J. Korean Math. Soc. **50** (2013), No. 0, pp. 1–

PRIME M-IDEALS, M-PRIME SUBMODULES, M-PRIME RADICAL AND M-BAER'S LOWER NILRADICAL OF MODULES

J. A. Beachy, M. Behboodi, and F. Yazdi

Abstract. Let M be a fixed left R-module. For a left R-module X, we introduce the notion of M-prime (resp. M-semiprime) submodule of X such that in the case $M = R$, which coincides with prime (resp. semiprime) submodule of X. Other concepts encountered in the general theory are M-m-system sets, M-n-system sets, M-prime radical and M-Baer's lower nilradical of modules. Relationships between these concepts and basic properties are established. In particular, we identify certain submodules of M, called "prime M-ideals", that play a role analogous to that of prime (two-sided) ideals in the ring R. Using this definition, we show that if M satisfies condition H (defined latter) and $\mathrm{Hom}_R(M, X) \neq 0$ for all modules X in the category $\sigma[M]$, then there is a one-to-one correspondence between isomorphism classes of indecomposable M-injective modules in $\sigma[M]$ and prime M-ideals of M. Also, we investigate the prime M-ideals, M-prime submodules and M-prime radical of Artinian modules.

1. Introduction

All rings in this paper are associative with identity and modules are unitary left modules. Let R be a ring and X be an R-module. If Y is a submodule (resp. proper submodule) of X we write $Y \leq X$ (resp. $Y \lneqq X$).

In the literature, there are many different generalizations of the notion of prime two-sided ideals to left ideals and also to modules. For instance, a proper left ideal L of a ring R is called prime if, for any elements a and b in R such that $aRb \subseteq L$, either $a \in L$ or $b \in L$. Prime left ideals have properties reminiscent of prime ideals in commutative rings. For example, Michler [19] and Koh [12] proved that the ring R is left Noetherian if and only if every prime left ideal is finitely generated. Moreover, Smith [20], showed

Received December 17, 2012; Revised July 11, 2013.
2010 *Mathematics Subject Classification.* 16S38, 16D50, 16D60, 16N60.
Key words and phrases. prime submodules, prime M-ideal, M-prime submodule, M-prime radical, M-injective module.
The research of the second author was in part supported by a grant from IPM (No. 90160034).

that if R is left Noetherian (or even if R has finite left Krull dimension) then a left R-module X is injective if and only if, for every essential prime left ideal L of R and homomorphism $\varphi : L \to X$, there exists a homomorphism $\theta : R \to X$ such that $\theta|_L = \varphi$. Let us mention another generalization of the notion of prime ideals to modules. Let X be a left R-module. If $X \neq 0$ and $\mathrm{Ann}_R(X) = \mathrm{Ann}_R(Y)$ for all nonzero submodules Y of X then X is called a *prime module.* A proper submodule P of X is called a *prime submodule* if X/P is a prime module, i.e., for every ideal $I \subseteq R$ and every submodule $Y \subseteq X$, if $IY \subseteq P$, then either $Y \subseteq P$ or $IX \subseteq P$. The notion of prime submodule was first introduced and systematically studied by Dauns [7] and recently has received some attention. Several authors have extended the theory of prime ideals of R to prime submodules (see [2, 3, 4, 7, 10, 15, 17, 18]). For example, the classical result of Cohens is extended to prime submodules over commutative rings, namely a finitely generated module is Noetherian if and only if every prime submodule is finitely generated (see [15, Theorem 8] and [11]) and also any Noetherian module contains only finitely many minimal prime submodules (see [18, Theorem 4.2]).

We assume throughout the paper ${}_RM$ is a fixed left R-module. The category $\sigma[M]$ is defined to be the full subcategory of R-Mod that contains all modules ${}_RX$ such that X is isomorphic to a submodule of an M-generated module (see [21] for more detail).

Let $\mathcal{C}$ be a class of modules in R-Mod, and let Ω be the set of kernels of R-homomorphisms from M in to $\mathcal{C}$. That is,

$$\Omega = \{K \subseteq M \mid \exists\, W \in \mathcal{C} \text{ and } f \in \mathrm{Hom}_R(M, W) \text{ with } K = \ker(f)\}.$$

Then the *annihilator of* $\mathcal{C}$ *in* M, denoted by $\mathrm{Ann}_M(\mathcal{C})$, is defined to be the intersection of all elements of Ω, i.e., $\mathrm{Ann}_M(\mathcal{C}) = \bigcap_{K\in\Omega} K$.

Let N be a submodule of M. Following Beachy [1], for each module ${}_RX$ we define

$$N \cdot X = \mathrm{Ann}_X(\mathcal{C}),$$

where $\mathcal{C}$ is the class of modules ${}_RW$ such that $f(N) = (0)$ for all $f \in \mathrm{Hom}_R(M, W)$. It follows immediately from the definition that

$$N \cdot X = (0) \text{ if and only if } f(N) = (0) \text{ for all } f \in \mathrm{Hom}_R(M, X).$$

Clearly the class $\mathcal{C}$ in definition of $N\cdot X$ is closed under formation of submodules and direct products, and so $N \cdot X$ is the smallest submodule $Y \subseteq X$ such that $N \cdot (X/Y) = (0)$.

The submodule N of M is called an *M-ideal* if there is a class $\mathcal{C}$ of modules in $\sigma[M]$ such that $N = \mathrm{Ann}_M(\mathcal{C})$. Note that although the definition of an M-ideal is given relative to the subcategory $\sigma[M]$, it is easy to check that N is an M-ideal if and only if $N = \mathrm{Ann}_M(\mathcal{C})$ for some class $\mathcal{C}$ in R-Mod (see [1, Page 4651]).

In this article for a left R-module X, we introduce the notions of M-prime submodule, M-semiprime submodule of X and prime M-ideal of M as follows:

Definition 1.1. Let X be an R-module. A proper submodule P of X is called an *M-prime submodule* if for all submodules $N \leq M$, $Y \leq X$, if $N \cdot Y \subseteq P$, then either $N \cdot X \subseteq P$ or $Y \subseteq P$. An R-module X is called an *M-prime module* if $(0) \subsetneqq X$ is an M-prime submodule. Also, a proper submodule P of X is called an *M-semiprime submodule* if for all submodules $N \leq M$, $Y \leq X$, if $N^2 \cdot Y \subseteq P$, then $N \cdot Y \subseteq P$, where $N^2 := N \cdot N$. An R-module X is called an *M-semiprime module* if $(0) \subsetneqq X$ is an M-semiprime submodule.

Definition 1.2. A proper M-ideal P of M is called a *prime M-ideal* (resp. *semiprime M-ideal*) if there exists an M-prime module (resp. M-semiprime module) ${}_RX$ such that $P = \mathrm{Ann}_M(X)$.

It is clear that in case $M = R$, the notion of an R-prime submodule (resp. R-semiprime submodule) reduces to the familiar definition of a prime submodule (resp. semiprime submodule). Also, the notion of an R-ideal (resp. prime R-ideal) of ${}_RR$ reduces to the familiar definition of an ideal (resp. a prime ideal) of R.

Recently, the idea of *M-prime module* was introduced and extensively studied by Beachy [1] by defining a module ${}_RX$ to be *M-prime* if $\mathrm{Hom}_R(M, X) \neq 0$, and $\mathrm{Ann}_M(Y) = \mathrm{Ann}_M(X)$ for all submodules $Y \subseteq X$ such that $\mathrm{Hom}_R(M, Y) \neq 0$. Also, he defined an M-ideal P to be *prime M-ideal* if there exists an M-prime module ${}_RX$ such that $P = \mathrm{Ann}_M(X)$. Clearly, our definition of M-prime module is slightly different than Beachy, and hence, for the sake of clarity, for the remainder of the paper we will use the term "Beachy-M-prime module" (resp. "Beachy-prime M-ideal") rather than "M-prime module" (resp. "prime M-ideal") of Beachy [1], respectively.

In ring theory, prime ideals are closely tied to m-system sets (a nonempty set $S \subseteq R$ is said to be an *m-system set* if for each pair a, b in S, there exists $r \in R$ such that $arb \in S$). The complement of a prime ideal is an m-system, and given an m-system set S, an ideal disjoint from S and maximal with respect to this property is always a prime ideal. Moreover, for an ideal I in a ring R, the set $\sqrt{I} := \{s \in R \mid \text{every m-system containing } s \text{ meets } I\}$ equals the intersection of all the prime ideals containing I. In particular, $\sqrt{I}$ is a semiprime ideal in R and $\sqrt{(0)}$ is called *Baer-McCoy radical* (or *prime radical*) of R (see for example [14, Chapter 4], for more details). In this paper, we extend these facts for M-prime submodules. Relationships between these concepts and basic properties are established. In Section 2, among other results, for an R-module X we define *M-Baer-McCoy radical* (or *M-prime radical*) of X, denoted $\mathrm{rad}_M(X) = \sqrt[M]{(0)}$, to be the intersection of all the M-prime submodules in X. Also, in Section 3, we extend the notion of nilpotent and strongly nilpotent element of modules to M-nilpotent and strongly M-nilpotent element of modules $X \in \sigma[M]$ for a fix module M. Also, for an R-module $X \in \sigma[M]$, we define *M-Baer's lower nilradical* of X, denoted by $M\text{-Nil}_*({}_RX)$, to be the set of all strongly M-nilpotent elements of X. In particular, it is shown that if M is projective in

$\sigma[M]$, then for each $X \in \sigma[M]$, $\text{Nil}_*(M) \cdot X \subseteq M\text{-Nil}_*({}_RX) \subseteq \text{rad}_M(X)$ (see Proposition 3.6).

