theorem are satisfied. □

Corollary 4 ([6], Main Theorem) *Let I be an ideal of the ring R, let $\phi : R \to R/I$ be the canonical projection, and let S be the set of elements inverted by ϕ. If the universal S-inverting ring is a flat right R-module, then S satisfies the left Ore condition.*

Proof. If R_S is the universal S-inverting ring, and $\theta : R \to R_S$ is the canonical S-inverting homomorphism, then by the universality of R_S there exists a homomorphism $\eta : R_S \to R/I$ with $\eta\theta = \phi$. Since ϕ maps R onto R/I, the remaining hypotheses of the theorem are certainly satisfied. $\square$

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