ON MAXIMAL TORSION RADICALS, IV

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ABSTRACT

It is well-known that if R is a left Noetherian ring, then there is a bijective correspondence between minimal prime ideals of R and maximal torsion radicals of R–Mod. Using the notion of a prime M-ideal, it is shown that this correspondence can be extended to the category $\sigma[M]$ of modules subgenerated by a module M, provided that M is a Noetherian quasi-projective generator in $\sigma[M]$. Furthermore, under this hypothesis the prime M-ideals are the fully invariant submodules P of M such that M/P is semi-compressible.

It will be assumed throughout that R is an associative ring with identity, and that M is a fixed nonzero left R-module. A module X in R–Mod, the category of unital left R-modules, is said to be *M-generated* if there exists an R-epimorphism from a direct sum of copies of M onto X. The category $\sigma[M]$ of modules *subgenerated* by M is defined to be the full subcategory of R–Mod that contains all modules ${}_RX$ such that X is isomorphic to a submodule of an M-generated module.

Our goal is to study an individual module ${}_RM$ by studying the category $\sigma[M]$, in which the module M plays a role analogous to that of ${}_RR$ in R–Mod. Since R is a projective generator in R–Mod, to obtain parallel results in $\sigma[M]$ in certain cases we will need to assume a condition that holds whenever M is a quasi-projective generator in $\sigma[M]$.

The reader is referred to [10] and [12] for results on the category $\sigma[M]$. It is an abelian category, and in R–Mod it is closed under formation of homomorphic images, submodules, and direct sums. Furthermore, it has enough injectives, since a module ${}_RX$ is injective in $\sigma[M]$ if and only if it is M-injective, and the M-injective envelope $\widehat{X}$ of X is its injective envelope in $\sigma[M]$.

The results in this paper concern the analog in $\sigma[M]$ of the notion of a prime ideal of the ring R. If $\mathcal{C}$ is any collection of modules in $\sigma[M]$, we define

$$\mathrm{Ann}_M(\mathcal{C}) = \{m \in M \mid f(m) = 0 \text{ for all } f \in \mathrm{Hom}_R(M,X), \text{ for all } X \text{ in } \mathcal{C}\} ,$$

and we say that a submodule N of M is an *M-ideal* if $N = \mathrm{Ann}_M(\mathcal{C})$ for some collection $\mathcal{C}$ of modules in $\sigma[M]$. Proposition 3.3 of [8] implies that N is an M-ideal if and only if $N = \mathrm{Ann}_M(M/N)$.

The following definitions have been introduced by the author [7]. The module ${}_RX$ in $\sigma[M]$ is said to be *M-prime* if $\mathrm{Hom}_R(M,X) \neq 0$ and $\mathrm{Ann}_M(Y) = \mathrm{Ann}_M(X)$ for all submodules $Y \subseteq X$ such that $\mathrm{Hom}_R(M,Y) \neq 0$. An M-ideal $P \subseteq M$ is said to be a *prime M-ideal* if $P = \mathrm{Ann}_M(X)$ for some M-prime module ${}_RX$. Recall that ${}_RX$ is said to be a prime module if $\mathrm{Ann}_M(Y) = \mathrm{Ann}_M(X)$ for all nonzero submodules $Y \subseteq X$. Thus if we take M to be ${}_RR$, then the notions of an M-ideal, an M-prime module, and a prime M-ideal are, respectively, the standard ones of an ideal of R, a prime R-module, and a prime ideal of R.

A subfunctor ρ of the identity on $\sigma[M]$ is called a *radical* of $\sigma[M]$ if $\rho(X/\rho(X)) = (0)$, for all modules ${}_RX$ in $\sigma[M]$; it is called a *torsion radical* if, in addition, $\rho(X_0) = X_0 \cap \rho(X)$, for all submodules $X_0 \subseteq X$ in $\sigma[M]$. For a class $\mathcal{C}$ of modules in $\sigma[M]$, the radical $\mathrm{rad}_{\mathcal{C}}$ of $\sigma[M]$ *cogenerated* by $\mathcal{C}$ is defined by setting $\mathrm{rad}_{\mathcal{C}}(X) = \mathrm{Ann}_X(\mathcal{C})$, for all modules ${}_RX$ in $\sigma[M]$. For a given radical ρ of $\sigma[M]$, a module ${}_RX$ is said to be *ρ-torsionfree* if $\rho(X) = (0)$, and *ρ-torsion* if $\rho(X) = X$. Since $X/\rho(X)$ is ρ-torsionfree, for all modules ${}_RX$, it follows that $\rho = \mathrm{rad}_{\mathcal{C}}(X)$, where $\mathcal{C}$ is the class of ρ-torsionfree modules, and $\rho(X)$ is the intersection of all submodules X_0 of X such that X/X_0 is ρ-torsionfree.

If ρ and γ are radicals of $\sigma[M]$ such that $\rho(X) \subseteq \gamma(X)$, for all modules ${}_RX$, then the notation $\rho \leq \gamma$ is used. With this notation, for any radical ρ we have $\rho \leq 1_{\sigma[M]}$, where $1_{\sigma[M]}$ is the identity functor on $\sigma[M]$. We say that ρ is a *proper* radical if $\rho \neq 1_{\sigma[M]}$. If the class $\mathcal{C}$ consists of a single module ${}_RW$, then the notation rad_W will be used in place of $\mathrm{rad}_{\mathcal{C}}$. Note that a module ${}_RV$ cogenerates W if and only if $\mathrm{rad}_V \leq \mathrm{rad}_W$, and that rad_W is the largest radical for which W is torsionfree. A proper radical ρ of $\sigma[M]$ is said to be a *maximal radical* if $\rho \leq \gamma$ implies $\rho = \gamma$ or $\gamma = 1_{\sigma[M]}$, for all radicals γ of $\sigma[M]$.

Recall that the module ${}_RX$ is *semi-compressible* if for each nonzero submodule $Y \subseteq X$ there is a monomorphism $f : X \to Y^k$, for some positive integer k. It is clear that a semi-compressible left R-module is a prime module, and Corollary 2.7 of [7] shows that if X is semi-compressible, then X is an M-prime module if and only if $\mathrm{Hom}_R(M, X) \neq 0$.