In Section 4, we rely on the prime M-ideals of M that play a role analogous to that of prime ideals in the ring R. The module ${}_RX$ is called *M-injective* if each R-homomorphism $f : K \to X$ defined on a submodule K of M can be extended to an R-homomorphism $\widehat{f} : M \to X$ with $f = \widehat{f}i$, where $i : K \to M$ is the natural inclusion mapping. We note that Baers criterion for injectivity shows that any R-injective module is injective in the category R-Mod of all left R-modules. It is well-known that if R is a commutative Noetherian ring, then there is a one-to-one correspondence between isomorphism classes of indecomposable injective R-modules and prime ideals of R. Gabriel showed in [8] that this one-to-one correspondence remains valid for any left Noetherian ring that satisfies what he called condition H. In current terminology, a module ${}_RX$ is said to be finitely annihilated if there is a finite subset $x_1, \ldots, x_n$ of X with $\text{Ann}_R(X) = \text{Ann}_R(x_1, \ldots, x_n)$. Then by definition the ring R satisfies condition H if and only if every cyclic left R-module is finitely annihilated. It follows immediately that, the ring R satisfies condition H if and only if every finitely generated left R-module is finitely annihilated. We note the stronger result due to Krause [13] that if R is left Noetherian, then there is a one-to-one correspondence between isomorphism classes of indecomposable injective left R-modules and prime ideals of R if and only if R is a left fully bounded ring (see [9, Theorem 8.12] for a proof). In [1, Theorem 6.7], Beachy shown that Gabriels correspondence can be extended to M-injective modules, provided that $\text{Hom}_R(M, X) \neq 0$ for all modules X in $\sigma[M]$. In Section 4, by using our definition of prime M-ideal, we show that also there is Gabriels correspondence between indecomposable M-injective modules in $\sigma[M]$ and our prime M-ideals.

Finally, in Section 5, we study the prime M-ideal, M-prime submodules and M-prime radical of Artinian modules. The *prime radical* of the module M, denoted by $P(M)$, is defined to be the intersection of all prime M-ideals of M. Recall that a proper submodule P of M is *virtually maximal* if the factor module M/P is a homogeneous semisimple R-module, i.e., M/P is a direct sum of isomorphic simple modules. It is shown that if M is an Artinian M-prime module, then M is a homogeneous semisimple module (see Proposition 5.1). In particular, if M is an Artinian R-module such that it is projective in $\sigma[M]$, then every prime M-ideal of M is virtually maximal and $M/P(M)$ is a Noetherian R-module (see Theorem 5.6). Moreover, either $P(M) = M$ or there exist primitive (prime) M-ideals $P_1, \ldots, P_n$ of M such that $P(M) = \bigcap_{i=1}^{n} P_i$ (see Theorem 5.7).

2. M-prime submodules and M-prime radical of modules

We begin this section with the following three useful lemmas.

Lemma 2.1 ([1, Proposition 1.6]). *Let N be a submodule of M. Then for any R-module X, $N \cdot X = (0)$ if and only if $N \subseteq Ann_M(X)$.*

Lemma 2.2 ([1, Proposition 1.9]). *Let N and K be submodules of M.*
(a) *If $N \subseteq K$, then $N \cdot X \subseteq K \cdot X$ for all submodules ${}_RX$.*
(b) *If K is an M-ideal, then so is $N \cdot K$.*
(c) *The submodule $N \cdot M$ is the smallest M-ideal that contains N.*
(d) *If N is an M-ideal, then $N \cdot K \subseteq N \cap K$.*

Lemma 2.3. *Let Y_1, Y_2 be submodules of ${}_RX$. If $Y_1 \subseteq Y_2$, then $N \cdot Y_1 \subseteq N \cdot Y_2$, for each submodule N of M.*

Proof. Suppose $N \leq M$ and Y_1, Y_2 be submodules of ${}_RX$ with $Y_1 \subseteq Y_2$. Then $N \cdot Y_1 = \mathrm{Ann}_{Y_1}(\mathcal{C})$ and $N \cdot Y_2 = \mathrm{Ann}_{Y_2}(\mathcal{C})$, where $\mathcal{C}$ is the class of modules ${}_RW$ such that $f(N) = (0)$ for all $f \in \mathrm{Hom}_R(M, W)$. On the other hand $N \cdot Y_i = \bigcap_{K \in \Omega_i} K$ $(i = 1, 2)$, where

$$\Omega_i = \{K \subseteq Y_i \ \mid \ \exists\, W \in \mathcal{C} \text{ and } f \in \mathrm{Hom}_R(Y_i, W) \text{ with } K = \ker(f)\}$$

Clearly, for each $f \in \mathrm{Hom}_R(Y_2, W)$, $f|_{Y_1} \in \mathrm{Hom}_R(Y_1, W)$, where $f|_{Y_1}$ is the restriction of f on Y_1. Since $\ker(f|_{Y_1}) \subseteq \ker(f)$, we conclude that for each $K \in \Omega_2$, there exists $K' \in \Omega_1$ such that $K' \subseteq K$. Thus $N \cdot Y_1 \subseteq N \cdot Y_2$. □

The following evident proposition offers several characterizations of an M-prime module.

Proposition 2.4. *Let X be a nonzero R-module. Then the following statements are equivalent.*
(1) *X is an M-prime module.*
(2) *For every submodule $N \subseteq M$ and every nonzero submodule $Y \subseteq X$, if $N \cdot Y = (0)$, then $N \cdot X = (0)$.*
(3) *For every M-ideal $N \subseteq M$ and every nonzero submodule $Y \subseteq X$, if $N \cdot Y = (0)$, then $N \cdot X = (0)$.*
(4) *For every nonzero submodules Y_1, $Y_2 \subseteq X$, $Ann_M(Y_1) = Ann_M(Y_2)$.*
(5) *Every nonzero submodule $Y \subseteq X$ is an M-prime module.*
(6) *For every nonzero submodule $Y \subseteq X$, $P = Ann_M(Y)$ is a prime M-ideal of M and $P = Ann_M(X)$.*

Proof. (1) ⇒ (2) ⇒ (3) is clear.

(3) ⇒ (4). Let Y_1, Y_2 be two nonzero submodules of X and let $N_1 := \mathrm{Ann}_M(Y_1)$, $N_2 := \mathrm{Ann}_M(Y_2)$. Thus by Lemma 2.1, $N_1 \cdot Y_1 = (0)$ and $N_2 \cdot Y_2 = (0)$. Since N_1, N_2 are M-ideals, $N_1 \cdot X = N_2 \cdot X = (0)$ by (3). Thus $N_1 \subseteq \mathrm{Ann}_M(X)$ and $N_2 \subseteq \mathrm{Ann}_M(X)$. On the other hand $\mathrm{Ann}_M(X) \subseteq N_1$ and $\mathrm{Ann}_M(X) \subseteq N_2$. Thus $N_1 = N_2 = \mathrm{Ann}_M(X)$.

(4) ⇒ (5). Let Y be a nonzero submodule of X. Assume that N is a submodule of M and Z be a nonzero submodule of Y such that $N \cdot Z = (0)$. So $N \subseteq \mathrm{Ann}_M(Z)$. By (4), $\mathrm{Ann}_M(Z) = \mathrm{Ann}_M(X)$ and so it follows that $N \subseteq \mathrm{Ann}_M(X)$ and hence $N \cdot X = (0)$. Since $N \cdot Y \subseteq N \cdot X$, so $N \cdot Y = (0)$. Thus Y is an M-prime module.

(5) ⇒ (1) and (5) ⇒ (6) ⇒ (4) are clear. □

Remark 2.5. Clearly every simple R-module X is an M-prime module. Now let R be a domain which is not a field and let M be a nonzero divisible R-module. Then every nonzero simple R-module X is an M-prime module, but X is not a Beachy-M-prime module, since $\mathrm{Hom}_R(M, X) = 0$.

The following lemma shows that in the case $\mathrm{Hom}_R(M, X) \neq 0$, if X is an M-prime module then X is also a Beachy-M-prime module.

Lemma 2.6 ([1, Proposition 2.2]). *Let X be an R-module such that $\mathrm{Hom}_R(M, X) \neq 0$. Then the following statements are equivalent.*

(1) *X is a Beachy-M-prime module.*

(2) *For every M-ideal N of M and every nonzero submodule Y of X with $M \cdot Y \neq (0)$, if $N \cdot Y = (0)$, then $N \cdot X = (0)$.*

(3) *For each $m \in M \setminus Ann_M(X)$ and each $0 \neq f \in Hom_R(M, X)$, there exists $g \in Hom_R(M, f(M))$ such that $g(m) \neq 0$.*

(4) *For any M-ideal $N \subseteq M$ and any M-generated submodule $Y \subseteq X$, if $N \cdot Y = (0)$, then $N \cdot X = (0)$.*

Proposition 2.7. *Let X be an R-module such that $Hom_R(M, X) \neq 0$. If X is an M-prime module then X is a Beachy-M-prime module.*

Proof. By Proposition 2.4 and Lemma 2.6, it is clear. □

The following example shows that the converse of Proposition 2.7 is not true in general.

Example 2.8. Let $R = \mathbb{Z}$. For each prime number p, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{p_\infty}, \mathbb{Z}_{p_\infty}) \neq 0$ and for each proper $\mathbb{Z}$-submodule $Y \subsetneqq \mathbb{Z}_{p_\infty}$, $\mathbb{Z}_{p_\infty} \cdot Y = (0)$, since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{p_\infty}, Y) = (0)$. Thus by Lemma 2.6, $\mathbb{Z}_{p_\infty}$ is a Beachy-$\mathbb{Z}_{p_\infty}$-prime module but it is not a $\mathbb{Z}_{p_\infty}$-prime module, since $\mathbb{Z}_{p_\infty} \cdot \mathbb{Z}_{p_\infty} \neq (0)$.

