On reduced rank of triangular matrix rings

Abigail C. Bailey and John A. Beachy

Abstract: We determine conditions under which a generalized triangular matrix ring has finite reduced rank, in the general torsion-theoretic sense. These are applied to characterize certain orders in Artinian rings, and to show that if each homomorphic image of a ring S has finite reduced rank, then so does the ring of lower triangular matrices over S.

The notion of the reduced rank of a module has proven to be a useful tool in the study of Noetherian rings. In [7], Goldie defined the reduced rank $\rho_R(M)$ of a finitely generated module M over a semiprime Goldie ring R to be the uniform dimension of $M/\gamma(M)$, where γ is the Goldie torsion radical defined by the set of regular elements of R. Equivalently, $\rho_R(M)$ is given by the length of the left Q-module $Q \otimes_R M$, where Q is the classical ring of left quotients of R, which is semisimple Artinian. The reduced rank of a finitely generated module M over a Noetherian ring R was defined by Goldie as follows: if N is the prime radical of R, then N is nilpotent, say $N^k = (0)$, and R/N is a semiprime Goldie ring, so it is possible to utilize the previous definition by setting $\rho_R(M) = \sum_{i=0}^{k-1} \rho_{R/N}(N^i M/N^{i+1}M)$.

A general definition of reduced rank has been introduced and studied in [2]. If R is any ring with prime radical N, let γ denote the torsion radical cogenerated by the injective envelope $E(R/N)$ of the left R-module R/N. Then a module ${}_RM$ is said to have finite reduced rank if the module of quotients $Q_\gamma(M)$ has finite length in the quotient category R–Mod$/\gamma$. This approach has the advantage of allowing one to work in a Grothendieck category, and so, for example, the Jordan–Hölder Theorem guarantees that the length of a composition series is well-defined, and since the quotient functor $Q_\gamma : R\text{–Mod} \to R\text{–Mod}/\gamma$ is exact, it follows immediately that this rank function is well-defined and additive on short exact sequences of modules.

The class of rings with finite reduced rank (in the general sense of [2]) is quite extensive. Lenagan [10] has shown that it contains all rings with Krull dimension (see [9] for the definition of Krull dimension). It has been shown in [2] that any left order in a left Artinian ring has finite reduced rank (extending a result of Warfield [14]), and that if R has finite reduced rank, then the polynomial ring $R[x_1, x_2, \ldots]$ in countably many commuting indeterminates has the same reduced rank. It has been shown in [3] that having finite reduced rank is a Morita invariant property.

In this paper we investigate the general notion of reduced rank for formal triangular matrix rings, after giving new conditions equivalent to the existence of finite reduced rank for a ring R. In particular, Theorem 2 shows that a ring of formal triangular matrices has finite reduced rank on the left if and only if the same condition holds for each component of the ring. As a consequence, we show that if a ring R has finite reduced rank for each factor ring (e.g. if R has Krull dimension), then the same condition holds for the ring of triangular matrices over R. We also give very simple conditions under which a triangular matrix ring is a left order in a left Artinian ring. The majority of these results are included in the first author's thesis [12].

Throughout the paper, R will be assumed to be an associative ring with identity element, and all modules will be assumed to be unital R-modules. The injective envelope of a module ${}_RM$ will be denoted by $E(M)$, and the direct sum of n isomorphic copies of M will be denoted by M^n. The reader is referred to [13] for definitions and results on quotient categories and torsion radicals, and to [5] and [6] for results on Noetherian rings and the rank of a module.