Theorem 1.3 of [2] shows that ρ is a maximal radical of R–Mod if and only if there exists a prime ideal P of R such that $\rho = \mathrm{rad}_{R/P}$, establishing the existence of a bijective correspondence between maximal radicals of R–Mod and prime ideals of R. In preparation for later results in this paper, it will be shown in Theorem 1.10 that, under the hypothesis that M is quasi-projective and $\mathrm{Hom}_R(M, X) \neq 0$ for all modules ${}_RX$ in $\sigma[M]$, there is a bijective correspondence between maximal radicals of $\sigma[M]$ and prime M-ideals. If M is also Noetherian, then Theorem 1.8 characterizes the prime M-ideals as the fully invariant submodules $P \subseteq M$ such that M/P is semi-compressible.

A proper torsion radical τ of $\sigma[M]$ is said to be a *maximal torsion radical* of $\sigma[M]$ if $\tau \leq \gamma$ implies $\tau = \gamma$ or $\gamma = 1_{\sigma[M]}$, for all torsion radicals γ of $\sigma[M]$. It was shown by the Walkers in [11] that if R is a commutative Noetherian ring, then there is a bijective correspondence between minimal prime ideals of R and maximal torsion radicals of R–Mod. In a series of papers ([2], [3], [4], [6]) the author investigated maximal torsion radicals and showed that this bijective correspondence remains valid for any left Noetherian ring. It will be shown in Corollary 2.6 that this correspondence holds in $\sigma[M]$, provided that M is a Noetherian quasi-projective module that is a generator in $\sigma[M]$.

1 Prime M-ideals when M is quasi-projective

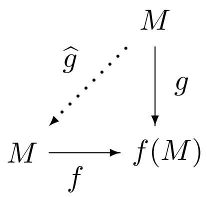
Let N be a submodule of M. As in Definition 1.5 of [7], for each module ${}_RX$ we define $N \cdot X = \mathrm{Ann}_X(\mathcal{C})$, where $\mathcal{C}$ is the class of modules ${}_RW$ such that $f(N) = (0)$ for all $f \in \mathrm{Hom}_R(M, W)$. It follows from the definition that $N \cdot X = (0)$ if and only if $f(N) = (0)$ for all $f \in \mathrm{Hom}_R(M, X)$, and thus $N \cdot X$ is the smallest submodule $Y \subseteq X$ such that $N \cdot (X/Y) = (0)$. Another way to view this definition is to note that if tr_M^N is the subfunctor of the identity defined by setting $\mathrm{tr}_M^N(X) = \sum_{f \in \mathrm{Hom}(M,X)} f(N)$ for all modules X in $\sigma[M]$, then $\rho(X) = N \cdot X$ defines the smallest radical ρ with $\mathrm{tr}_M^N \leq \rho$.

Lemma 1.1. *Let ${}_RM$ be quasi-projective.*

(a) If $f \in \mathrm{Hom}_R(M, X)$ and $N \subseteq M$ is any M-ideal such that $f(N) = (0)$, then $N \subseteq \mathrm{Ann}_M(f(M))$.

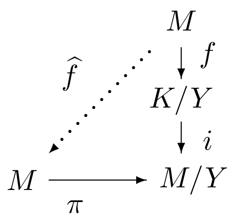
(b) If N and K are submodules of M, then $N \cdot K = \mathrm{tr}_M^N(K)$.

Proof. (a) Let $g \in \mathrm{Hom}_R(M, f(M))$. Then since M is quasi-projective we can lift g to $\widehat{g} \in \mathrm{End}_R(M)$ with $f\widehat{g} = g$, as in the following diagram.



Because N is an M-ideal, it is a fully invariant submodule of M, so $\widehat{g}(N) \subseteq N$, and therefore, by assumption, $g(N) = f(\widehat{g}(N)) \subseteq f(N) = (0)$. This shows that $N \subseteq \mathrm{Ann}_M(f(M))$.

(b) Let $Y = \mathrm{tr}_M^N(K) = \sum_{f \in \mathrm{Hom}(M,K)} f(N)$. Since $Y \subseteq N \cdot K$ by definition, we only need to show that $N \cdot K \subseteq Y$. If $f \in \mathrm{Hom}_R(M, K/Y)$, let $i : K/Y \to M/Y$ be the natural inclusion and let $\pi : M \to M/Y$ be the natural projection. Since M is quasi-projective, there exists $\widehat{f} \in \mathrm{End}_R(M)$ with $\pi\widehat{f} = if$, as in the following diagram.



Since $\pi\widehat{f}(M) = if(M) \subseteq K/Y$, we have $\widehat{f}(M) \subseteq K$. It follows from the definition of Y that $\widehat{f}(N) \subseteq Y$, so $if(N) = \pi\widehat{f}(N) = (0)$. Thus $f(N) = (0)$ for all homomorphisms $f \in \mathrm{Hom}_R(M, K/Y)$, so $N \cdot (K/Y) = (0)$, and the definition of $N \cdot K$ implies that $N \cdot K \subseteq Y$. $\square$

We can now prove Theorem 5.7 of [7] with the hypothesis that M is quasi-projective, rather than assuming that M is a projective object in $\sigma[M]$. Note that condition (3) is new.

Theorem 1.2. *If ${}_RM$ is quasi-projective, then the following conditions are equivalent for an M-ideal $P \subseteq M$:*

(1) P is a prime M-ideal;

(2) $N \cdot K \subseteq P$ implies $N \subseteq P$ or $K \subseteq P$, for all M-ideals N and all M-generated submodules $K \subseteq M$;

(3) $N \cdot f(M) \subseteq P$ implies $N \subseteq P$ or $f(M) \subseteq P$, for all M-ideals N and all $f \in \mathrm{End}_R(M)$;

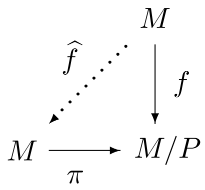
(4) M/P is an M-prime module.

