

# Universal localization at semiprime ideals

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If  $R$  is a commutative Noetherian ring and  $P \subseteq R$  is a prime ideal, then a ring  $R_P$  and ring homomorphism  $\lambda : R \rightarrow R_P$  can be constructed for which every element of the complement  $C(P)$  of  $P$  is inverted by  $\lambda$ .

The construction uses ordered pairs  $(c, a)$  (*think*  $c^{-1}a$ ) where  $c \in C(P)$  and  $a \in R$ , subject to the equivalence relation  $(c, a) \sim (d, b)$  if there exists  $c' \in C(P)$  with  $c'(ad - bc) = 0$ .

The ideal  $PR_P$  generated by  $P$  is the unique maximal ideal of  $R_P$ .

David Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, p. 57:

“A *local ring* is a ring with just one maximal ideal. Ever since Krull’s paper (1938) local rings have occupied a central position in commutative algebra. The technique of *localization* reduces many problems in commutative algebra to problems about local rings. This often turns out to be extremely useful. Most of the problems with which commutative algebra has been successful are those that can be reduced to the local case.”

# Motivation

Properties of  $R_P$ :

**(1).**  $\lambda : R \rightarrow R_P$  is universal with respect to the property that if  $c \in C(P)$  then  $\lambda(c)$  has an inverse in  $R_P$ .

That is, if  $\phi : R \rightarrow T$  inverts  $C(P)$ , then there exists a unique ring homomorphism  $\phi'$  such that the following diagram commutes.

$$\begin{array}{ccc} R & \xrightarrow{\lambda} & R_P \\ & \searrow \phi & \vdots \phi' \\ & & T \end{array}$$

Properties of  $R_P$ :

**(2).** The ideal  $PR_P$  is the unique maximal ideal of  $R$ , and  $R_P/PR_P$  is isomorphic to  $Q(R/P)$ , the quotient field of  $R/P$ .

**(3).** The functor  $R_P \otimes_R \_ : R\text{-Mod} \rightarrow R_P\text{-Mod}$  takes short exact sequences to short exact sequences.

**(4).** For any  $R$ -module  $M$ , the kernel of the mapping  $M \rightarrow R_P \otimes_R M$  is the  $C(P)$ -torsion submodule  $\{m \in M \mid cm = 0 \text{ for some } c \in C(P)\}$ .

# The noncommutative case

In a noncommutative ring  $R$  an ideal  $P$  is prime if  $AB \subseteq P$  implies  $A \subseteq P$  or  $B \subseteq P$ , for all ideals  $A, B$  of  $R$ .

Example 1. Let  $R = \begin{bmatrix} Z & Z \\ Z & Z \end{bmatrix}$  and  $P = \begin{bmatrix} 2Z & 2Z \\ 2Z & 2Z \end{bmatrix}$ . Then  $P$  is prime since the ideals of  $R$  are in one-to-one correspondence with the ideals of the  $Z$ .

Note that  $R/P \cong \begin{bmatrix} Z/2Z & Z/2Z \\ Z/2Z & Z/2Z \end{bmatrix}$ . This factor ring has divisors of zero, but at least it is a full matrix ring over a field.

The logical candidate for a localization of  $R$  at  $P$  is

$\begin{bmatrix} Z_{(2)} & Z_{(2)} \\ Z_{(2)} & Z_{(2)} \end{bmatrix}$ , which can be constructed by inverting all scalar matrices with an odd entry.

# The analog of a field of quotients

The ring  $R$  is called a *left order* in the extension ring  $Q$  if

- (i) each  $a \in R$  that is not a divisor of zero has an inverse in  $Q$ , and
- (ii) each  $q \in Q$  can be written in the form  $c^{-1}a$  for  $a, c \in R$ .