Any injective module ${}_RX$ is the cogenerator of a quotient category $R\text{–Mod}/\sigma$, which consists of all modules ${}_RM$ such that M and $E(M)/M$ can be embedded in a direct product of copies of X, together with all R-homomorphisms between such modules. The quotient category determines (and is determined by) the torsion radical σ, and the quotient functor $Q_\sigma : R\text{–Mod} \to R\text{–Mod}/\sigma$. For any module ${}_RM$ the submodule $\sigma(M)$ is the intersection of all kernels of homomorphisms $f \in \mathrm{Hom}_R(M, X)$, and so, in particular, $\sigma(R) = \mathrm{Ann}_R(X)$. Note that $\sigma(R)M \subseteq \sigma(M)$. The module ${}_RM$ is called σ-torsionfree if $\sigma(M) = (0)$, and σ-torsion if $\sigma(M) = M$; a submodule $M' \subseteq M$ is called σ-closed if M/M' is σ-torsionfree, and σ-dense if M/M' is σ-torsion. Thus the σ-closed left ideals of R are the left annihilators of subsets of X.

In defining the quotient functor $Q_\sigma : R\text{–Mod} \to R\text{–Mod}/\sigma$ on the module ${}_RM$, the first step is to factor out the torsion submodule $\sigma(M)$. Then for $M' = M/\sigma(M)$, $Q_\sigma(M) = Q_\sigma(M')$ is defined by the identity $Q_\sigma(M')/M' = \sigma(E(M')/M')$. For any R-homomorphism $f : M \to N$ there is a functorial morphism $Q_\sigma(f) : Q_\sigma(M) \to Q_\sigma(N)$. The subobjects of $Q_\sigma(M)$ correspond to the σ-closed submodules of M, and so for the ring R, the subobjects of $Q_\sigma(R)$ in $R\text{–Mod}/\sigma$ correspond to the left annihilators of subsets of ${}_RX$.

In the quotient category $R\text{–Mod}/\sigma$, the object $Q_\sigma(M)$ has finite length if and only if M satisfies both ascending and descending chain conditions on σ-closed submodules. In this case, $M/\sigma(M)$ must have finite uniform dimension (by Lemma 1 of [1] this is true whenever $Q_\sigma(M)$ has Krull dimension in the quotient category), and if I is any σ-closed semiprime ideal of R such that $Q_\sigma(R/I)$ has finite length, then R/I must be a left Goldie ring. We note that Theorem 1.4 of [11] shows that Hopkins' Theorem holds for quotient categories: if σ is any torsion radical and $Q_\sigma(R)$ satisfies the descending chain condition in $R\text{–Mod}/\sigma$, then it must also satisfy the ascending chain condition.

We need to recall two theorems from [2].

Theorem A [2]. Let N be the prime radical of the ring R. Then the ring of quotients $Q_\gamma(R)$ has finite length in the quotient category $R\text{–Mod}/\gamma$ cogenerated by $E(R/N)$ if and only if R/N is a left Goldie ring, $N^k \cdot E(R/N) = (0)$ for some integer $k > 0$, and for any left annihilator A of a subset of $E(R/N)$, the left R-module R/A has finite uniform dimension.

Definition [2]. Let R be a ring with prime radical N, and let γ denote the torsion radical cogenerated by $E(R/N)$. The module ${}_RM$ is said to have finite reduced rank if the module of quotients $Q_\gamma(M)$ has finite length in the quotient category R–Mod$/\gamma$. In this case the length will be denoted by $\rho_R(M)$ (or simply $\rho(M)$, when the context is clear). The left reduced rank of the ring R is defined to be the reduced rank of the module ${}_RR$.

If I is an ideal of R, then the set of elements of R which are regular modulo I will be denoted by $C(I)$. The module M is said to be $C(I)$-torsion if for each $m \in M$ there exists $c \in C(I)$ such that $cm = 0$, and then a torsion radical σ can be defined for any module M by letting $\sigma(M)$ be the sum in M of all $C(I)$-torsion submodules. It is important to note that M is $C(I)$-torsionfree if and only if for each element $0 \neq m \in M$ and each $c \in C(I)$ there exists $r \in R$ such that $crm \neq 0$. (It may also happen that $cm = 0$ for some $0 \neq m \in M$ and $c \in C(I)$.)