Proof. (1) $\Longrightarrow$ (2): Suppose that P is a prime M-ideal, so that there exists an M-prime module ${}_RX$ with $P = \mathrm{Ann}_M(X)$. Let N and K be M-ideals such that K is M-generated, and suppose that $N \cdot K \subseteq P$. If $K \not\subseteq P$, then since K is M-generated there exists $f \in \mathrm{Hom}_R(M, K)$ with $f(M) \not\subseteq P$. Since $\mathrm{Ann}_M(X) = P$, there exists $g \in \mathrm{Hom}_R(M, X)$ with $gf(M) \neq (0)$, and then $\mathrm{Ann}_M(gf(M)) = P$ since X is M-prime. Since $N \cdot K \subseteq P$ by assumption, we have $\mathrm{tr}_M^N(K) \subseteq P$, and so $f(N) \subseteq P$, and then $g(f(N)) = (0)$ since $P = \mathrm{Ann}_M(X)$. It follows from Lemma 1.1 (a) that $N \subseteq \mathrm{Ann}_M(gf(M)) = P$.

(2) $\Longrightarrow$ (3): This follows immediately, since $f(M)$ is certainly M-generated.

(3) $\Longrightarrow$ (4): Assume that condition (3) holds. To show that M/P is M-prime, let K/P be a nonzero submodule of M/P for which $\mathrm{Hom}_R(M, K/P) \neq 0$, and suppose that $f : M \to K/P$ is a nonzero R-homomorphism. Then $P \subseteq \mathrm{Ann}_M(K/P) \subseteq \mathrm{Ann}_M(f(M))$, and so to show that M/P is M-prime it suffices to check that $\mathrm{Ann}_M(f(M)) = P$. Let $\mathrm{Ann}_M(f(M)) = N$, and suppose that $N \neq P$.

If $\pi : M \to M/P$ is the canonical projection, then since M is quasi-projective there exists $\widehat{f} \in \mathrm{End}_R(M)$ with $\pi\widehat{f} = f$, as in the following diagram.



Then $\widehat{f}(M) \not\subseteq P$ since $\pi\widehat{f} = f \neq 0$, so applying condition (2) to N and $\widehat{f}$ shows that $N \cdot \widehat{f}(M) \not\subseteq P$, because by assumption we also have $N \not\subseteq P$. Since $N \cdot \widehat{f}(M) = \mathrm{tr}_M^N(\widehat{f}(M))$ by Lemma 1.1 (b), there must exist $g \in \mathrm{End}_R(M)$ such that $g(M) \subseteq \widehat{f}(M)$ and $g(N) \not\subseteq P$. It follows that $\pi g(N) \subseteq \pi\widehat{f}(M) = f(M)$ but $\pi g(N) \neq 0$, which contradicts the fact that $N = \mathrm{Ann}_M(f(M))$. We conclude that $\mathrm{Ann}_M(f(M)) = P$, showing that M/P is an M-prime module.

(4) $\Longrightarrow$ (1): This follows immediately, since P is an M-ideal and therefore $P = \mathrm{Ann}_M(M/P)$. □

In the second section of the paper we require the existence of minimal prime M-ideals. Condition (3) of Theorem 1.2 makes it possible to use the standard proof for the existence of minimal prime ideals in the ring R.

Corollary 1.3. *If ${}_RM$ is quasi-projective, then every prime M-ideal contains a minimal prime M-ideal.*

Proof. Given a prime M-ideal $P \subseteq M$, we can apply Zorn's lemma to the set $\mathcal{P}$ of prime M-ideals contained in P, directed by reverse inclusion.

Suppose that $\{Q_\alpha\}_{\alpha \in I}$ is a chain of prime M-ideals in $\mathcal{P}$ with $\cap_{\alpha \in I} Q_\alpha = Q$. Proposition 5.1 of [7] shows that if M is quasi-projective, then a submodule of M is an M-ideal if and only if it is fully invariant in M. Since an intersection of fully invariant submodules is fully invariant, it follows that Q is an M-ideal. To show that Q is a prime M-ideal, suppose that N is a M-ideal and $f \in \mathrm{End}_R(M)$ with $N \cdot f(M) \subseteq Q$. If $f(M) \not\subseteq Q$, then there exists $\alpha \in I$ such that $f(M) \not\subseteq Q_\alpha$. Then $f(M) \not\subseteq Q_\beta$, for all $\beta \geq \alpha$, and this implies that $N \subseteq Q_\beta$, for all $\beta \geq \alpha$, since each Q_β is a prime M-ideal. Thus $N \subseteq \cap_{\alpha \in I} Q_\alpha = Q$, showing that Q is a prime M-ideal.

Since each chain of ideals in $\mathcal{P}$ has a lower bound in $\mathcal{P}$, it follows that $\mathcal{P}$ contains a minimal element, which is then minimal in the set of all prime M-ideals. $\square$

If R is a prime left Goldie ring, then every left ideal $A \subseteq R$ is faithful, and it follows immediately from the descending chain condition on left annihilators that R can be embedded in a finite direct sum of copies of A. Thus the left module ${}_RR$ is semi-compressible. Theorem 1.7 will extend this to certain quasi-projective M-prime modules that satisfy "Goldie-like" conditions.

The definition of an M-prime module is closely related to another definition in the literature. In [9] a nonzero module ${}_RX$ is called prime if $\text{rad}_Y = \text{rad}_X$ for all nonzero submodules Y of X. It is shown in Proposition 2.3 of [9] that a nonzero module X satisfies this definition if and only if it is cogenerated by each of its nonzero submodules. Note that, in particular, any semi-compressible module satisfies this condition. Proposition 2.6 of [7] shows that such modules are "universally" M-prime, since each nonzero submodule of X cogenerates X if and only if X is M-prime for each module ${}_RM$ such that $\text{Hom}_R(M, X) \neq 0$.

Definition 1.4. *Let $P \subset M$ be an M-ideal. Then P is said to be a* completely prime *M-ideal if each nonzero submodule of M/P cogenerates M/P.*

Proposition 1.5. *If M is quasi-projective, then the following conditions are equivalent for an M-ideal $P \subset M$:*

(1) P is a completely prime M-ideal;

(2) P is a prime M-ideal such that $\text{Hom}_R(M, K/P) \neq 0$ for all nonzero submodules $K/P \subseteq M/P$.