To put the product  $ac^{-1}$  into standard form we need to be able to find  $a_1$  and  $c_1$  with  $ac^{-1} = c_1^{-1}a_1$ , or  $c_1a = a_1c$ . This is the *left Ore condition*. Then  $c^{-1}a \cdot d^{-1}b$  can be put into standard form by finding  $a_1$  and  $d_1$  with  $d_1a = a_1d$ , so that  $(c^{-1}a)(d^{-1}b) = c^{-1}(ad^{-1})b = c^{-1}(d_1^{-1}a_1)b = (c^{-1}d_1^{-1})(a_1b) = (d_1c)^{-1}(a_1b)$ .

# Goldie's theorem

Goldie's theorem (1958) shows that  $R$  is a left order in a full ring of  $n \times n$  matrices over a skew field if and only if  $R$  is a prime ring with a.c.c. on left annihilators and finite uniform dimension.

(These finiteness conditions always hold when  $R$  is left Noetherian.) This ring of quotients is called the *classical ring of left quotients of  $R$*  and is denoted by  $Q_{cl}(R)$ .

We are now ready to look at noncommutative localization. We focus on prime ideals of  $R$  for which  $Q_{cl}(R/P)$  exists. We would like to invert  $C(P)$ , which we must now define as the set of elements that are regular modulo  $P$  (not as the complement of  $P$ ). Equivalently, these are the elements inverted by the canonical homomorphism  $R \rightarrow R/P \rightarrow Q_{cl}(R/P)$ .

In Example 1, where  $P$  is the set of  $2 \times 2$  matrices with even entries,  $C(P)$  is the set of matrices whose determinant is odd.



# Ore localization

If  $P$  is a prime Goldie ideal for which  $C(P)$  satisfies the left Ore condition and is left reversible (if  $ac = 0$  for  $c \in C(P)$ , then  $c'a = 0$  for some  $c' \in C(P)$ ) then the construction of a localization  $R_P$  goes through much as in the commutative case, and all four of the properties listed above still hold.

Even in very nice cases the left Ore condition may not hold.

Example 2.

$$R = \begin{bmatrix} Z & 0 \\ Z & Z \end{bmatrix}, P_1 = \begin{bmatrix} 2Z & 0 \\ Z & Z \end{bmatrix}, P_2 = \begin{bmatrix} Z & 0 \\ Z & 2Z \end{bmatrix}.$$

Then  $R/P_i \cong Z/2Z$  and so  $Q_{cl}(R/P_i) = R/P_i$  is a field, making  $P_i$  as nice a prime Goldie ideal as possible.

# Checking the Ore condition

The ideal  $P_1$  satisfies the left Ore condition:

given  $a = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \in R$  and  $c = \begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix} \in C(P_1)$  we

need to solve  $c'a = a'c$  with  $c' \in C(P)$ .

$$\begin{bmatrix} c_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix}.$$

The ideal  $P_2$  does *not* satisfy the left Ore condition:

given  $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in R$  and  $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in C(P_2)$  the equation

$$\begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

has no solution with  $c_{22}$  odd.

# Category theoretic localization

In the absence of the left Ore condition we could focus on the last two conditions that hold in the commutative case.

**(3).** The functor  $R_P \otimes_R \_ : R\text{-Mod} \rightarrow R_P\text{-Mod}$  takes short exact sequences to short exact sequences.

**(4).** For any  $R$ -module  $M$ , the kernel of the mapping  $M \rightarrow R_P \otimes_R M$  is the  $C(P)$ -torsion submodule  $\{m \in M \mid cm = 0 \text{ for some } c \in C(P)\}$ .

The  $C(P)$ -torsion submodule is changed to  $\{m \in M \mid \forall r \in R, crm = 0 \text{ for some } c \in C(P)\}$ . This determines a closure operator, a module of quotients, and an exact functor into an abelian category consisting of these quotient modules.

# Category theoretic localization

This construction can be done more generally relative to any torsion theory, and there is a large body of work studying it. The best answer to its origin probably lies in Gabriel's thesis *Des catégories abéliennes*, Bull. Soc. Math. France **90** 323–448, published in 1962.

In the case of a prime Goldie ideal, this quotient functor does produce a ring  $R_{C(P)}$ , but properties (1) and (2) from the commutative case may be lost.