If $C(I)$ is a left Ore set, that is, if for each $c \in C(I)$ and each $r \in R$ there exist $c' \in C(I)$ and $r' \in R$ such that $r'c = c'r$, then $\sigma(M) = \{m \in M \mid cm = 0 \text{ for some } c \in C(I)\}$. Furthermore, in this case a module ${}_RM$ is $C(I)$-torsionfree if and only if $cm = 0$ implies $m = 0$, for all $m \in M$ and $c \in C(I)$. Note that the Ore condition holds if and only if R/Rc is $C(I)$-torsion for all $c \in C(I)$. The set $C(I)$ is said to be a left denominator set if it is a left Ore set and in addition, if $rc = 0$ for any $r \in R$ and $c \in C(I)$, then $c'r = 0$ for some $c' \in C(I)$. If $C(I)$ is a left denominator set, then the ring of quotients $Q_\sigma(R)$ is constructed by inverting the elements of $C(I)$, and $Q_\sigma(M) \cong Q_\sigma(R) \otimes_R M$.

If R is a ring with finite reduced rank, then by Theorem A the ring R/N is a left Goldie ring. It can be shown that the torsion radicals defined by $E(R/N)$ and $C(N)$ must be equal (see Proposition 1 of [4]). This equality will be exploited in the remainder of the paper. The next theorem shows that the above definition of reduced rank coincides with Goldie's original definition of reduced rank for left Noetherian rings.

Theorem B [2]. Let R be a ring with finite reduced rank on the left, such that $N^k \cdot E(R/N) = (0)$ for the positive integer k. Then the module ${}_RM$ has finite reduced rank if and only if $N^{i-1}M/N^iM$ has finite reduced rank as an R/N-module for $1 \leq i \leq k$. In this case $\rho_R(M) = \sum_{i=0}^{k-1} \rho_{R/N}(N^iM/N^{i+1}M)$.

LEMMA 1 *Let N be the prime radical of the ring R, and let $I \subseteq N$ be an ideal of R. Let $R/I = \overline{R}$, $N/I = \overline{N}$, let γ be the torsion radical of R–*Mod *defined by $C(N)$, and let $\overline{\gamma}$ be the torsion radical of $\overline{R}$–Mod defined by $C(\overline{N})$. If ${}_RM$ is a left $\overline{R}$-module, then the set of $\overline{\gamma}$-closed submodules of M coincides with the set of γ-closed submodules of M.*

Proof. A γ-closed submodule of M is an intersection of kernels of elements of $\mathrm{Hom}_R(M, E(R/N))$. The $\overline{R}$-injective envelope of $\overline{R}/\overline{N}$ can be identified with $\{x \in E_R(R/N) \mid Ix = (0)\}$. It follows from the fact that $I \cdot M = (0)$ that $\mathrm{Hom}_R(M, E(R/N)) = \mathrm{Hom}_{\overline{R}}(M, E(\overline{R}/\overline{N}))$. Thus a submodule of M is $\overline{\gamma}$-closed if and only if it is γ-closed. $\square$

THEOREM 1 *Let N be the prime radical of the ring R. The following conditions are equivalent:*

(1) *R has finite reduced rank on the left;*

(2) *R/N is a left Goldie ring, N^k is $C(N)$-torsion for some integer $k > 0$, and N^i/N^{i+1} has finite reduced rank as an R/N-module, for $i = 1, \ldots, k-1$;*

(3) *there exists an ideal $I \subseteq N$ such that R/I has finite reduced rank on the left, I^t is $C(N)$-torsion for some integer $t > 0$, and I^j/I^{j+1} has finite reduced rank as an R/I-module, for $j = 1, \ldots, t-1$.*

Proof. (1) $\Longrightarrow$ (2): If R has finite reduced rank on the left, then it follows from Theorem A that R/N is left Goldie, and that N^k is $C(N)$-torsion for some integer $k > 0$, since $N^k \cdot E(R/N) = (0)$ implies that N^k is $C(N)$-torsion. The third condition follows immediately from Theorem B.