Lemma 2.9 ([1, Proposition 5.5]). *Assume that M is projective in $\sigma[M]$, and let N be any submodule of M. The following conditions hold for any module ${}_RX$ in $\sigma[M]$ and any submodule $Y \subseteq X$.*

(a) *$N \cdot X = \sum_{f \in Hom_R(M,X)} f(N)$.*

(b) *$N \cdot (X/Y) = (0)$ if and only if $N \cdot X \subseteq Y$.*

(c) *If $N = Ann_M(X/Y)$, then $Ann_M(X/(N \cdot X)) = N$.*

Proposition 2.10. *Assume that M is projective in $\sigma[M]$, and let ${}_RX \in \sigma[M]$. Then*

(i) *For a submodule $P \subsetneqq X$, if P is an M-prime submodule of X, then X/P is an M-prime module.*

(ii) *For an M-ideal $P \subsetneqq M$, the following conditions are equivalent.*

(1) *P is a prime M-ideal.*

(2) *P is an M-prime submodule of M.*

(3) *M/P is an M-prime module.*

Proof. (i). Let N be a submodule of M and Y/P be a nonzero submodule of X/P such that $N \cdot (Y/P) = (0)$. By Lemma 2.9(b), $N \cdot Y \subseteq P$. Since P is an

M-prime submodule, either $N \cdot X \subseteq P$ or $Y \subseteq P$. If $Y \subseteq P$, then $Y/P = (0)$, a contradiction. Thus $N \cdot X \subseteq P$ and so $N \cdot (X/P) = (0)$ by Lemma 2.9(b). Thus by Proposition 2.4, X/P is an M-prime module.

(ii) (1) $\Rightarrow$ (2). Suppose that P is a prime M-ideal and $N \cdot K \subseteq P$, for an M-ideal N and submodule K of M with $K \nsubseteq P$. By assumption there is an M-prime module X with $P = \mathrm{Ann}_M(X)$, and so there exists $f \in \mathrm{Hom}_R(M/P, X)$ with $f((K+P)/P) \neq (0)$. Since $N \cdot K \subseteq P$, we have $N \cdot K \subseteq P \cap K$. Now Lemma 2.9(b) implies that $N \cdot (K/(P \cap K)) = (0)$ and hence $N \cdot f((K+P)/P) = (0)$ (since $(K+P)/P \cong K/(P \cap K)$). Since X is an M-prime module, $N \cdot X = (0)$ by Proposition 2.4, and so $N \subseteq P$ (since $P = \mathrm{Ann}_M(X)$).

(2) $\Rightarrow$ (3). Let N be an M-ideal and K/P be a nonzero submodule of M/P such that $N \cdot (K/P) = (0)$. Since M is projective in $\sigma[M]$, so $N \cdot K \subseteq P$ by Lemma 2.9(b). Now by (2) either $N \subseteq P$ or $K \subseteq P$. Since $K/P \neq (0)$, so $K \nsubseteq P$ and hence $N \subseteq P$. On the other hand $N \cdot M = N$, since N is an M-ideal. Thus $N \cdot M \subseteq P$ and hence by Lemma 2.9(b), $N \cdot (M/P) = (0)$. Now M/P is an M-prime module by Proposition 2.4.

(3) $\Rightarrow$ (1). Since P is an M-ideal, $P = \mathrm{Ann}_M(M/P)$ and since M/P is an M-prime module, we conclude that P is a prime M-ideal. □

The following example shows that even in the case the R-module M is projective in $\sigma[M]$, an M-prime module need not be a Beachy-M-prime module.

Example 2.11. Let $R = \mathbb{Z}$ and $M = \mathbb{Q}$ as $\mathbb{Z}$-module. Then it is easy to check that $\mathbb{Q}$ is projective in $\sigma[\mathbb{Q}]$. Clearly, for each prime number p, $\mathbb{Z}_p$ is a $\mathbb{Q}$-prime module, but it is not a Beachy-$\mathbb{Q}$-prime module, since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}_p) = (0)$.

Now we have to adapt the notion of an M-m-system set to modules ${}_RX$ (Behboodi in [2], has generalized the notion of m-system of rings to modules).

Definition 2.12. Let X be an R-module. A nonempty set $S \subseteq X \setminus \{0\}$ is called an M-m-system if, for each submodule $N \subseteq M$, and for all submodules $Y, Z \subseteq X$, if $(Y+Z) \cap S \neq \emptyset$ and $(Y + N \cdot X) \cap S \neq \emptyset$, then $(Y + N \cdot Z) \cap S \neq \emptyset$.

Corollary 2.13. *Let X be an R-module. Then a submodule $P \subsetneqq X$ is M-prime if and only if $X \setminus P$ is an M-m-system.*

Proof. ($\Rightarrow$). Suppose $S = X \setminus P$. Let N be a submodule of M and Y, Z be submodules of X such that $(Y+Z) \cap S \neq \emptyset$ and $(Y + N \cdot X) \cap S \neq \emptyset$. If $(Y + N \cdot Z) \cap S = \emptyset$ then $Y + N \cdot Z \subseteq P$. Hence $N \cdot Z \subseteq P$ and since P is an M-prime submodule, $Z \subseteq P$ or $N \cdot X \subseteq P$. It follows that $(Y+Z) \cap S = \emptyset$ or $(Y + N \cdot X) \cap S = \emptyset$, a contradiction. Therefore, $S \subseteq X \setminus \{0\}$ is an M-m-system set.

($\Leftarrow$). Let $S = X \setminus P$ be an M-m-system in X. Suppose $N \cdot Z \subseteq P$, where N is a submodule of M and Z is a submodule X. If $Z \nsubseteq P$ and $N \cdot X \nsubseteq P$, then $Z \cap S \neq \emptyset$ and $(N \cdot X) \cap S \neq \emptyset$. Thus $(N \cdot Z) \cap S \neq \emptyset$, a contradiction. Therefore, P is an M-prime submodule of X. □

Proposition 2.14. *Let X be an R-module, P be a proper submodule of X and $S := X \setminus P$. Then the following statements are equivalent.*

(1) *P is an M-prime submodule.*

(2) *S is an M-m-system.*

(3) *For every submodule $N \leq M$ and for every submodule $Z \leq X$, if $Z \cap S \neq \emptyset$ and $(N \cdot X) \cap S \neq \emptyset$, then $(N \cdot Z) \cap S \neq \emptyset$.*

Proof. (1) $\Leftrightarrow$ (2) is by Corollary 2.13.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1). Suppose that $N \leq M$ and $Z \leq X$ such that $N \cdot Z \subseteq P$. If $N \cdot X \nsubseteq P$ and $Z \nsubseteq P$, then $(N \cdot X) \cap S \neq \emptyset$ and $Z \cap S \neq \emptyset$. It follows that $(N \cdot Z) \cap S \neq \emptyset$ by (3), i.e., $N \cdot Z \nsubseteq P$, a contradiction. □

Proposition 2.15. *Let X be an R-module, $S \subseteq X$ be an M-m-system and P be a submodule of X maximal with respect to the property that P is disjoint from S. Then P is an M-prime submodule of X.*

Proof. Suppose $N \cdot Z \subseteq P$, where $N \leq M$ and $Z \leq X$. If $Z \nsubseteq P$ and $N \cdot X \nsubseteq P$, then by the maximal property of P, we have, $(P + Z) \cap S \neq \emptyset$ and $(P + N \cdot X) \cap S \neq \emptyset$. Thus $(P + N \cdot Z) \cap S \neq \emptyset$ and it follows that $P \cap S \neq \emptyset$, a contradiction. Thus P must be an M-prime submodule. □

Next we need a generalization of the notion of $\sqrt{Y}$ for any submodule Y of X. We adopt the following:

Definition 2.16. Let X be an R-module. For a submodule Y of X, if there is an M-prime submodule containing Y, then we define

$$\sqrt[M]{Y} = \{x \in X : \text{every } M\text{-m-system containing } x \text{ meets } Y\}.$$

If there is no M-prime submodule containing Y, then we put $\sqrt[M]{Y} = X$.

Theorem 2.17. *Let X be an R-module and $Y \leq X$. Then either $\sqrt[M]{Y} = X$ or $\sqrt[M]{Y}$ equals the intersection of all M-prime submodules of X containing Y.*

Proof. Suppose that $\sqrt[M]{Y} \neq X$. This means that

$$\{P : P \text{ is an } M\text{-prime submodule of } X \text{ and } Y \subseteq P\} \neq \emptyset.$$

We first prove that

$$\sqrt[M]{Y} \subseteq \bigcap\{P : \mid P \text{ is an } M\text{-prime submodule of } X \text{ and } Y \subseteq P\}.$$

Let $x \in \sqrt[M]{Y}$ and P be any M-prime submodule of X containing Y. Consider the M-m-system $X \setminus P$. This M-m-system cannot contain x, for otherwise it meets Y and hence also P. Therefore, we have $x \in P$. Conversely, assume $x \notin \sqrt[M]{Y}$. Then, by Definition 2.16, there exists an M-m-system S containing x which is disjoint from Y. By Zorn's Lemma, there exists a submodule $P \supseteq Y$ which is maximal with respect to being disjoint from S. By Proposition 2.15, P is an M-prime submodule of X, and we have $x \notin P$, as desired. □

Also, the following evident proposition offers several characterizations of M-semiprime modules.

Proposition 2.18. *Let X be an R-module. Then the following statements are equivalent.*

(1) *X is an M-semiprime module.*

(2) *For every submodule $N \subseteq M$ and every submodule $Y \subseteq X$, if $N^2 \cdot Y = (0)$, then $N \cdot Y = (0)$.*

(3) *Every nonzero submodule $Y \subseteq X$ is an M-semiprime module.*

(4) *For every nonzero submodule $Y \subseteq X$, $P = Ann_M(Y)$ is a semiprime M-ideal.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) is clear.