For the prime ideal  $P_2$  in Example 2 the torsion-theoretic construction produces  $R_{C(P_2)} = \begin{bmatrix} Z_{(2)} & Z_{(2)} \\ Z_{(2)} & Z_{(2)} \end{bmatrix}$ , not a bad answer, but its unique maximal ideal is not closely connected to  $P_2$ .

# Universal localization

We now turn to properties (1) and (2) of the commutative case:

**(1).**  $\lambda : R \rightarrow R_P$  is universal with respect to the property that if  $c \in C(P)$  then  $\lambda(c)$  has an inverse in  $R_P$ .

**(2).** The ideal  $PR_P$  is the unique maximal ideal of  $R$ , and  $R_P/PR_P$  is isomorphic to  $Q(R/P)$ .

A ring satisfying (1) can be defined, but it may be the zero ring.

A new approach inverting matrices rather than elements was introduced by Cohn in 1973, in *Inversive localization in Noetherian rings*, Commun. Pure Appl. Math. **26**, 679-691.

It's convenient to generalize to a semiprime ideal  $S$  for which  $Q_{cl}(R/S)$  exists and is semisimple Artinian, i.e. for which  $R/S$  is a semiprime left Goldie ring. In this case we say that  $S$  is a semiprime Goldie ideal.

# Definition of the universal localization

Let  $S$  be a semiprime Goldie ideal, and let  $\Gamma(S)$  be the set of all matrices inverted by the canonical mapping  $R \rightarrow R/S \rightarrow Q_{cl}(R/S)$ .

**Definition.** *The universal localization of  $R$  at a semiprime Goldie ideal  $S$  is the ring universal with respect to inverting all matrices in  $\Gamma(S)$ .*

That is, if  $\phi : R \rightarrow T$  inverts all matrices in  $\Gamma(S)$ , then there exists a unique ring homomorphism  $\phi'$  such that the following diagram commutes.

$$\begin{array}{ccc} R & \xrightarrow{\lambda} & R_{\Gamma(S)} \\ & \searrow \phi & \vdots \phi' \\ & & T \end{array}$$

Remark: If  $S$  is left localizable, then  $R_{\Gamma(S)}$  coincides with  $R_S$ .

**Theorem.** *Let  $S$  be a semiprime Goldie ideal of  $R$ .*

(a) *The universal localization of  $R$  at  $S$  exists.*

(b) *The canonical mapping  $\lambda : R \rightarrow R_{\Gamma(S)}$  is an epimorphism in the category of rings.*

(c) *The ring  $R_{\Gamma(S)}$  is flat as a right module over  $R$  if and only if  $S$  is a left localizable ideal.*

Cohn's construction of  $R_{\Gamma(S)}$ :

For each  $n$  and each  $n \times n$  matrix  $[c_{ij}]$  in  $\Gamma(S)$ ,

take a set of  $n^2$  symbols  $[d_{ij}]$ ,

and take a ring presentation of  $R_{\Gamma(S)}$  consisting of all of the elements of  $R$ , as well as all of the elements  $d_{ij}$  as generators;

as defining relations take all of the relations holding in  $R$ ,

together with all of the relations  $[c_{ij}][d_{ij}] = I$  and  $[d_{ij}][c_{ij}] = I$  which define all of the inverses of the matrices in  $\Gamma(S)$ .

**Proposition.** Each element of  $R_{\Gamma(S)}$  is an entry in a matrix of the form  $(\lambda(C))^{-1}$ , for some  $C \in \Gamma(S)$ .



# A characterization of $R_{\Gamma(P)}$

**Theorem.** *Let  $S$  be a semiprime Goldie ideal of  $R$ .*

- (a)  $R_{\Gamma(S)}$  modulo its Jacobson radical is naturally isomorphic to  $Q_{cl}(R/S)$ .
- (b)  $R_{\Gamma(S)}$  is universal with respect to the property in (a).