(2) $\Longrightarrow$ (3): This follows immediately by letting $I = N$.

(3) $\Longrightarrow$ (1): Let ρ denote the reduced rank of a left R-module, and let $\overline{\rho}$ denote the reduced rank of a left R/I-module. Since I^k is $C(N)$-torsion, it follows that $\rho(R) = \rho(R/I^k)$ whenever either term is finite. Following the proof of Theorem B found in [2], we will show that $Q_\gamma(R)$ has finite length in the quotient category R–Mod$/\gamma$. Since length is additive on short exact sequences in the quotient category, we have $\rho(R) = \rho(R/I) + \sum_{j=1}^{t-1} \rho(I^j/I^{j+1})$ whenever these

terms are finite. It follows from Lemma 1 that $\rho(R) = \overline{\rho}(R/I) + \sum_{j=1}^{t-1} \overline{\rho}(I^j/I^{j+1})$, and therefore $\rho(R)$ is finite. $\square$

COROLLARY 1 *If S and T are rings with finite reduced rank on the left, then the product $S \times T$ has finite reduced rank on the left.*

Proof. Let $N(S)$ and $N(T)$ be the prime radicals of S and T, respectively. Since the direct product of two semiprime left Goldie rings is left Goldie and the prime radical of $S \times T$ is $N(S) \times N(T)$, the first part of condition (2) of Theorem 1 is satisfied. Suppose that $N(S)^{k_1}$ and $N(T)^{k_2}$ are torsion relative to $C(N(S))$ and $C(N(T))$, respectively. Since an element of $S \times T$ is regular modulo the prime radical if and only each component is regular modulo the appropriate prime radical of S or T, it follows that $N(S \times T)^k$ is torsion relative to $C(N(S \times T))$, for $k = \max\{k_1, k_2\}$. To check the third part of condition (2) of Theorem 1 we need to check that the factors of the powers of the prime radical of $S \times T$ have finite reduced rank. The classical ring of left quotients of $(S \times T)/N(S \times T)$ is isomorphic to $Q = Q_{cl}(S/N(S)) \times Q_{cl}(T/N(T))$, and so we can calculate the reduced rank of a left $(S \times T)/N(S \times T)$ module by tensoring with Q. The desired conclusion follows from the fact that tensoring respects ring direct products. $\square$

THEOREM 2 *Let S and T be rings, let ${}_TM_S$ be a bimodule, and let $R = \begin{bmatrix} S & 0 \\ M & T \end{bmatrix}$ be the ring of formal triangular matrices. Then R has finite reduced rank on the left if and only if S and T each have finite reduced rank on the left, and ${}_TM$ has finite reduced rank.*

Proof. Let $N(S)$ and $N(T)$ be the prime radicals of S and T, respectively. Then $N = \begin{bmatrix} N(S) & 0 \\ M & N(T) \end{bmatrix}$ is the prime radical of R. Let σ and τ be the torsion radicals of S–Mod and T–Mod defined by $C(N(S))$ and $C(N(T))$, respectively, and let γ be the torsion radical of R–Mod defined by $C(N)$.

We first assume that R has finite reduced rank on the left. To show that the set of σ-closed left ideals of S has finite length, by Theorem 1.4 of [11] it suffices to show that S satisfies the descending chain condition on σ-closed left ideals. Let $A_1 \supseteq A_2 \supseteq \cdots$ be a descending chain of σ-closed left ideals of S, and consider the corresponding descending chain $I_1 \supseteq I_2 \supseteq \cdots$ of left ideals of R, where $I_k = \begin{bmatrix} A_k & 0 \\ M & T \end{bmatrix}$. Given a nonzero element $\begin{bmatrix} x & 0 \\ y & z \end{bmatrix}$ in $R \backslash I_k$ and an element $\begin{bmatrix} c & 0 \\ n & d \end{bmatrix}$ in $C(N)$, we have $x \in S \backslash A_k$ and $c \in C(N(S))$, so there exists $s \in S$ with $csx \notin A_k$, since A_k is σ-closed. Then