Proof. (1) $\Longrightarrow$ (2): If P is a completely prime M-ideal, then M/P is certainly an M-prime module, and so P is a prime M-ideal. Furthermore, if K/P is a nonzero submodule of M/P, then $\text{Ann}_M(K/P) = P$ since K/P cogenerates M/P, and this guarantees that $\text{Hom}_R(M, K/P) \neq 0$.

(2) $\Longrightarrow$ (1): By Theorem 1.2 (4), the hypothesis that M is quasi-projective implies that M/P is an M-prime module. It then follows from Proposition 2.8 of [7] that $h(M/P)$ cogenerates M/P, for each endomorphism h of M/P.

Let K/P be a nonzero submodule of M/P. By hypothesis there exists a nonzero R-homomorphism $f : M \to K/P$, and since $P = \mathrm{Ann}_M(K/P)$, we have $P \subseteq \ker(f)$, and so f induces a nonzero endomorphism $\overline{f}$ of M/P. Since $\overline{f}$ is nonzero, its image cogenerates M/P, and therefore K/P cogenerates M/P. □

To prove Theorem 1.7 we need the following lemma. In [10] a submodule of M is called an *M-annihilator* if it is the intersection of kernels of a set of endomorphisms in $\mathrm{End}_R(M)$. If the set of such submodules satisfies the ascending chain condition, then M is said to have ACC on M-annihilators.

Lemma 1.6. *Let ${}_RM$ be a quasi-projective module that is cogenerated by each of its nonzero submodules, and assume that M has* ACC *on M-annihilators. If $f \in \mathrm{End}_R(M)$ and $f(K) \neq (0)$ for the submodule $K \subseteq M$, then $K \cap \ker(f)$ is not an essential submodule of K.*

Proof. Let $f \in \mathrm{End}_R(M)$, with $f(K) \neq (0)$ for the submodule $K \subseteq M$. Since M has ACC on M-annihilators, the nonempty set

$$\{h \in \mathrm{End}_R(M) \mid h(K) \neq (0) \text{ and } \ker(h) \supseteq \ker(f)\}$$

must contain an element whose kernel is maximal in the set. If we show that $K \cap \ker(h)$ is not essential in K, then it follows that $K \cap \ker(f) \subseteq K \cap \ker(h)$ is not essential in K. Therefore we can assume without loss of generality that $\ker(f)$ is maximal among R-homomorphisms such that $f(K) \neq (0)$.

Let $\pi_1 : M \to M/(K \cap \ker(f))$ and $\pi_2 : M/(K \cap \ker(f)) \to M/\ker(f)$ be the natural projections, and let $i : K/(K \cap \ker(f)) \to M/(K \cap \ker(f))$ be the inclusion. Let $\overline{f}$ be the factorization of f through $\pi_2\pi_1$, so that we have $\overline{f}\pi_2\pi_1 = f$. Then $\pi_2 i$ is a monomorphism, so $\overline{f}\pi_2 i$ is also a monomorphism, and thus the module $K/(K \cap \ker(f))$ is isomorphic to a nonzero submodule of M. By hypothesis, M is cogenerated by each of its nonzero submodules. It follows that there exists a homomorphism $g \in \mathrm{Hom}_R(M, K/(K \cap \ker(f)))$ with $gf(K) \neq (0)$. Since M is quasi-projective, the homomorphism $ig : M \to M/(K \cap \ker(f))$ can be lifted to $\widehat{g} \in \mathrm{End}_R(M)$ with $\pi_1\widehat{g} = ig$. This leads to the following commutative diagram.

$$\begin{array}{ccccccc}
K \subseteq M & \xrightarrow{f} & M & \xrightarrow{\widehat{g}} & M & \xrightarrow{f} & M \\
 & & g \downarrow & & \pi_1 \downarrow & & \uparrow \overline{f} \\
 & & K/(K \cap \ker(f)) & \xrightarrow{i} & M/(K \cap \ker(f)) & \xrightarrow{\pi_2} & M/\ker(f)
\end{array}$$

Because $\overline{f}\pi_2 i$ is a monomorphism and $gf(K) \neq (0)$, we have

$$f\widehat{g}f(K) = \overline{f}\pi_2\pi_1\widehat{g}f(K) = \overline{f}\pi_2 igf(K) \neq (0) \ ,$$

and then since $\ker(f) \subseteq \ker(f\widehat{g}f)$, it follows from the maximality of $\ker(f)$ that $\ker(f\widehat{g}f) = \ker(f)$. We claim that $(K \cap \ker(f)) \cap \widehat{g}f(K) = (0)$, which will show that $K \cap \ker(f)$ is not essential in K, since $\widehat{g}f(K) \neq (0)$. (We note that $\widehat{g}(M) \subseteq K$, since we have $\pi_1\widehat{g}(M) = ig(M) \subseteq K/(K \cap \ker(f))$.) If $y \in (K \cap \ker(f)) \cap \widehat{g}f(K)$, then $y = \widehat{g}f(x)$ for some $x \in K$. But then $f(y) = 0$ since $y \in \ker(f)$, and so $f\widehat{g}f(x) = f(y) = 0$ implies $f(x) = 0$ since $\ker(f\widehat{g}f) = \ker(f)$. We conclude that $y = \widehat{g}f(x) = 0$, completing the proof. □

Theorem 1.7. *Let ${}_RM$ be a quasi-projective module that is cogenerated by each of its nonzero submodules. If M has* ACC *on M-annihilators and finite uniform dimension, then M is semi-compressible.*

Proof. Let N be any nonzero submodule of M. Since N cogenerates M, there exists a nonzero homomorphism $f_1 : M \to N$. If f_1 is a monomorphism, then we are done. If not, then $K_1 \neq (0)$ for $K_1 = \ker(f_1)$, and so there exists $f_2 : M \to N$ with $f_2(K_1) \neq (0)$. If $K_2 = K_1 \cap \ker(f_2)$, then it follows from Lemma 1.6 that K_2 is not an essential submodule of K_1, since M has ACC on M-annihilators. Since M has finite uniform dimension, the preceding remarks show that the uniform dimension of K_2 is strictly less than the uniform dimension of K_1. Therefore the descending chain we are constructing must terminate in (0) after some finite number k of steps, at which point $\cap_{i=1}^{k} \ker(f_i) = (0)$. The homomorphisms $\{f_i\}_{i=1}^{k}$ can then be combined to give the necessary embedding $M \to N^k$. □

The next theorem is the main goal of this section, as it will be used in the next section to show that maximal torsion radicals of $\sigma[M]$ correspond to minimal prime M-ideals.