(4) $\Rightarrow$ (1). Suppose $(0) \neq Y \leq X$ and $N \leq M$ such that $N^2 \cdot Y = (0)$. It follows that $N^2 \subseteq \mathrm{Ann}_M(Y)$ and since $P = \mathrm{Ann}_M(Y)$ is a semiprime M-ideal, there exists an M-semiprime module Z such that $\mathrm{Ann}_M(Y) = \mathrm{Ann}_M(Z)$. Thus $N^2 \cdot Z = (0)$ and so $N \cdot Z = (0)$, i.e., $N \subseteq \mathrm{Ann}_M(Z) = \mathrm{Ann}_M(Y)$. Thus $N \cdot Y = (0)$. Therefore X is an M-semiprime module. □

Proposition 2.19. *Let X be an R-module. Then any intersection of M-semiprime submodules of X is an M-semiprime submodule.*

Proof. Suppose $Z_i \leq X$ $(i \in I)$ be M-semiprime submodules of X and put $Z = \bigcap_{i \in I} Z_i$. Suppose $Y \leq X$ and $N \leq M$ such that $N^2 \cdot Y \subseteq Z$. It follows that $N^2 \cdot Y \subseteq Z_i$ for each i. Since each Z_i is an M-semiprime submodule, $N \cdot Y \subseteq Z_i$ for each i. Thus $N \cdot Y \subseteq Z$ and so Z is an M-semiprime submodule. □

We recall the definition of the notion of n-system in a ring R. A nonempty set $T \subseteq R$ is said to be an *n-system set* if for each a in T, there exists $r \in R$ such that $ara \in T$ (see for example [14, Chapter 4], for more details). The complement of a semiprime ideal is an n-system set, and if T is an n-system in a ring R such that $a \in T$, then there exists an m-system $S \subseteq T$ such that $a \in S$ (see [14, Lemma 10.10]). This notion of n-system of rings has also generalized by Behboodi in [2] for modules. Now we have to adapt the notion of an M-n-system set to modules ${}_RX$.

Definition 2.20. Let X be an R-module. A nonempty set $T \subseteq X \setminus \{0\}$ is called an M-*n*-system if, for every submodule $N \subseteq M$, and for all submodules $Y, Z \subseteq X$, if $(Y + N \cdot Z) \cap T \neq \emptyset$, then $(Y + N^2 \cdot Z) \cap T \neq \emptyset$.

Proposition 2.21. *Let X be an R-module. Then a submodule $P \subsetneqq X$ is an M-semiprime submodule if and only if $X \setminus P$ is an M-n-system.*

Proof. ($\Rightarrow$). Let $T = X \setminus P$. Suppose N is a submodule of M and Y, Z are submodules of X such that $(Y + N \cdot Z) \cap T \neq \emptyset$. If $(Y + N^2 \cdot Z) \cap T = \emptyset$, then $(Y + N^2 \cdot Z) \subseteq P$. Since P is M-semiprime submodule, $(Y + N \cdot Z) \subseteq P$. Thus $(Y + N \cdot Z) \cap T = \emptyset$, a contradiction. Therefore, T is an M-n-system set in X.

($\Leftarrow$). Suppose that $T = X \setminus P$ is an M-n-system in X. Suppose $N^2 \cdot Z \subseteq P$, where $N \leq M$, $Z \leq X$, but $N \cdot Z \nsubseteq P$. It follows that $(N \cdot Z) \cap T \neq \emptyset$ and so $(N^2 \cdot Z) \cap T \neq \emptyset$, a contradiction. Therefore, P is an M-semiprime submodule of X. □

The proof of the next proposition is similar to the proof of Proposition 2.14.

Proposition 2.22. *Assume that P be a proper submodule of X and $T := X \setminus P$. Then the following statements are equivalent.*

(1) *P is an M-semiprime submodule.*

(2) *T is an M-n-system set.*

(3) *For every submodule $N \leq M$ and for every submodule $Z \leq X$, if $(N \cdot Z) \cap T \neq \emptyset$, then $(N^2 \cdot Z) \cap T \neq \emptyset$.*

Lemma 2.23 ([1, Proposition 5.6])**.** *Assume that M is projective in $\sigma[M]$, and let K, N be submodules of M. Then $(K \cdot N) \cdot X = K \cdot (N \cdot X)$ for any module ${}_RX$ in $\sigma[M]$.*

Proposition 2.24. *Assume that M is projective in $\sigma[M]$, and let $X \in \sigma[M]$. Then any M-prime submodule of X is an M-semiprime submodule.*

Proof. Let $P \subsetneqq X$ be an M-prime submodule of X and $N \leq M$, $Y \leq X$ such that $N^2 \cdot Y \subseteq P$. Since M is projective in $\sigma[M]$, so $N^2 \cdot Y = (N \cdot N) \cdot Y = N \cdot (N \cdot Y)$ by Lemma 2.23. Hence $N \cdot (N \cdot Y) \subseteq P$. Now by assumption, $N \cdot X \subseteq P$ or $N \cdot Y \subseteq P$. If $N \cdot Y \subseteq P$, then P is an M-semiprime submodule. If $N \cdot X \subseteq P$, then $N \cdot Y \subseteq N \cdot X \subseteq P$. Thus P is an M-semiprime submodule. □

Corollary 2.25. *Assume that M is projective in $\sigma[M]$ and $X \in \sigma[M]$. Then any intersection of M-prime submodules of X is an M-semiprime submodule.*

Proof. It follows by Proposition 2.19 and Proposition 2.24. □

Corollary 2.26. *Assume that M is projective in $\sigma[M]$, and let $X \in \sigma[M]$. Then for each submodule Y of X, either $\sqrt[M]{Y} = X$ or $\sqrt[M]{Y}$ is an M-semiprime submodule of X.*

Proof. By Theorem 2.17 and Corollary 2.25, it is clear. □

Definition 2.27. Let M be an R-module. For any module X, we define $\text{rad}_M(X) = \sqrt[M]{(0)}$. This is called *$M$-Baer-McCoy radical* or *M-prime radical* of X. Thus if X has an M-prime submodule, then $\text{rad}_M(X)$ is equal to the intersection of all the M-prime submodules in X but, if X has no M-prime submodule, then $\text{rad}_M(X) = X$.

The following two propositions have been established in [2] for prime radical of modules. Now by the same method as [2], we extend these facts to M-prime radical of modules.

Proposition 2.28. *Let X be an R-module and $Y \leq X$. Then $rad_M(Y) \subseteq rad_M(X)$.*

Proof. Let P be any M-prime submodule of X. If $Y \subseteq P$, then $\mathrm{rad}_M(Y) \subseteq P$. If $Y \nsubseteq P$, then it is easy to check that $Y \cap P$ is an M-prime submodule of Y, and hence $\mathrm{rad}_M(Y) \subseteq (Y \cap P) \subseteq P$. Thus in any case, $\mathrm{rad}_M(Y) \subseteq P$. It follows that $\mathrm{rad}_M(Y) \subseteq \mathrm{rad}_M(X)$. □

Lemma 2.29. *Assume that M is projective in $\sigma[M]$, and let X be an R-module in $\sigma[M]$ such that $X = \bigoplus_{\lambda\in\Lambda} X_\lambda$ is a direct sum of submodules X_λ ($\lambda \in \Lambda$). Then for every submodule $N \subseteq M$, we have*

$$N \cdot X = \bigoplus_{\lambda\in\Lambda} N \cdot X_\lambda.$$

Proof. Since for every $\lambda \in \Lambda$, $X_\lambda \subseteq X$, $N \cdot X_\lambda \subseteq N \cdot X$ for every $\lambda \in \Lambda$. It follows that $\bigoplus_\Lambda N \cdot X_\lambda \subseteq N \cdot X$. On the other hand, since M is projective in $\sigma[M]$, so $N \cdot X = \sum_{f\in \mathrm{Hom}_R(M,X)} f(N)$ and for every $\lambda \in \Lambda$, $N \cdot X_\lambda = \sum_{f\in\mathrm{Hom}_R(M,X_\lambda)} f(N)$ by Lemma 2.9 (a). Now let $x \in N \cdot X$. Thus $x = \sum_{i=1}^{t} f_i(n_i)$ where $t \in \mathbb{N}$, $n_i \in N$ and $f_i \in \mathrm{Hom}_R(M, X)$. Since $f_i(n_i) \in X$, so for every $1 \le i \le t$, $f_i(n_i) = \{x_\lambda^{(i)}\}_\Lambda$, where $x_\lambda^{(i)} \in X_\lambda$. Thus $x = \{x_\lambda^{(1)} + \cdots + x_\lambda^{(t)}\}_\Lambda = \{\pi_\lambda f_1(n_1) + \cdots + \pi_\lambda f_t(n_t)\}_\Lambda$, where $\pi_\lambda : X \longrightarrow X_\lambda$ is the canonical projection for every $\lambda \in \Lambda$. It is clear that by Lemma 2.9, $\sum_{i=1}^{t} \pi_\lambda f_i(n_i) \in N \cdot X_\lambda$ for every $\lambda \in \Lambda$. Thus $x \in \bigoplus_\Lambda N \cdot X_\lambda$. □

We note that, since in Lemma 2.29, we assume that M is projective in $\sigma[M]$, so our product coincides with the product defined in [6, Definition 1.1]. Thus Lemma 2.29 is also proved in [6, Proposition 1.3 (8)]