$$\begin{bmatrix} c & 0 \\ n & d \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & 0 \\ y & z \end{bmatrix} = \begin{bmatrix} csx & 0 \\ nsx & 0 \end{bmatrix} \notin I_k\,,$$

and therefore I_k is a γ-closed left ideal of R. By assumption the descending chain $I_1 \supseteq I_2 \supseteq \cdots$ must terminate, and therefore the descending chain $A_1 \supseteq A_2 \supseteq \cdots$ must also terminate. This shows that S has finite reduced rank on the left.

Similarly, a descending chain $B_1 \supseteq B_2 \supseteq \cdots$ of τ-closed left ideals of T gives rise to a descending chain $J_1 \supseteq J_2 \supseteq \cdots$ of γ-closed left ideals of R, where $J_k = \begin{bmatrix} S & 0 \\ M & B_k \end{bmatrix}$. As above, the assumption that R has finite reduced rank forces the chain $B_1 \supseteq B_2 \supseteq \cdots$ to terminate, and hence T has finite reduced rank on the left.

Finally, the τ-closed T-submodules of M correspond exactly to the γ-closed left subideals of the left ideal $I = \begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix}$ of R. Since R has finite reduced rank, this set of τ-closed T-submodules must have finite length, showing that ${}_TM$ has finite reduced rank. We conclude that if R has finite reduced rank on the left, then so do S and T, and, furthermore, M has finite reduced rank as a left T-module.

To prove the converse, suppose that S and T have finite reduced rank on the left, and that ${}_TM$ has finite reduced rank. We will apply condition (3) of Theorem 1 to the ideal $I = \begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix}$. We have $I \subseteq N$, and $I^2 = (0)$, so I^2 is certainly torsion relative to $C(N)$. Since $R/I \cong S \times T$, it has finite reduced rank on the left by Corollary 1. The submodule structure of I as a left R/I-module is precisely that of ${}_TM$, and so R/I and I have finite reduced rank over R/I. □

If the ring R has Krull dimension on the left, then so does R/I, for each ideal I of R. Thus R/I has finite reduced rank for each ideal I. The next result examines this question for rings with finite reduced rank.

COROLLARY 2 *If each factor ring of the ring S has finite reduced rank on the left, then the same property holds for the ring $R = \begin{bmatrix} S & 0 \\ S & S \end{bmatrix}$ of lower triangular matrices over S.*

.

Proof. By Proposition 4.1 (c) of [8], if $\widehat{I}$ is an ideal of R, then it has the form $\widehat{I} = \begin{bmatrix} I & 0 \\ K & J \end{bmatrix}$, where I, J, and K are ideals of S with $I + J \subseteq K$. It can be shown that $R/\widehat{I} \cong \begin{bmatrix} S/I & 0 \\ S/K & S/J \end{bmatrix}$. By assumption, S/I and S/J have finite reduced rank on the left. Furthermore, by assumption S/K has finite reduced rank on the left, and since $J \subseteq K$, we may view S/K as a left S/J-module. It follows from Lemma 1 that ${}_{S/J}S/K$ has finite reduced rank. Since the conditions of Theorem 2 are satisfied, we conclude that $R/\widehat{I}$ has finite reduced rank on the left, as required. $\square$

Example 1 Each factor ring of the triangular matrix ring $R = \begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Z} \end{bmatrix}$ has finite reduced rank on the left, but R does not have Krull dimension on the left. We note that R is a standard example of a ring that is right Noetherian but not left Noetherian. It fails to have Krull dimension on the left because $\begin{bmatrix} 0 & 0 \\ \mathbb{Q} & 0 \end{bmatrix} / \begin{bmatrix} 0 & 0 \\ \mathbb{Z} & 0 \end{bmatrix}$ does not have finite uniform dimension. Since $\mathbb{Q}$ is the ring of quotients of $\mathbb{Z}$ and $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ has dimension one as a vector space, it follows that ${}_{\mathbb{Z}}\mathbb{Q}$ has finite reduced rank. Theorem 2 implies that R has finite reduced rank.