Theorem 1.8. *Let $_RM$ be Noetherian and quasi-projective. Then the following conditions are equivalent for a proper submodule $P \subsetneq M$:*

(1) P is a completely prime M-ideal;

(2) P is a fully invariant submodule of M such that M/P is semi-compressible.

Proof. (1) $\Longrightarrow$ (2): Since P is an M-ideal, it is a fully invariant submodule of M, and therefore M/P is also quasi-projective. Since P is a completely prime M-ideal, by definition each nonzero submodule of M/P cogenerates M/P. Furthermore, M/P is Noetherian since M is Noetherian, and so M/P has finite uniform dimension and ACC on M/P-annihilators. Thus M/P satisfies the hypotheses of Theorem 1.7, and so it is semi-compressible.

(2) $\Longrightarrow$ (1): Proposition 5.1 of [7] shows that since M is quasi-projective, a submodule $P \subseteq M$ is an M-ideal if and only if it is fully invariant in M. If M/P is a semi-compressible module, then each nonzero submodule of M/P certainly cogenerates M/P, and so Proposition 1.5 implies that P is a completely prime M-ideal. □

We next investigate the connection between prime M-ideals and maximal radicals of $\sigma[M]$. Theorem 1.10 will require the hypothesis that $\mathrm{Hom}_R(M,X) \neq 0$ for all nonzero modules $_RX$ in $\sigma[M]$. If this condition fails, say $\mathrm{Hom}_R(M,X) = 0$, then $\mathrm{rad}_X(M) = M$ even though rad_X is a proper radical of $\sigma[M]$, and so any connection between rad_X and submodules of M is lost.

Lemma 1.9. *Let $_RM$ be quasi-projective. Then the following conditions are equivalent for an M-ideal P:*

(1) P is a prime M-ideal;

(2) for any module X in $\sigma[M]$, if $\mathrm{Hom}_R(M,X) \neq 0$ and X is cogenerated by M/P, then $\mathrm{Ann}_M(X) = P$;

(3) if ρ is a radical of $\sigma[M]$ such that $\mathrm{rad}_{M/P} \leq \rho$, then either $\rho(M) = P$ or $\rho(M) = M$;

(4) P is a maximal ρ-closed M-ideal for $\rho = \mathrm{rad}_{M/P}$;

(5) there exists a radical ρ of $\sigma[M]$ such that P is a maximal ρ-closed M-ideal.

Proof. (1) $\Longrightarrow$ (2): Let X be a module in $\sigma[M]$ such that $\text{Hom}_R(M,X) \neq 0$ and M/P cogenerates X. Since P is an M-ideal and M/P cogenerates X, we have $P = \text{Ann}_M(M/P) \subseteq \text{Ann}_M(X)$. By assumption there exists a nonzero homomorphism $f : M \to X$, and then since M/P cogenerates X there exists a homomorphism $g \in \text{Hom}_R(X, M/P)$ with $gf \neq 0$. Then $gf(\text{Ann}_M(X)) = (0)$, so because M is quasi-projective, it follows from Lemma 1.1 (a) that $\text{Ann}_M(X) \subseteq \text{Ann}_M(gf(M))$. Since P is a prime M-ideal by assumption, it follows from Theorem 1.2 that M/P is an M-prime module. Thus $\text{Ann}_M(gf(M)) \neq M$ since $gf \neq 0$, and so we must have $\text{Ann}_M(gf(M)) = P$. This shows that $\text{Ann}_M(X) = P$.

(2) $\Longrightarrow$ (3): Let ρ be a radical of $\sigma[M]$ with $\text{rad}_{M/P} \leq \rho$. If $\rho(M) \neq M$, then $\text{Hom}_R(M, M/\rho(M)) \neq 0$, and by assumption $\text{rad}_{M/P}(M/\rho(M)) \subseteq \rho(M/\rho(M) = 0$. Thus M/P cogenerates $M/\rho(M)$, and therefore $\rho(M) = \text{Ann}_M(M/\rho(M)) = P$ by hypothesis.

(3) $\Longrightarrow$ (1): Let K be any nonzero submodule of M/P such that $\text{Hom}_R(M,K)$ is nonzero. Then $\rho = \text{rad}_K$ defines a radical of $\sigma[M]$ for which $\text{rad}_{M/P} \leq \rho$ and $\rho(M) \neq M$, so $\rho(M) = P$, showing that $\text{Ann}_M(K) = P$. It follows that M/P is an M-prime module, and hence P is a prime M-ideal.

(2) $\Longrightarrow$ (4): Let $\rho = \text{rad}_{M/P}$, and suppose that Q is a ρ-closed M-ideal with $P \subseteq Q \subset M$. Then M/Q is a ρ-torsionfree module with $\text{Hom}_R(M, M/Q) \neq 0$, so by hypothesis $Q = \text{Ann}_M(M/Q) = P$.

(4) $\Longrightarrow$ (5): This follows trivially by taking $\rho = \text{rad}_{M/P}$.

(5) $\Longrightarrow$ (1): To show that M/P is an M-prime module, let K/P be a nonzero submodule of M/P with $\text{Hom}_R(M, K/Q) \neq 0$. Then M/P is ρ-torsionfree since M/P is ρ-closed, and so K/P is also ρ-torsionfree, which implies that $\text{Ann}_M(K/P)$ is a ρ-closed M-ideal that contains P. By hypothesis we must have $\text{Ann}_M(K/P) = P$. Since M/P is an M-prime module, it follows that P is a prime M-ideal. □

Theorem 1.10. *Let $_RM$ be quasi-projective, such that* $\text{Hom}_R(M,X) \neq 0$ *for all nonzero modules $_RX$ in $\sigma[M]$. Then there is a bijective correspondence between maximal radicals of $\sigma[M]$ and prime M-ideals.*

Proof. Suppose that $P \subseteq M$ is a prime M-ideal and ρ is a proper radical of $\sigma[M]$ with $\mathrm{rad}_{M/P} \leq \rho$. Since M is quasi-projective it follows from Lemma 1.9 (3) that either $\rho(M) = M$ or $\rho(M) = P$. The first case is impossible since ρ is proper and $\mathrm{Hom}_R(M, X) \neq 0$ for all nonzero modules $X \in \sigma[M]$, and the second case implies that $\rho \leq \mathrm{rad}_{M/P}$. This shows that $\mathrm{rad}_{M/P}$ is a maximal radical of $\sigma[M]$.