Proposition 2.30. *Assume that M is projective in $\sigma[M]$, and let X be an R-module in $\sigma[M]$ such that $X = \bigoplus_{\lambda\in\Lambda} X_\lambda$ is a direct sum of submodules X_λ ($\lambda \in \Lambda$). Then*

$$rad_M(X) = \bigoplus_{\lambda\in\Lambda} rad_M(X_\lambda).$$

Proof. By Proposition 2.28, $\mathrm{rad}_M(X_\lambda) \subseteq \mathrm{rad}_M(X)$ for all $\lambda \in \Lambda$. Thus $\bigoplus_\Lambda \mathrm{rad}_M(X_\lambda) \subseteq \mathrm{rad}_M(X)$. Now let $x \notin \bigoplus_\Lambda \mathrm{rad}_M(X_\lambda)$, for some $x \in X$. Then there exists $\mu \in \Lambda$ such that $\pi_\mu(x) \notin \mathrm{rad}_M(X_\mu)$, where $\pi_\mu : X \to X_\mu$ denotes the canonical projection. Thus there exists an M-prime submodule Y_μ of X_μ such that $\pi_\mu(x) \notin Y_\mu$. Let $Z = Y_\mu \bigoplus (\bigoplus_{\lambda\ne\mu} X_\lambda)$. It is easy to check by Lemma 2.29 that Z is an M-prime submodule of X and $x \notin Z$. Thus $x \notin \mathrm{rad}_M(X)$. It follows that $\mathrm{rad}_M(X) \subseteq \bigoplus_\Lambda \mathrm{rad}_M(X_\lambda)$. □

3. M-Baer's lower nilradical of modules

We recall the definition of the nilpotent element in a module. An element x of an R-module X is called *nilpotent* if $x = \sum_{i=1}^{r} a_i x_i$ for some $a_i \in R$, $x_i \in X$ and $r \in \mathbb{N}$, such that ${a_i}^k x_i = 0 (1 \le i \le r)$ for some $k \in \mathbb{N}$ and x is called *strongly nilpotent* if $x = \sum_{i=1}^{r} a_i x_i$, for some $a_i \in R$, $x_i \in X$ and $r \in \mathbb{N}$, such that for every i $(1 \le i \le r)$ and every sequence $a_{i1}, a_{i2}, a_{i3}, \ldots$ where $a_{i1} = a_i$

and $a_{in+1} \in a_{in}Ra_{in}(\forall n)$, we have $a_{ik}Rx_i = 0$ for some $k \in \mathbb{N}$ (see [4]). It is clear that every strongly nilpotent element of a module X is a nilpotent element but the converse is not true (see the example 2.3 [4]). In case that R is commutative ring, nilpotent and strongly nilpotent are equal.

This notion has been generalized to modules over a projective module M in $\sigma[M]$.

Definition 3.1. Assume that M is projective in $\sigma[M]$, and let X be an R-module in $\sigma[M]$. Then an element $x \in X$ is called *M-nilpotent* if $x = \sum_{i=1}^{n} r_i f_i(m_i)$ for some $r_i \in R$, $m_i \in M$, $n \in \mathbb{N}$ and $f_i \in \mathrm{Hom}_R(M, Rx_i)$, where $x_i \in X$ such that ${r_i}^k f_i(m_i) = 0 (1 \leq i \leq n)$ for some $k \in \mathbb{N}$. Also, an element $x \in X$ is called *strongly M-nilpotent* if $x = \sum_{i=1}^{n} r_i f_i(m_i)$ for some $r_i \in R$, $m_i \in M$, $n \in \mathbb{N}$ and $f_i \in \mathrm{Hom}_R(M, Rx_i)$, where $x_i \in X$ such that for every i$(1 \leq i \leq n)$ and every sequence $r_{i1}, r_{i2}, r_{i3}, \ldots$, where $r_{i1} = r_i$ and $r_{it+1} \in r_{it}Rr_{it}$ $(\forall t)$, we have $r_{ik}Rf_i(m_i) = 0$ for some $k \in \mathbb{N}$.

Proposition 3.2. *Let X be an R-module. Then an element $x \in X$ is strongly nilpotent if and only if x is strongly R-nilpotent.*

Proof. $(\Rightarrow)$. Suppose that $x \in X$ is strongly nilpotent. Then $x = \sum_{i=1}^{n} r_i x_i$ for some $r_i \in R$, $x_i \in X$, $n \in \mathbb{N}$ such that for every i $(1 \leq i \leq n)$ and for every sequence $r_{i1}, r_{i2}, r_{i3}, \ldots$, where $r_{i1} = r_i$ and $r_{it+1} \in r_{it}Rr_{it}$ $(\forall t)$, we have $r_{ik}Rx_i = 0$ for some $k \in \mathbb{N}$. Now consider $f_i : R \to Rx_i$ such that $f_i(r) = rx_i$. Then $f_i(1) = x_i$ and it follows that $x = \sum_{i=1}^{n} r_i x_i = \sum_{i=1}^{n} r_i f_i(1)$. Since $r_{ik}Rx_i = 0$ $(1 \leq i \leq n)$ for some $k \in \mathbb{N}$,we conclude that $r_{ik}Rf_i(1) = 0$ $(1 \leq i \leq n)$ for some $k \in \mathbb{N}$, i.e., x is an strongly R-nilpotent element of X.

$(\Leftarrow)$. Assume that $x \in X$ is strongly R-nilpotent. Thus $x = \sum_{i=1}^{n} r_i f_i(a_i)$ for some $r_i, a_i \in R$, $n \in \mathbb{N}$ and $f_i \in \mathrm{Hom}_R(R, Rx_i)$, where $x_i \in X$ such that for every i $(1 \leq i \leq n)$ and for every sequence $r_{i1}, r_{i2}, r_{i3}, \ldots$, where $r_{i1} = r_i$ and $r_{it+1} \in r_{it}Rr_{it}$ $(\forall t)$, we have $r_{ik}Rf_i(a_i) = 0$ for some $k \in \mathbb{N}$. Since $f_i(a_i) \in Rx_i \subseteq X$, we conclude that x is a strongly nilpotent element of X. □

Proposition 3.3. *Let X be an R-module. Then an element $x \in X$ is nilpotent if and only if x is R-nilpotent.*

Proof. $(\Rightarrow)$. Assume that $x \in X$ is nilpotent. Thus $x = \sum_{i=1}^{n} r_i x_i$ for some $r_i \in R$, $x_i \in X$, $n \in \mathbb{N}$ such that ${r_i}^k x_i = 0 (1 \leq i \leq n)$ for some $k \in \mathbb{N}$. Now consider $f_i : R \to Rx_i$ such that $f_i(r) = rx_i$, so $f_i(1) = x_i$. It follows that $x = \sum_{i=1}^{n} r_i x_i = \sum_{i=1}^{n} r_i f_i(1)$. Since ${r_i}^k x_i = 0$ $(1 \leq i \leq n)$ for some $k \in \mathbb{N}$, so ${r_i}^k f_i(1) = 0$ $(1 \leq i \leq n)$ for some $k \in \mathbb{N}$, i.e., x is an R-nilpotent element of X.

$(\Leftarrow)$. Assume that $x \in X$ is an R-nilpotent element. Thus $x = \sum_{i=1}^{n} r_i f_i(a_i)$ for some $r_i, a_i \in R$, $n \in \mathbb{N}$ and $f_i \in \mathrm{Hom}_R(R, Rx_i)$, where $x_i \in X$ such that ${r_i}^k f_i(a_i) = 0 (1 \leq i \leq n)$ for some $k \in \mathbb{N}$. Since $f_i(a_i) \in Rx_i \subseteq X$, we conclude that x is a nilpotent element of X. □

Proposition 3.4. *Assume that R is a commutative ring, M is projective in $\sigma[M]$ and $X \in \sigma[M]$. Then an element $x \in X$ is M-nilpotent if and only if x is strongly M-nilpotent.*

Proof. ($\Rightarrow$). Assume that $x \in X$ is M-nilpotent. Thus $x = \sum_{i=1}^{n} r_i f_i(m_i)$ for some $r_i \in R$, $m_i \in M$, $n \in \mathbb{N}$ and $f_i \in \mathrm{Hom}_R(M, Rx_i)$, where $x_i \in X$ such that ${r_i}^k f_i(m_i) = 0$ $(1 \leq i \leq n)$ for some $k \in \mathbb{N}$. Consider sequence $r_{i1}, r_{i2}, r_{i3}, \ldots$, where $r_{i1} = r_i$ and $r_{it+1} \in r_{it}Rr_{it}$ for every $1 \leq i \leq n$ and $(\forall t)$. Thus there exists an element $r_{ik} = {r_{i1}}^k r'$ (where $r' \in R$) such that $r_{ik}Rf_i(m_i) = {r_{i1}}^k r' R f_i(m_i) = 0$ (since R is commutative and ${r_{i1}}^k f_i(m_i) = 0$). Thus $x \in X$ is a strongly M-nilpotent element.

($\Leftarrow$). Suppose that $x \in X$ is a strongly M-nilpotent element. Thus $x = \sum_{i=1}^{n} r_i f_i(m_i)$ for some $r_i \in R$, $m_i \in M$, $n \in \mathbb{N}$ and $f_i \in \mathrm{Hom}_R(M, Rx_i)$, where $x_i \in X$ such that for every i $(1 \leq i \leq n)$ and for every sequence $r_{i1}, r_{i2}, r_{i3}, \ldots$, where $r_{i1} = r_i$ and $r_{it+1} \in r_{it}Rr_{it}$ $(\forall t)$, we have $r_{ik}Rf_i(m_i) = 0$ for some $k \in \mathbb{N}$. Consider sequence $r_{i1}, r_{i2}, r_{i3}, \ldots$, where $r_{i1} = r_i$ and $r_{i2} = {r_{i1}}^2 = r_{i1}1r_{i1} \in r_{i1}Rr_{i1}$, $r_{i3} = {r_{i1}}^4 = r_{i1}1r_{i1}1r_{i1}1r_{i1} \in r_{i2}Rr_{i2}, \ldots$. By assumption, we have $r_{ik}Rf_i(m_i) = 0$ for some $k \in \mathbb{N}$. Since $r_{ik} = {r_{i1}}^{k'}$ for some $k' \in \mathbb{N}$, so ${r_{i1}}^{k'} R f_i(m_i) = r_{ik}Rf_i(m_i) = 0$. Now for $r = 1$, we have ${r_{i1}}^{k'} 1 f_i(m_i) = 0$. Thus x is an M-nilpotent element. $\square$

We recall the definition of Baer's lower nilradical in a module. For any module X, $\mathrm{Nil}_*({}_RX)$ is the set of all strongly nilpotent elements of X. In case that R is a commutative ring, $\mathrm{Nil}_*({}_RX)$ is the set of all nilpotent elements of X.