**Theorem.** *Let  $R$  be left Noetherian, let  $N$  be the prime radical of  $R$ , and let  $K = \ker(\lambda)$ , for  $\lambda : R \rightarrow R_{\Gamma(N)}$ .*

- (a) *The kernel  $K$  is the intersection of all ideals  $I \subseteq N$  such that  $C(N) \subseteq C(I)$ .*
- (b) *The ring  $R/K$  is a left order in a left Artinian ring, and  $R_{\Gamma(N)}$  is naturally isomorphic to  $Q_{cl}(R/K)$ .*

**Definition.** Let  $S$  be a semiprime Goldie ideal of  $R$ , with  $\lambda : R \rightarrow R_{\Gamma(S)}$ . The  $n^{\text{th}}$  symbolic power of  $S$ , denoted by  $S^{(n)}$ , is defined as  $S^{(n)} = \lambda^{-1}(R_{\Gamma(S)}\lambda(S^n)R_{\Gamma(S)})$ .

**Theorem.** If  $R$  is left Noetherian, then the following conditions hold for the symbolic powers of the semiprime ideal  $S$ .

- (a)  $S^{(n)}$  is the intersection of all ideals  $I$  such that  $S^n \subseteq I \subseteq S$  and  $C(S) \subseteq C(I)$ .
- (b)  $C(S)$  is a left Ore set modulo  $S^{(n)}$ .
- (c)  $R_{\Gamma(S)}\lambda(S^n)R_{\Gamma(S)} = (J(R_{\Gamma(S)}))^n$ , for all  $n > 0$ .
- (d)  $R/S^{(n)}$  is a left order in the Artinian ring  $R_{\Gamma(S)}/(J(R_{\Gamma(S)}))^n$ .

# Goldie's localization

In two papers in 1967 and 1968, Goldie defined a localization at a prime ideal  $P$  of a Noetherian ring  $R$  by first factoring out the intersection  $\bigcap_{n=1}^{\infty} P^{(n)}$  of the symbolic powers. He then took the inverse limit of the Artinian quotient rings  $Q_{cl}(R/P^{(n)})$ , and finally defined an appropriate subring of this inverse limit.

**Theorem.** Let  $P$  be a prime ideal of the Noetherian ring  $R$ . Then Goldie's localization of  $R$  at  $P$  is isomorphic to  $R_{\Gamma(P)} / \bigcap_{n=1}^{\infty} J^n$ , where  $J$  is the Jacobson radical of  $R_{\Gamma(P)}$ .

**Theorem.** *If  $K \subseteq S$  is a finitely generated left ideal of  $R$  with  $SK = K$ , then  $K \subseteq \ker(\lambda)$ .*

Proof. Let  $K = \sum_{i=1}^n Rx_i$ , for  $x_1, \dots, x_n \in R$ .

Since  $K = SK$ , we have  $K = \sum_{i=1}^n Sx_i$ . Let  $\mathbf{x} = (x_1, \dots, x_n)$ . We can write  $\mathbf{x}^t = C\mathbf{x}^t$ , where the  $n \times n$  matrix  $C$  has entries in  $S$ . Thus  $(I_n - C)\mathbf{x}^t = \mathbf{0}^t$ . But  $I_n - C$  is invertible modulo  $S$ , so it certainly belongs to  $\Gamma(S)$ . Therefore the entries of  $\mathbf{x}$  must belong to  $\ker(\lambda)$ , and so  $K \subseteq \ker(\lambda)$ .

**Corollary.**  $R_{\Gamma(P)}$  can be determined for a prime ideal of an hereditary Noetherian prime ring, since in HNP rings each prime ideal is either localizable or idempotent.

## Example 2 again

$$\text{Example 2: } R = \begin{bmatrix} Z & 0 \\ Z & Z \end{bmatrix}, P_2 = \begin{bmatrix} Z & 0 \\ Z & 2Z \end{bmatrix}, K = \begin{bmatrix} Z & 0 \\ Z & 0 \end{bmatrix}.$$

Then  $K^2 = K$ , so  $P_2 K = K$ , and therefore  $K \subseteq \ker \lambda$ , for  $\lambda : R \rightarrow R_{\Gamma(P_2)}$ . It follows easily that  $K = \ker \lambda$  and  $R_{\Gamma(P_2)}$  is isomorphic to  $Z_{(2)}$ . (We could have predicted this since  $C(P_2)$  satisfies the right Ore condition.)