We next show that the correspondence that assigns to a prime M-ideal P the maximal radical $\mathrm{rad}_{M/P}$ is bijective.

Let ρ be a maximal radical of $\sigma[M]$. Then ρ is not the identity on $\sigma[M]$, so there exists a nonzero ρ-torsionfree module X in $\sigma[M]$. By assumption, $\mathrm{Hom}_R(M, X) \neq 0$, so it follows that M has a proper ρ-closed submodule, and thus $\rho(M) \neq M$. Letting $\rho(M) = P$, it follows that P is an M-ideal with $\rho \leq \mathrm{rad}_{M/P}$, since M/P is ρ-torsionfree. The maximality of ρ implies that $\rho = \mathrm{rad}_{M/P}$. If K is any nonzero submodule of M/P, then rad_K is a proper radical of $\sigma[M]$ with $\mathrm{rad}_{M/P} \leq \mathrm{rad}_K$. The maximality of $\mathrm{rad}_{M/P}$ implies that $\mathrm{rad}_{M/P} = \mathrm{rad}_K$, and therefore K cogenerates M/P. It follows that M/P is a prime module, and thus $P = \mathrm{Ann}_M(M/P)$ is a prime M-ideal. □

Recall that a module ${}_RM$ is said to be *finitely annihilated* if there exist finitely many elements $m_1, \ldots, m_k \in M$ such that $\mathrm{Ann}_R(M) = \mathrm{Ann}_R(m_1, \ldots, m_k)$.

Corollary 1.11. *Let ${}_RM$ be quasi-projective, such that $\mathrm{Hom}_R(M, X) \neq 0$ for all nonzero modules ${}_RX$ in $\sigma[M]$. If M is finitely annihilated, then there is a bijective correspondence between prime M-ideals and prime ideals of $R/\mathrm{Ann}_R(M)$.*

Proof. If M is finitely annihilated, then $\sigma[M] = R/\mathrm{Ann}_R(M)$-$Mod$. Thus the maximal radicals of $\sigma[M]$ correspond to the maximal radicals of $R/\mathrm{Ann}_R(M)$-Mod, which in turn correspond to the prime ideals of $R/\mathrm{Ann}_R(M)$. □

2 Maximal torsion radicals in $\sigma[M]$

Proposition 9.3 of [12] shows that every torsion radical of $\sigma[M]$ has the form rad_W, where ${}_RW$ is an M-injective module in $\sigma[M]$. Since each nonzero M-injective module in $\sigma[M]$ is M-generated, it follows that rad_W is the identity on $\sigma[M]$ if and only if $W = (0)$. In fact, this remark also implies that a torsion radical τ of R–Mod restricts to the identity on $\sigma[M]$ if and only if $\tau(M) = M$.

If $\widehat{X}$ is used to denote the M-injective envelope of X, which is constructed as the largest M-generated submodule of the R-injective envelope $E(X)$ of X, then $\tau = \mathrm{rad}_{\widehat{X}}$ is the largest torsion radical τ for which X is τ-torsionfree.

In Theorem 1.10 we needed to assume not only that ${}_RM$ is quasi-projective, but that $\mathrm{Hom}_R(M, X) \neq 0$ for all nonzero modules ${}_RX$ in $\sigma[M]$. This condition certainly holds if M is a quasi-projective generator in $\sigma[M]$. In this section we only need a weaker condition, which we will state as the assumption that every prime M-ideal is completely prime. Recall that by Proposition 1.5 this is equivalent to the condition that $\mathrm{Hom}_R(M, K/P) \neq 0$ for all prime M-ideals P and all nonzero submodules $K/P \subseteq M/P$.

Theorem 2.1. *Let ${}_RM$ be quasi-projective, such that each prime M-ideal is completely prime. Let P be a proper M-ideal, and let $\tau = \mathrm{rad}_{\widehat{M/P}}$ be the torsion radical determined by the M-injective envelope $\widehat{M/P}$ of M/P. Then the following conditions are equivalent:*

(1) P is a prime M-ideal;

(2) P contains all proper τ-closed M-ideals.

Proof. (1) $\Longrightarrow$ (2): Let N be a proper τ-closed M-ideal of M, and let $\pi : M \to M/N$ be the canonical projection. Then M/N is cogenerated by $\widehat{M/P}$ since it is τ-torsionfree, and so there exists a nonzero homomorphism $f : M/N \to \widehat{M/P}$. Since M/P is an essential submodule of $\widehat{M/P}$, we have $f\pi(M) \cap (M/P) \neq (0)$. By hypothesis P is a completely prime M-ideal, and so it follows from Proposition 1.5 that $\mathrm{Hom}_R(M, f\pi(M) \cap (M/P)) \neq 0$. Therefore

$$\mathrm{Ann}_M(f\pi(M)) \subseteq \mathrm{Ann}_M(f\pi(M) \cap (M/P)) = P \ ,$$

since M/P is an M-prime module. Since $f\pi(N) = (0)$, it follows from Lemma 1.1 (a) that $N \subseteq \mathrm{Ann}_M(f\pi(M))$, and therefore $N \subseteq P$ (as required).

(2) $\Longrightarrow$ (1): Since P is a maximal τ-closed M-ideal, this implication follows from Lemma 1.9. □

The following characterization of prime M-ideals via torsion radicals of $\sigma[M]$ should be compared to Lemma 1.9.

Corollary 2.2. *Let $_RM$ be quasi-projective, such that each prime M-ideal is completely prime. The following conditions are equivalent for a proper M-ideal P:*

(1) P is a prime M-ideal;

(2) $\mathrm{Ann}_M(X) \subseteq P$ for all modules $_RX$ in $\sigma[M]$ cogenerated by $\widehat{M/P}$;

(3) P is a maximal τ-closed M-ideal for some torsion radical τ of $\sigma[M]$.