Definition 3.5. Assume that M is projective in $\sigma[M]$. For any module X in $\sigma[M]$, we define M-$Nil_*({}_RX)$ to be the set of all strongly M-nilpotent elements of X. This is called *M-Baer's lower nilradical* of X.

Proposition 3.6. *Assume that M is projective in $\sigma[M]$. Then for any module X in $\sigma[M]$*

$$Nil_*(M) \cdot X \subseteq M{-}Nil_*({}_RX) \subseteq rad_M(X).$$

Proof. Since M is projective in $\sigma[M]$, by Lemma 2.9 (a),

$$\mathrm{Nil}_*(M) \cdot X = \sum_{f \in \mathrm{Hom}_R(M,X)} f(\mathrm{Nil}_*(M)).$$

Now let $x \in \mathrm{Nil}_*(M) \cdot X$. Thus $x = \sum_{i=1}^{s} f_i(m_i)$ for some $m_i \in \mathrm{Nil}_*(M)$, $s \in (N)$ and $f_i \in \mathrm{Hom}_R(M, X)$. Since $m_i \in \mathrm{Nil}_*(M)$, so $m_i = \sum_{j=1}^{t} r_{i_j} n_{i_j}$ for some $r_{i_j} \in R$, $n_{i_j} \in M$, $t \in \mathbb{N}$ such that for every j $(1 \leq j \leq t)$ and for every sequence $r_{i_{j1}}, r_{i_{j2}}, r_{i_{j3}}, \ldots$, where $r_{i_{j1}} = r_{i_j}$ and $r_{i_{ju+1}} \in r_{i_{ju}}Rr_{i_{ju}}$ $(\forall u)$, we have $r_{i_{jk_i}}Rn_{i_j} = 0$ for some $k_i \in \mathbb{N}$. Thus $x = \sum_{i=1}^{s} f_i(m_i) = \sum_{i=1}^{s} f_i(\sum_{j=1}^{t} r_{i_j} n_{i_j}) = \sum_{i=1}^{s} \sum_{j=1}^{t} r_{i_j} f_i(n_{i_j})$. Since $r_{i_{jk_i}}Rn_{i_j} = 0$, we conclude that $0 = f_i(r_{i_{jk_i}}Rn_{i_j}) = r_{i_{jk_i}}Rf_i(n_{i_j})$ for some $k_i \in \mathbb{N}$, where $(1 \leq i \leq s)$ and $(1 \leq j \leq t)$. Thus $x \in M$-$\mathrm{Nil}_*({}_RX)$.

Let $x \in M\text{-Nil}_*({}_RX)$ and $x \notin \text{rad}_M(X) = \sqrt[M]{(0)}$. So $x = \sum_{i=1}^n a_i f_i(m_i)$ for some $a_i \in R$, $m_i \in M$, $n \in \mathbb{N}$ and $f_i \in \text{Hom}_R(M, Rx_i)$ such that for every i$(1 \leq i \leq n)$ and for every sequence $a_{i1}, a_{i2}, a_{i3}, \ldots$, where $a_{i1} = a_i$ and $a_{iu+1} \in a_{iu}Ra_{iu}$ $(\forall u)$, we have $a_{ik}Rf_i(m_i) = 0$ for some $k \in \mathbb{N}$. Without loss of generality, we can assume that $a_1 f_1(m_1) \notin \text{rad}_M(X)$. Thus there exists an M-m-system S such that $a_1 f_1(m_1) \in S$ and $0 \notin S$. On the other hand $a_1 f_1(m_1) \in Ra_1(Rm_1) \cdot (Rx_1)$. Thus $Ra_1(Rm_1) \cdot (Rx_1) \ \cap S \neq \emptyset$ and hence $Ra_1(Rm_1) \cdot X \ \cap S \neq \emptyset$. Therefore, if we put $N = Ra_1(Rm_1)$, $Y = (0)$ and $Z = Ra_1(Rm_1) \cdot (Rx_1)$, then $(Ra_1(Rm_1))^2 \cdot (Rx_1) \cap S \neq \emptyset$ by Proposition 2.14. Since M is projective in $\sigma[M]$, by Lemma 2.9(a) and Lemma 2.23, we conclude that

$$\begin{aligned}(Ra_1(Rm_1))^2 \cdot (Rx_1) &= (Ra_1(Rm_1) \cdot Ra_1(Rm_1)) \cdot (Rx_1) \\ &= (Ra_1(Rm_1)).(Ra_1(Rm_1) \cdot (Rx_1)) \\ &= \sum_{f \in \text{Hom}_R(M, Ra_1(Rm_1) \cdot (Rx_1))} f(Ra_1(Rm_1)).\end{aligned}$$

Assume that $s_1 = 1$, $a_{11} = a_1$ and $a_1 f_1(t_1 a_1 s_2 m_1) \in (Ra_1(Rm_1))^2 \cdot (Rx_1) \cap S$, where $s_2, t_1 \in R$. Since $a_1 f_1(t_1 a_1 s_2 m_1) = s_2 a_1 t_1 a_1 f_1(m_1)$ and $a_{12} = a_1 t_1 a_1$, so $s_2 a_{12} f_1(m_1) \in Ra_{12}(Rm_1) \cdot (Rx_1) \cap S$. It follows that $Ra_{12}(Rm_1) \cdot (Rx_1) \cap S \neq \emptyset$ and so

$$(Ra_{12}(Rm_1))^2 \cdot (Rx_1) \ \cap S \neq \emptyset.$$

Thus there exists $s_3 a_{13} f_1(m_1) \in (Ra_{12}(Rm_1))^2 \cdot (Rx_1) \cap S$, where $s_3 \in R$, and $a_{13} := a_{12} t_2 s_2 a_{12}$ for some $t_2 \in R$. We can repeat this argument to get sequences $\{s_u\}_{u \in N}$ and $\{a_{1u}\}_{u \in \mathbb{N}}$ in R, where $a_{11} = a_1$ and $a_{1u+1} \in a_{1u}Ra_{1u}$ $(\forall u)$, such that $s_u a_{1u} f_1(m_1) \in S$ for all $u \geq 1$. Now by our hypothesis $a_{1k}Rf_1(m_1) = 0$ for some $k \in \mathbb{N}$, and so $s_k a_{1k} f_1(m_1) = 0 \in S$, a contradiction. □

In case $M = R$, by Proposition 3.6, $\text{Nil}_*(R) \cdot X \subseteq R\text{-Nil}_*({}_RX) \subseteq \text{rad}_R(X)$. Since by Proposition 3.2, $R\text{-Nil}_*({}_RX)$ is the set of all strongly R-nilpotent elements of X, so we have $R\text{-Nil}_*({}_RX) = \text{Nil}_*({}_RX)$ (see also, [2, Lemma 3.2]).

Corollary 3.7. *Assume that M is projective in $\sigma[M]$. Then*

$$Nil_*(M) = Nil_*(M) \cdot M = M - Nil_*(M).$$

Proof. By Proposition 3.6, $\text{Nil}_*(M) \cdot M \subseteq M\text{-Nil}_*(M)$. Also, we have $\text{Nil}_*(M) \cdot M = \sum_{f \in \text{Hom}_R(M,M)} f(\text{Nil}_*(M))$, by Lemma 2.9 (a). Since $1_M \in \text{Hom}_R(M, M)$, so $\text{Nil}_*(M) \subseteq \text{Nil}_*(M) \cdot M$. On the other hand, if $x \in M\text{-Nil}_*(M)$, then $x = \sum_{i=1}^n r_i f_i(m_i)$ for some $r_i \in R$, $m_i \in M$, $n \in \mathbb{N}$ and $f_i \in \text{Hom}_R(M, Rx_i)$, where $x_i \in M$ such that for every i $(1 \leq i \leq n)$ and for every sequence $r_{i1}, r_{i2}, r_{i3}, \ldots$, where $r_{i1} = r_i$ and $r_{it+1} \in r_{it}Rr_{it}$ $(\forall t)$, we have $r_{ik}Rf_i(m_i) = 0$ for some $k \in \mathbb{N}$. Since $f_i(m_i) \in Rx_i \subseteq M$, it follows that x is a strongly nilpotent element of M. So $x \in \text{Nil}_*(M)$. It follows that $M\text{-Nil}_*(M) \subseteq \text{Nil}_*(M)$

and $\mathrm{Nil}_*(M) \subseteq \mathrm{Nil}_*(M)\cdot M \subseteq M\text{-}\mathrm{Nil}_*(M) \subseteq \mathrm{Nil}_*(M)$. Thus $\mathrm{Nil}_*(M) = \mathrm{Nil}_*(M)\cdot M = M\text{-}\mathrm{Nil}_*(M)$. □

Corollary 3.8. *Assume that M is projective in $\sigma[M]$. Then $rad_R(M) \subseteq rad_M(M)$.*