An alternate approach:

For  $P_1 = \begin{bmatrix} 2Z & 0 \\ Z & Z \end{bmatrix}$ , we can invert the scalar matrices in

$C(P_1 \cap P_2)$  to obtain  $R_{P_1 \cap P_2} = \begin{bmatrix} Z_{(2)} & 0 \\ Z_{(2)} & Z_{(2)} \end{bmatrix}$  with maximal ideal

$\widehat{P}_2 = \begin{bmatrix} Z_{(2)} & 0 \\ Z_{(2)} & 2Z_{(2)} \end{bmatrix}$ . Factoring out  $\bigcap_{i=n}^{\infty} \widehat{P}_2^n$  yields  $R_{\Gamma(P_2)} \cong Z_{(2)}$ .

Example 3. If  $P_3 = \begin{bmatrix} Z & 0 \\ Z & 3Z \end{bmatrix}$ , then  $R_{\Gamma(P_1 \cap P_3)} = \begin{bmatrix} Z_{(2)} & 0 \\ Q & Z_{(3)} \end{bmatrix}$ .

This ring is no longer Noetherian.

It seems to be harder to produce such an example for a prime ideal. Bill Blair and I found a complicated example, in which the ring is finitely generated as a module over its center.

**Theorem.** [with Christine Leroux] *If  $R$  is finitely generated as a module over its Noetherian center, and  $P$  is a prime ideal that does not contain the intersection of powers of any other prime ideal in the clique of  $P$ , then  $R_{\Gamma(P)}$  is a homomorphic image of an Ore localization, and therefore Noetherian.*

# Some questions

Question: Are there conditions under which the kernel of  $R \rightarrow R_{\Gamma(P)}$  is the intersection of the symbolic powers of  $P$ , as in the commutative case? Fully bounded Noetherian?

Question: Is there a natural class of rings for which the universal localization works well? In particular, are there (weak) finiteness conditions that might be preserved? Piecewise Noetherian?

Question: The “top” of  $R_{\Gamma(P)}$  is well-behaved since  $R/P^{(n)}$  is a left order in  $R_{\Gamma(P)}/(J(R_{\Gamma(P)}))^n$ . Is it possible to use universal localization to give new proofs of some results for modules over Noetherian rings?

e.g.  $Q_{cl}(R/P) \otimes_R (M/PM) \cong R_{\Gamma(P)} \otimes_R M / J(R_{\Gamma(P)}) \otimes_R M$

# Another construction of the universal localization

Let  $S$  be a semiprime Goldie ideal of  $R$ . Each element in the universal localization  $R_{\Gamma(S)}$  has the form  $\lambda(a)\lambda(C)^{-1}\lambda(b)^t$  for some  $a, b \in R^n$  and some matrix  $C \in \Gamma_n(S)$ .

Instead of modeling elements of the form  $c^{-1}a$  where  $c \in C(S)$ , via ordered pairs  $(c, a)$ , we model elements of the form  $q_{ij} = e_i C^{-1} e_j^t$ , where  $C \in \Gamma(S)$  and  $e_i, e_j$  are unit vectors.

Let  $X$  be a left  $R$ -module. To construct a module of quotients, consider ordered triples

$$(a, C, x^t)$$

where  $a \in R^n$ ,  $C \in \Gamma_n(S)$ , and  $x \in X^n$ , for all positive integers  $n$ .



# The first equivalence relation

Model: If  $C, U, V$  are matrices that are already invertible, then  $aU(VCU)^{-1}Vx^t = aC^{-1}x^t$ .

## Definition:

$(aU, VCU, Vx^t) \equiv (a, C, x^t)$  if  $U, V$  are invertible matrices.  
Write  $(a : C : x^t)$  for the congruence classes.