Proof. (1) $\iff$ (2): Let τ be the torsion radical defined by the M-injective envelope $\widehat{M/P}$ of M/P. If P is a prime M-ideal and $_RX$ is cogenerated by $\widehat{M/P}$, then $\mathrm{Ann}_M(X)$ is τ-closed, and so it follows from Theorem 2.1 that $\mathrm{Ann}_M(X) \subseteq P$.

Conversely, if $\mathrm{Ann}_M(X) \subseteq P$ for all modules $_RX$ in $\sigma[M]$ cogenerated by $\widehat{M/P}$, then P contains every τ-closed M-ideal, and so it follows from Theorem 2.1 that P is a prime M-ideal.

(1) $\iff$ (3): This follows immediately from Theorem 2.1 and Lemma 1.9. □

Definition 2.3. *A proper torsion radical μ of $\sigma[M]$ is said to be* maximal *if $\mu \leq \tau$ implies $\mu = \tau$, for all proper torsion radicals τ of $\sigma[M]$.*

In addition to studying the correspondence between maximal torsion radicals of $\sigma[M]$ and minimal prime M-ideals, it is also of interest to know when every proper torsion radical of $\sigma[M]$ is contained in a maximal torsion radical of $\sigma[M]$. In the following lemma it is convenient to combine the two conditions.

Lemma 2.4. *If every proper torsion radical of $\sigma[M]$ is contained in a maximal torsion radical, and the maximal torsion radicals of $\sigma[M]$ correspond to the minimal prime M-ideals, then every nonzero M-injective module contains a submodule whose left annihilator in M is a minimal prime M-ideal.*

The converse holds if M is a quasi-projective module such that each prime M-ideal is completely prime.

Proof. Suppose that every proper torsion radical of $\sigma[M]$ is contained in a maximal torsion radical, and the maximal torsion radicals of $\sigma[M]$ correspond to the minimal prime M ideals of M. If $_RW$ is a nonzero module that is injective in $\sigma[M]$, then rad_W is a proper torsion radical of $\sigma[M]$, so by assumption there exists a prime M-ideal P that is minimal among prime M-ideals and has the property that $\text{rad}_W \leq \mu$, for the torsion radical $\mu = \text{rad}_{\widehat{M/P}}$ determined by the M-injective envelope $\widehat{M/P}$. Then P must be rad_W-closed since it is μ-closed, and therefore P is the left annihilator of a submodule of W.

Conversely, let M be a quasi-projective module such that each prime M-ideal is completely prime. Assume that each nonzero M-injective module contains a submodule whose left annihilator in M is a minimal prime M-ideal. If τ is a proper torsion radical of $\sigma[M]$, then $\tau = \text{rad}_W$ for a nonzero M-injective module W. If P is a minimal prime M-ideal which is the annihilator in M of a submodule of W, then $\text{rad}_W(M/P) = 0$ implies that $\text{rad}_W \leq \mu$, for $\mu = \text{rad}_{\widehat{M/P}}$.

If P is any minimal prime M-ideal, let $\mu = \text{rad}_{\widehat{M/P}}$. If $\mu \leq \tau$ for a proper torsion radical τ, then, as above, there is a minimal prime M-ideal P' such that $\tau \leq \mu'$, where $\mu' = \text{rad}_{\widehat{M/P'}}$. But then P' is μ-closed, and so it follows from Theorem 2.1 that $P' \subseteq P$, since by assumption M/P is an M-prime module. Since P is minimal we must have $P' = P$, and thus $\tau = \mu$. It follows that every minimal prime M-ideal defines a maximal torsion radical of $\sigma[M]$, and then every proper torsion radical of $\sigma[M]$ is contained in a maximal torsion radical of $\sigma[M]$.

As above, if $\text{rad}_{\widehat{M/P}} = \text{rad}_{\widehat{M/P'}}$, where P and P' are minimal prime M-ideals, then $P = P'$, and this establishes the one-to-one correspondence between minimal prime M-ideals and maximal torsion radicals of $\sigma[M]$. □

The next theorem is our main result.

Theorem 2.5. *Let ${}_RM$ be quasi-projective, such that each prime M-ideal is completely prime. If M is Noetherian, then every proper torsion radical of $\sigma[M]$ is contained in a maximal torsion radical of $\sigma[M]$, and the maximal torsion radicals of $\sigma[M]$ are in one-to-one correspondence with the minimal prime M-ideals.*

Proof. Since M is quasi-projective, Theorem 1.2 shows that an M-ideal P is a prime M-ideal if and only if M/P is an M-prime module. Let ${}_RX$ be a nonzero M-injective module. By Lemma 2.4 it suffices to show that X contains a submodule whose left annihilator in M is a minimal prime M-ideal.

Proposition 3.4 of [7] states that if M is Noetherian and ${}_RX$ is a module with $\mathrm{Hom}_R(M,X) \neq 0$, then X has an associated prime M-ideal. Because $\mathrm{Hom}_R(M,X) \neq 0$ for any nonzero M-injective module, it follows that X has an associated prime M-ideal P. Since $P = \mathrm{Ann}_M(Y)$ for a nonzero submodule $Y \subseteq X$, we have $\mathrm{rad}_X(M/P) = (0)$. By Corollary 1.3 there is a minimal prime M-ideal Q that is contained in P. Let K be the preimage in M of $\mathrm{rad}_X(M/Q)$, so that $K/Q = \mathrm{rad}_X(M/Q)$. Since $Q \subseteq P$ and $\mathrm{rad}_X(M/P) = (0)$, it follows that M/Q is not rad_X-torsion, and so $K \neq M$. If $K \neq Q$, then by Theorem 1.8 there is an embedding $M/Q \to (K/Q)^k$ for some positive integer k, since M/Q is semi-compressible. This forces M/Q to be rad_X-torsion, a contradiction. We conclude that M/Q is rad_X-torsionfree, and therefore Q is the annihilator of a nonzero submodule of X. This completes the proof. □