Proof. By Proposition 3.6, we have $M\text{-}\mathrm{Nil}_*(M) \subseteq \mathrm{rad}_M(M)$. On the other hand $\mathrm{Nil}_*(M) = M\text{-}\mathrm{Nil}_*(M)$ by Corollary 3.7. Thus $\mathrm{Nil}_*(M) \subseteq \mathrm{rad}_M(M)$. Since M is projective in $\sigma[M]$, $\mathrm{rad}_R(M) = \mathrm{Nil}_*(M)$ by [2, Theorem 3.8]. Thus $\mathrm{rad}_R(M) = \mathrm{Nil}_*(M) \subseteq \mathrm{rad}_M(M)$. □

Proposition 3.9. *Assume that M is projective in $\sigma[M]$. If $X \in \sigma[M]$ such that $rad_M(X) = M\text{-}Nil_*(X)$, then $rad_M(Y) = M\text{-}Nil_*(Y)$ for any direct summand Y of X.*

Proof. Suppose that $X = Y \oplus Z$, where Z, Y are submodules of X. By Proposition 3.6, $M\text{-}\mathrm{Nil}_*(Y) \subseteq \mathrm{rad}_M(Y)$. Let $x \in \mathrm{rad}_M(Y)$. By Proposition 2.28, $x \in \mathrm{rad}_M(X)$. By hypothesis $x \in M\text{-}\mathrm{Nil}_*(X)$. Thus $x = \sum_{i=1}^n r_i f_i(m_i)$ for some $r_i \in R$, $m_i \in M$, $n \in \mathbb{N}$ and $f_i \in \mathrm{Hom}_R(M, Rx_i)$, where $x_i \in X$ such that for every i $(1 \le i \le n)$ and for every sequence $r_{i1}, r_{i2}, r_{i3}, \ldots$, where $r_{i1} = r_i$ and $r_{it+1} \in r_{it}Rr_{it}$ $(\forall t)$, we have $r_{ik}Rf_i(m_i) = 0$ for some $k \in \mathbb{N}$. Since $x_i \in X$, there exist elements $y_i \in Y$, $z_i \in Z$ such that $x_i = y_i + z_i$ for each i $(1 \le i \le n)$. On the other hand, $f_i(m_i) \in Rx_i$ for each i, and hence $f_i(m_i) = a_i(y_i + z_i)$ for some $a_i \in R$ $(1 \le i \le n)$. It is clear that $x = r_1a_1y_1 + r_2a_2y_2 + \cdots + r_na_ny_n$, and $r_{ik}Ra_iy_i = 0$ for some $k \in \mathbb{N}$ $(1 \le i \le n)$. Now for each i $(1 \le i \le n)$, we consider $g_i : M \xrightarrow{f_i} Rx_i \subseteq X \xrightarrow{\pi_i} Ry_i \subseteq Y$, where π_i is the natural projection map such that $g_i(m_i) = \pi_i f_i(m_i) = \pi_i(a_i(y_i + z_i)) = a_iy_i$. Thus $x = r_1a_1y_1 + r_2a_2y_2 + \cdots + r_na_ny_n = \sum_{i=1}^n r_ig_i(m_i)$, where $g_i \in \mathrm{Hom}_R(M, Ry_i)$ and $r_{ik}Ra_iy_i = r_{ik}Rg_i(m_i) = 0$. It follows that $x \in M\text{-}\mathrm{Nil}_*(Y)$. Thus $\mathrm{rad}_M(Y) = M\text{-}\mathrm{Nil}_*(Y)$. □

4. M-injective modules and prime M-ideals

The module ${}_RX$ is said to be M*-generated* if there exists an R-epimorphism from a direct sum of copies of M onto X. Equivalently, for each nonzero R-homomorphism $f : X \to Y$ there exists an R-homomorphism $g : M \to X$ with $fg \neq 0$. The *trace* of M in X is defined to be

$$tr^M(X) = \sum_{f \in \mathrm{Hom}_R(M,X)} f(M)$$

and thus X is M-generated if and only if $tr^M(X) = X$.

We recall the definition of prime M-ideal. The proper M-ideal P is said to be a prime M-ideal if there exists an M-prime module ${}_RX$ such that $P = \mathrm{Ann}_M(X)$.

Proposition 4.1. *Let M an R-module with $Hom_R(M, X) \neq 0$ for every $X \in \sigma[M]$ and P be a proper M-ideal. Then P is a prime M-ideal if and only if P is a Beachy-prime M-ideal.*

Proof. Assume that P is a prime M-ideal. Thus there exists M-prime module X such that $P = \mathrm{Ann}_M(X)$. Since $P \neq M$, $\mathrm{Hom}_R(M, X) \neq 0$. Thus by Proposition 2.7, X is a Beachy-M-prime module. Thus P is a Beachy-prime M-ideal.

Conversely, let P be a Beachy-prime M-ideal. Thus there exists a Beachy-M-prime module X in $\sigma[M]$ such that $P = \mathrm{Ann}_M(X)$. Since $\mathrm{Hom}_R(M, X) \neq 0$, so $X \neq (0)$. Now assume that Y is a nonzero submodule of X. So $Y \in \sigma[M]$ and $\mathrm{Hom}_R(M, Y) \neq 0$ by assumption. Therefore, $\mathrm{Ann}_M(X) = \mathrm{Ann}_M(Y)$ by the definition of Beachy-M-prime module. Thus by Proposition 2.4, X is an M-prime module and hence P is a prime M-ideal. □

The module ${}_RX$ in $\sigma[M]$ is said to be *finitely M-generated* if there exists an epimorphism $f : M^n \to X$, for some positive integer n. It is said to be *finitely M-annihilated* if there exists a monomorphism $g : M/\mathrm{Ann}_M(X) \to X^m$, for some positive integer m. Also, the module ${}_RM$ is said to *satisfy condition H* if every finitely M-generated module is finitely M-annihilated. Note that if $M = R$ and R is a fully bounded Noetherian ring, then M satisfies condition H. The same is true if M is an Artinian module, since then M/K has the finite intersection property.

In [1, Theorem 6.7], it is shown that if M is a Noetherian module such that M satisfies condition H and $\mathrm{Hom}_R(M, X) \neq 0$ for all modules X in $\sigma[M]$, then there is a one-to-one correspondence between isomorphism classes of indecomposable M-injective modules in $\sigma[M]$ and Beachy-prime M-ideals. Next, in the main result of this section, we show this fact is also true for a Noetherian module with condition H and the assumption $\mathrm{Hom}_R(M, X) \neq 0$ for all modules X in $\sigma[M]$ via prime M-ideals.

Corollary 4.2. *Let M be a Noetherian R-module. If M satisfies condition H and $Hom_R(M, X) \neq 0$ for all modules X in $\sigma[M]$, then there is a one-to-one correspondence between isomorphism classes of indecomposable M-injective modules in $\sigma[M]$ and prime M-ideals.*

Proof. By [1, Theorem 6.7] and Proposition 4.1, it is clear. □

5. Prime M-ideals and M-prime radical of Artinian modules

Let M be an R-module. Recall that a proper submodule P of M is *virtually maximal* if the factor module M/P is a homogeneous semisimple R-module, i.e., M/P is a direct sum of isomorphic simple modules. Clearly, every virtually maximal submodule of M is prime. Also, every maximal submodule of M is virtually maximal and for $M = R$ and R commutative, this is equivalent to the notion of maximal ideal in R.

We recall that $\mathrm{Soc}(M)$ is sum of all minimal submodules of M. If M has no minimal submodule, then $\mathrm{Soc}(M) = (0)$.

Proposition 5.1. *Let M be an Artinian R-module. If M is an M-prime module, then M is a homogeneous semisimple module.*

Proof. Since M is an Artinian R-module, $\mathrm{Soc}(M) \neq (0)$. Hence there exist simple submodule Rm of M where $0 \neq m \in M$. Since M is an M-prime module, $\mathrm{Ann}_M(Rm) = \mathrm{Ann}_M(M) = (0)$ by Proposition 2.4. Thus $(0) = \mathrm{Ann}_M(Rm) = \bigcap_{f \in \mathrm{Hom}_R(M,Rm)} \ker(f)$. Since $Rm \cong M/\ker(f)$ for every $f \in \mathrm{Hom}_R(M, Rm)$, (0) is an intersection of maximal submodules and since M is Artinian, (0) must be a finite intersection of maximal submodules. It follows that M is isomorphic to a finite direct sum of copies of Rm. Thus M is a homogeneous semisimple module. □

An M-ideal P is said to be a *primitive M-ideal* if $P = \mathrm{Ann}_M(S)$ for a simple module ${}_RS$ (see [1, Definition 3.5]).

Proposition 5.2. *Let P be a proper M-ideal. If P is a primitive M-ideal, then P is a prime M-ideal.*

Proof. If P is a primitive M-ideal, then $P = \mathrm{Ann}_M(S)$ for a simple R-module S. Since S has no any proper submodule, S is an M-prime module by Proposition 2.4. Thus P is a prime M-ideal. □

Proposition 5.3. *Let M be an M-prime module with $Soc(M) \neq (0)$. Then (0) is a primitive M-ideal.*

Proof. Since $\mathrm{Soc}(M) \neq (0)$, there exists a simple submodule Rm of M where $0 \neq m \in M$. Since M is an M-prime module, so $\mathrm{Ann}_M(Rm) = \mathrm{Ann}_M(M) = (0)$. Therefore, (0) is a primitive M-ideal. □

Proposition 5.4. *Assume that M is projective in $\sigma[M]$. If M is an Artinian R-module, then every prime M-ideal of M is virtually maximal.*

Proof. Suppose that $P \subsetneqq M$ is a prime M-ideal. Since M is projective in $\sigma[M]$, M/P is an M-prime module by Proposition 2.10. Since M/P is also an Artinian module, $\mathrm{Soc}(M/P) \neq (0)$ and hence there exists a simple submodule $R\bar{m}$ of M/P where $0 \neq \bar{m} \in M/P$. Since M/P is an M-prime module, $\mathrm{Ann}_M(R\bar{m}) = \mathrm{Ann}_M(M/P) = P$. On the other hand, $P = \mathrm{Ann}_M(R\bar{m}) = \bigcap_{f \in \mathrm{Hom}_R(M,R\bar{m})} \ker(f)$. Since $R\bar{m} \cong M/\ker(f)$ for every $f \in \mathrm{Hom}_R(M, R\bar{m})$, P must be an intersection of maximal submodules. Since M/P is Artinian, P must be a finite intersection of maximal submodules, and so M/P is isomorphic to a finite direct sum of copies of $R\bar{m}$. Thus M/P is a homogeneous semisimple module, i.e., P is a virtually maximal submodule of M. □

Definition 5.5. The *prime radical* of the module M, denoted by $P(M)$, is defined to be the intersection of all prime M-ideals.