It can be shown that any nonzero ideal I of R must contain the ideal $\begin{bmatrix} 0 & 0 \\ \mathbb{Q} & 0 \end{bmatrix}$. The factor ring R/I is therefore a factor ring of $\mathbb{Q} \times \mathbb{Z}$, so it is Noetherian and hence has finite reduced rank (on the left).

As an application of the above results, we consider the question of when a formal triangular matrix ring is a left order in a left Artinian ring. Recall that a ring R is said to satisfy the regularity condition if $C(N) \subseteq C(0)$.

THEOREM 3 *Let S and T be rings, with prime radicals $N(S)$ and $N(T)$, respectively, and let ${}_TM_S$ be a bimodule. Let $R = \begin{bmatrix} S & 0 \\ M & T \end{bmatrix}$ be the ring of formal triangular matrices. Then R is a left order in a left Artinian ring if and* *only if S and T are left orders in left Artinian rings, and M is a left T module with finite reduced rank such that ${}_TM$ is $C(N(T))$-torsionfree and for all $m \in M$ and $c \in C(N(S))$, $mc = 0$ implies $m = 0$.*

Proof. We will use Theorem 4 of [2], which shows that a ring is a left order in a left Artinian ring if and only if it satisfies the regularity condition and has finite reduced rank on the left.

First suppose that R is a left order in a left Artinian ring. Then R has finite reduced rank on the left, and so Theorem 2 implies that the same is true for S and T, and that ${}_TM$ has finite reduced rank. Let $c \in C(N(S))$, and suppose that $cx = 0$ for $x \in S$. Then

$$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

so the regularity condition in R implies that $x = 0$, since $\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$ belongs to $C(N)$. A parallel argument shows that if $xc = 0$ then $x = 0$, and so S satisfies the regularity condition. Similarly, it can be shown that T must satisfy the regularity condition.

Now suppose that $mc = 0$ for some $m \in M$. Then

$$\begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and so the regularity condition in R implies that $m = 0$, and thus the required condition holds for M_S. A similar argument shows that if $d \in C(N(T))$ and $dm = 0$ for some $m \in M$, then $m = 0$, which implies that M is a $C(N(T))$-torsionfree module over T (on the left).

Conversely, suppose that the given conditions hold for S, T, and M. Then ${}_SS$, ${}_TT$, and ${}_TM$ have finite reduced rank, and so Theorem 2 implies that R has finite reduced rank on the left. By Theorem 4 of [2], it only remains to show that R satisfies the regularity condition. Suppose that $\begin{bmatrix} c & 0 \\ n & d \end{bmatrix} \in C(N)$, and $\begin{bmatrix} x & 0 \\ y & z \end{bmatrix} \in R$. If

$$\begin{bmatrix} c & 0 \\ n & d \end{bmatrix} \begin{bmatrix} x & 0 \\ y & z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

then $cx = 0$ implies that $x = 0$, since $c \in C(N(S))$ and S satisfies the regularity condition. Similarly, $dz = 0$ implies $z = 0$, since $d \in C(N(T))$ and T satisfies the regularity condition. It follows that $dy = 0$. Since T is a left order in a left Artinian ring, the left Ore condition holds for $C(N(T))$, and so the assumption that M is $C(N(T))$-torsionfree implies that $y = 0$. If

$$\begin{bmatrix} x & 0 \\ y & z \end{bmatrix} \begin{bmatrix} c & 0 \\ n & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

a parallel argument shows that $x = z = 0$ since the regularity conditions holds in S and T. Finally, by hypothesis $yc = 0$ implies $y = 0$, since $c \in C(N(S))$. We conclude that the regularity condition hold for R, and this completes the proof. $\square$