Ultimately this congruence relation does not suffice since we cannot identify triples of different sizes.

# Addition of congruence classes

Model for addition: Suppose  $C, D$  are already invertible.

$$\begin{aligned} [a \ b] \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}^{-1} \begin{bmatrix} x^t \\ y^t \end{bmatrix} &= [a \ b] \begin{bmatrix} C^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} x^t \\ y^t \end{bmatrix} = \\ [aC^{-1} \ bD^{-1}] \begin{bmatrix} x^t \\ y^t \end{bmatrix} &= aC^{-1}x^t + bD^{-1}y^t \end{aligned}$$

**Definition:**

$$(a : C : x^t) + (b : d : y^t) = \left( [a \ b] : \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix} : \begin{bmatrix} x^t \\ y^t \end{bmatrix} \right)$$

This is a commutative, associative binary operation.

# Constructing a group

Model: Suppose  $C$  is already invertible.

We certainly must have  $aC^{-1}0 = 0$  and  $0C^{-1}x^t = 0$ .

## Definition:

Let  $K$  be the subsemigroup generated by all congruence classes of the form  $(0 : C : x^t)$  and  $(a : C : 0^t)$ . Then we define

$(a : C : x^t) \sim (b : D : y^t)$  if there exist  $z_1, z_2 \in K$  with  
 $(a : C : x^t) + z_1 = (b : D : y^t) + z_2$

**Proposition.** *The equivalence relation  $\sim$  defines a congruence, and modding out by it produces an abelian group.*

# A substitute for the Ore condition

Model:

If  $C, C_1$  are invertible matrices such that  $C_1A = A_1C$  for matrices  $A, A_1$ , then  $AC^{-1} = C_1^{-1}A_1$  and so  $aAC^{-1}x^t = aC_1^{-1}A_1x^t$ .

**Lemma:** Let  $a \in R^m$ ,  $C \in \Gamma_n(S)$ ,  $x \in X^n$ , and let  $A$  be any  $m \times n$  matrix. If there exist  $C_1 \in \Gamma_m(S)$  and an  $m \times n$  matrix  $A_1$  such that  $C_1A = A_1C$ , then

$$(aA : C : x^t) \sim (a : C_1 : A_1x^t).$$

Note that  $C$  and  $C_1$  can have different size. This lemma is the key to relatively simple proofs of the following results.

# Scalar multiplication

Model for scalar multiplication: Suppose  $C, D$  are invertible.

$$\begin{aligned} [a \ 0] \begin{bmatrix} C & -r^t b \\ 0 & D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ y^t \end{bmatrix} &= \\ [a \ 0] \begin{bmatrix} C^{-1} & C^{-1} r^t b D^{-1} \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ y^t \end{bmatrix} &= \\ [aC^{-1} & aC^{-1} r^t b D^{-1}] \begin{bmatrix} 0 \\ y^t \end{bmatrix} &= aC^{-1} r^t \cdot bD^{-1} y^t \end{aligned}$$

**Definition:**

$$(a : C : r^t) \cdot (b : D : y^t) = \left( [a \ 0] : \begin{bmatrix} C & -r^t b \\ 0 & D \end{bmatrix} : \begin{bmatrix} 0 \\ y^t \end{bmatrix} \right)$$

## Theorem.

(a) *The above addition and multiplication define a ring of quotients  $\Gamma^{-1}R$  and a module of quotients  $\Gamma^{-1}X$ .*

(b) *Each element of  $\Gamma^{-1}R$  is an entry in the inverse of a matrix in  $\Gamma(S)$ .*

**Theorem.**  $\Gamma^{-1}R \cong R_{\Gamma(S)}$  and  $\Gamma^{-1}X \cong R_{\Gamma(S)} \otimes_R X$ .

It is possible to give a rather complicated condition (Malcolmson's criterion) describing when  $(1 : 1 : x) \sim (1 : 1 : y)$  but unfortunately it doesn't seem to give any useful information about the kernel of the mapping  $X \rightarrow \Gamma^{-1}X$ .