We note that each prime M-ideal is the annihilator of an M-prime module in M. It follows that $P(M) = \text{rad}_{\mathcal{C}}(M)$, where $\mathcal{C}$ is the class of all M-prime left R-modules. If ${}_RX$ is any module with a submodule Y such that X/Y is an M-prime module, then $\text{rad}_{\mathcal{C}}(X) \subseteq Y$. In this case it follows from [1, Lemma 1.8] that $P(M) \cdot X \subseteq Y$.

Theorem 5.6. *Assume that M is projective in $\sigma[M]$. If M is an Artinian R-module, then every prime M-ideal of M is virtually maximal and $M/P(M)$ is a Noetherian R-module.*

Proof. If M does not contain any prime M-ideal, then $P(M) = M$. Suppose that M contains a prime M-ideal. By Proposition 5.4, every prime M-ideal of M is virtually maximal. Let N be minimal in the collection $\mathcal{S}$ of M-ideals of M which are finite intersections of primes. If P is any prime M-ideal of M, then $P \cap N \in \mathcal{S}$ and $P \cap N \subseteq N$. Thus $N = P \cap N \subseteq P$ by minimality of N in $\mathcal{S}$. It follows that $N = P(M)$. On the other hand, for each prime M-ideal, the factor module M/P is a homogeneous semisimple module with DCC. So M/P is Noetherian. Thus M/P is Noetherian for every prime M-ideal P of M. Since $P(M)$ is a finite intersection of prime M-ideals, $M/P(M)$ is also a Noetherian R-module. □

The following theorem is a generalization of [2, Theorem 2.11].

Theorem 5.7. *Assume that M is projective in $\sigma[M]$. If M be an Artinian R-module, then $P(M) = M$ or there exist primitive M-ideals $P_1, \dots, P_n$ of M such that $P(M) = \bigcap_{i=1}^{n} P_i$.*

Proof. Let P be a prime M-ideal of M. Since M is projective in $\sigma[M]$, so M/P is an M-prime module by Proposition 2.10 (ii). Since M/P is an Artinian R-module, $\text{Soc}(M/P) \neq (0)$. Thus there exists a simple submodule $R\bar{m}$ of M/P where $0 \neq \bar{m} \in M/P$. Since M/P is an M-prime module, $\text{Ann}_M(R\bar{m}) = \text{Ann}_M(M/P)$. On the other hand, $\text{Ann}_M(M/P) = P$, since P is an M-ideal. Thus P is a primitive M-ideal. Since P is arbitrary prime M-ideal, so every prime M-ideal of M is primitive M-ideal. On the other hand by Proposition 5.2, we have that every primitive M-ideals is prime M-ideal. Thus $P(M)$ is the intersection all of primitive M-ideal of M. Now let N be minimal in the collection $\mathcal{S}$ of M-ideals of M which are finite intersections of primes. If Q is any prime M-ideal of M, then $Q \cap N \in \mathcal{S}$ and $Q \cap N \subseteq N$. Thus $N = Q \cap N \subseteq Q$ by minimality of N in $\mathcal{S}$. It follows that $N = P(M)$. Thus $P(M)$ is a finite intersection of prime M-ideals and it follows that $P(M)$ is a finite intersection of primitive M-ideals. So there exist primitive M-ideals $P_1, \dots, P_n$ of M such that $P(M) = \bigcap_{i=1}^{n} P_i$. Since P_i is an M-ideal for every $1 \leq i \leq n$, $P_i \cdot M = P_i$ and so $P(M) = \bigcap_{i=1}^{n} P_i \cdot M = \bigcap_{i=1}^{n} P_i$. □

Corollary 5.8. *Assume that M is projective in $\sigma[M]$. If M be an Artinian M-prime module, then $P(M) = (0)$.*

Proof. By Proposition 5.3, (0) is a primitive M-ideal of M. It follows that $P(M) = (0)$ by Theorem 5.7. □

Minimal M-prime submodules are defined in a natural way. By Zorn's Lemma one can easily see that each M-prime submodule of a module X contains a minimal M-prime submodule of X. In [18, Theorem 5.2], it is shown that every Noetherian module contain only finitely many minimal prime submodules. It is easy to show that if X is a Noetherian module, then X contain only finitely many minimal M-prime submodules.

We conclude this paper with the following interesting result, which is a generalization of [2, Theorem 2.1].

Theorem 5.9. *Let X be a Noetherian R-module. If every M-prime submodule of X is virtually maximal, then $X/rad_M(X)$ is an Artinian R-module.*

Proof. By our hypotheses, for each M-prime submodule P of X, X/P is a homogeneous semisimple R-module. Since X is a Noetherian R-module, X/P is also Noetherian. This implies that X/P is an Artinian R-module. On the other hand $\mathrm{rad}_M(X) = P_1 \cap \cdots \cap P_n$ where $P_1, \ldots, P_n$ are all minimal M-prime submodules of M. Thus $X/P_1 \oplus \cdots \oplus X/P_n$ is also an Artinian R-module. It follows that $X/\mathrm{rad}_M(X)$ is an Artinian R-module. □

Acknowledgments. The authors owe a great debt to the referee who has carefully read an earlier version of this paper and made significant suggestions for improvement. We would like to express our deep appreciation for the referee's work.

References

[1] J. A. Beachy, *M-injective modules and prime M-ideals*, Comm. Algebra **30** (2002), no. 10, 4649–4676.

[2] M. Behboodi, *On the prime radical and Baer's lower nilradical of modules*, Acta Math. Hungar. **122** (2009), no. 3, 293–306.

[3] M. Behboodi, *A generalization of the classical krull dimension for modules*, J. Algebra **305** (2006), no. 2, 1128–1148.

[4] M. Behboodi, *A generalization of Baer's lower nilradical for modules*, J. Algebra Appl. **6** (2007), no. 2, 337–353.

[5] M. Behboodi and H. Koohy, *Wealy prime modules*, Vietnam J. Math. **32** (2004), no. 2, 185–195.

[6] J. C. Perez and J. R. Montes, *Prime submodules and local Gabriel correspondence in $\sigma[M]$*, Comm. Algebra **40** (2012), no. 1, 213–232.

[7] J. Dauns, *Prime modules*, J. Reine Angew. Math. **298** (1978), 156–181.

[8] P. Gabriel, *Des categories Abeliennes*, Bull. Soc. Math. France **90** (1964), 323–448.

[9] K. R. Goodearl and R. B. Warfield, *An Introduction to Non-Commutative Noetherian Rings*, London Math. Soc. Student Texts 16, Camberidge University Press, Cambrige, 1989.

[10] J. Jenkins and P. F. Smith, *On the prime radical of a module over a commutative ring*, Comm. Algebra **20** (1992), no. 12, 3593–3602.

[11] H. I. Karakas, *On Noetherian module*, METU J. Pure Appl. Sci. **5** (1972), no. 2, 165–168.

[12] K. Koh, *On prime one-sided ideals*, Canad. Math. Bull. **14** (1971), 259–260.

[13] G. Krause, *On fully left bounded left Noetherian ring*, J. Algebra **23** (1972), 88–99.
[14] T. Y. Lam, *A First Course in Noncommutative Rings*, Springer-Verlag New York, Inc 1991.
[15] C. P. Lu, *Prime submodules of modules*, Comment. Math. Univ. St. Paul. **33** (1984), no. 1, 61–69.
[16] S. H. Man, *On commutative Noetherian rings which have the s.p.a.r. property*, Arch. Math. (Basel) **70** (1998), no. 1, 31–40.
[17] R. L. McCasland, M. E. Moore, *On radicals of submodules*, Comm. Algebra **19** (1991), 1327–1341.
[18] R. L. McCasland and P. F. Smith, *Prime submodules of Noetherian modules*, Rocky Mountain J. Math. **23** (1993), no. 3, 1041–1062.
[19] G. Michler, *Prime right ideals and right Noetherian rings*, Proc. Symposium on Theory of Rings, 1971, in Ring theory (R. Gordon, ed.), 251–255, Academic Press, New York, 1972.
[20] P. F. Smith, *The injective test lemma in fully bounded rings*, Comm. Algebra **9** (1981), no. 17, 1701–1708.
[21] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach Reading 1991.

J. A. Beachy
Department of Mathematical Sciences
Northern Illinois University
DeKalb, IL, 60115-2888, USA
E-mail address: beachy@math.niu.edu

M. Behboodi
Department of Mathematical Sciences
Isfahan University of Technology
Isfahan, 84156-83111, Iran
E-mail address: mbehbood@cc.iut.ac.ir

F. Yazdi
School of Mathematics
Institute for Research in Fundamental Sciences (IPM)
Tehran, 19395-5746, Iran
E-mail address: f.yazdi@math.iut.ac.ir