COROLLARY 3 *Let* $R = \begin{bmatrix} S & 0 \\ M & T \end{bmatrix}$ *be the ring of formal triangular matrices defined in Theorem 3. Then* R *is an order in an Artinian ring if and only if* S *and* T *are orders in Artinian rings,* ${}_TM$ *and* M_S *have finite reduced rank,* ${}_TM$ *is* $C(N(T))$*-torsionfree, and* M_S *is* $C(N(S))$*-torsionfree.*

Example 2 Let k be a field. The ring $R = \begin{bmatrix} k & 0 \\ k[x] & k[x] \end{bmatrix}$ is a left order in a left Artinian ring, but not a right order in a right Artinian ring. This is a well-known example, and it is easy to check that the regularity condition holds. Theorem 3 explains the known result: the reduced rank of $k[x]$ on the left is 1, as a module over $k[x]$, since $k[x]$ is an integral domain, but on the right, as a module over k, the reduced rank of $k[x]$ is just its dimension as a vector space over k, which is not finite.

Example 3 The ring $R = \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}_p & \mathbb{Z}_p \end{bmatrix}$, where p is a prime number, fails to be an order in an Artinian ring on either side. This ring is also a standard example, and again Theorem 3 explains the behavior of the ring. We note that $\mathbb{Z}_p$ has finite reduced rank on the left, as a module over $\mathbb{Z}_p$, and on the right, as a module over $\mathbb{Z}$, but the regularity condition fails since $\mathbb{Z}_p$ is not a torsionfree on the right, as a $\mathbb{Z}$-module.

References

[1] J. A. Beachy, Injective modules with both ascending and descending chain conditions on annihilators, Comm. Algebra **6** (1978), 1777–1788.

[2] ________, Rings with finite reduced rank, Comm. Algebra,1G **10** (1982), 1517–1536.

[3] ________, Maximal torsion radicals over rings with finite reduced rank, In *Proc. of the Twelfth Hudson Symp.*, *Lecture Notes in Math*, **951**, pages 41–46, Springer–Verlag, (1982).

[4] J. A. Beachy and W. D. Blair, Localization at semiprime ideals, J. Algebra **38** (1976), 309–314.

[5] A. W. Chatters, A. W. Goldie, C. R. Hajarnavis and T. H. Lenagan, Reduced rank in Noetherian rings, J. Algebra **61** (1979), 582-589.

[6] A. W. Chatters and C. R. Hajarnavis, Rings with chain conditions, Pitman, Boston/London/Melbourne, 1980.

[7] A. W. Goldie, Torsion-free modules and rings, J. Algebra **1** (1964), 268–287.

[8] K. R. Goodearl, *Ring Theory. Nonsingular Rings and Modules*, (Marcel Dekker, 1976).

[9] R. Gordon and J. C. Robson, Krull dimension, Memoirs Amer. Math. Soc. **133** (1973).

[10] T. H. Lenagan, Reduced rank in rings with Krull dimension, Ring Theory, Proceedings of the 1978 Antwerp Conference, Marcel Dekker, Inc., New York/Basel, 1979, pp. 123–131.

[11] R. W. Miller and M. Teply, The descending chain condition relative to a torsion theory, Pacific J. Math. **83** (1979), 207–219.

[12] A. C. Pearson (Bailey), Noncommutative piecewise Noetherian rings, Ph. D. thesis, Northern Illinois University, 2011.

[13] B. Stenström, Rings of Quotients, Band 217, Springer-Verlag, Berlin/New York, 1975.

[14] R. B. Warfield, Bezout rings and serial rings, Comm. Algebra, **7** (1979), 533–545.

2000 Mathematics Subject Classification: 16P60, 16S50, 16S90

Keywords: Reduced rank, triangular matrix ring.

Abigail C. Bailey
Department of Mathematical Sciences
Harper College
Palatine, IL 60067, USA

John A. Beachy
Department of Mathematical Sciences
Northern Illinois University
DeKalb, IL 60115, USA