Corollary 2.6. *Let ${}_RM$ be Noetherian quasi-projective generator of $\sigma[M]$. Then every proper torsion radical of $\sigma[M]$ is contained in a maximal torsion radical of $\sigma[M]$, and the maximal torsion radicals of $\sigma[M]$ are in one-to-one correspondence with the minimal prime M-ideals, which are the minimal fully invariant submodules $P \subseteq M$ for which M/P is semi-compressible.*

Proof. Since M is a generator of $\sigma[M]$, we have $\mathrm{Hom}_R(M,X) \neq 0$ for each nonzero module in $\sigma[M]$. Since M is quasi-projective, Proposition 1.5 then implies that every prime M-ideal is completely prime. Since M is Noetherian and quasi-projective, it follows from Theorem 1.8 that the minimal prime M-ideals P are the minimal fully invariant submodules $P \subseteq M$ for which M/P is semi-compressible. □

The next example exhibits a projective module ${}_RM$ of finite length such that $\sigma[M] = R\text{–}Mod$ has two maximal torsion radicals and yet M has only one minimal prime M-ideal. The example shows that in the statement of Theorem 2.5 the hypothesis that each prime M-ideal is completely prime is necessary.

Example 2.1.

Let F be a field, and let $R = \begin{bmatrix} F & 0 \\ F & F \end{bmatrix}$ be the ring of lower triangular 2×2 matrices over F. Let M be the left ideal $\begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix}$, and let N be the subset $\begin{bmatrix} 0 & 0 \\ F & 0 \end{bmatrix}$. The lattice of submodules of the projective module M is just $(0) \subset N \subset M$, and (0) and N are M-ideals since they are fully invariant. Since M/N is a simple R-module, it is an M-prime module, and it follows that N is a prime M-ideal.

The module M is also an M-prime module, since $\text{Hom}_R(M, N) = 0$ and therefore $\text{Ann}_M(N) = M$. It follows that (0) is also a prime M-ideal, but it is not completely prime since N does not cogenerate M. Finally, (0) is the one and only minimal prime M-ideal.

There are two isomorphism classes of simple left R-modules, represented by N and M/N, and it can be shown that the injective envelopes in $\sigma[M]$ are $\widehat{N} = M$ and $\widehat{M/N} = M/N$, since M and M/N are injective left R-modules. The torsion radicals rad_M and $\text{rad}_{M/N}$ are both maximal, and so there is no one-to-one correspondence between maximal torsion radicals and minimal prime M-ideals.

If we let $N(R)$ denote the prime radical of R, then Proposition 2.7 of [5] shows that every proper torsion radical of R–Mod is contained in a maximal torsion radical, and the maximal torsion radicals of R–Mod correspond to the minimal prime ideals of R if and only if $R/N(R)$ satisfies the same condition and $N(R)$ is right T-nilpotent. We recall that an ideal A is right T-nilpotent if for each sequence $\{a_i\}_{i=1}^{\infty}$ of elements of A there exists an integer n such that $a_n a_{n-1} \cdots a_1 = 0$. Equivalently, A is right T-nilpotent if and only if $\{x \in X \mid Ax = (0)\} \neq (0)$ for any nonzero left R-module X, and this occurs if and only if $\text{Hom}_R(R/A, X) \neq 0$ for any nonzero left R-module X. With this characterization of T-nilpotence, Proposition 2.7 of [5] can be extended to $\sigma[M]$, letting $\text{Rad}(M)$ denote the intersection of all prime M-ideals of M.

Proposition 2.7. *If ${}_RM$ is quasi-projective, then the following conditions are equivalent:*

(1) $\text{Hom}_R(M, X) \neq 0$ for all nonzero modules ${}_RX$ in $\sigma[M]$, every proper torsion radical of $\sigma[M]$ is contained in a maximal torsion radical of $\sigma[M]$, and the maximal torsion radicals of $\sigma[M]$ correspond to the minimal prime M-ideals of M;

(2) $\text{Hom}_R(M/\text{Rad}(M), X) \neq 0$ for all modules ${}_RX$ in $\sigma[M]$, every proper torsion radical of $\sigma[M/\text{Rad}(M)]$ is contained in a maximal torsion radical of $\sigma[M/\text{Rad}(M)]$, and the maximal torsion radicals of $\sigma[M/\text{Rad}(M)]$ correspond to the minimal prime M-ideals of $M/\text{Rad}(M)$.

Proof. (1) $\Longrightarrow$ (2): We first show that $M/\text{Rad}(M)$ is a quasi-projective module with $\text{Hom}_R(M/\text{Rad}(M), X) \neq 0$ for all modules ${}_RX$ in $\sigma[M]$. Since $\text{Rad}(M)$ is the intersection of all prime M-ideals of M, it is also the intersection of the annihilators in M of all M-prime modules in $\sigma[M]$. Thus $\text{Rad}(M)$ is a fully invariant submodule of M, and so $M/\text{Rad}(M)$ is quasi-projective since M is quasi-projective.

For any nonzero module ${}_RX$ in $\sigma[M]$, by Lemma 2.4 there is a submodule Y of $\widehat{X}$ whose annihilator in M is a minimal prime M-ideal P. Then $X \cap Y \neq 0$, so by assumption $\text{Hom}_R(M, X \cap Y) \neq 0$. If $f \in \text{Hom}_R(M, X \cap Y)$, then $f(P) = (0)$, so $f(\text{Rad}(M)) = (0)$, which implies that $\text{Hom}_R(M/\text{Rad}(M), X \cap Y) \neq 0$.

If X is an $M/\text{Rad}(M)$-injective module, then by assumption its M-injective envelope $\widehat{X}$ has a submodule Y whose annihilator $\text{Ann}_M(Y)$ is a minimal prime M-ideal P. Without loss of generality we can assume that X is M-generated. Since M/P is a actually a factor module of $M/\text{Rad}(M)$, it follows that X is $M/\text{Rad}(M)$-generated, and thus X belongs to $\sigma[M/\text{Rad}(M)]$.

(2) $\Longrightarrow$ (1): If $\text{Hom}_R(M/\text{Rad}(M), X) \neq 0$ for all modules ${}_RX$ in $\sigma[M]$, then certainly $\text{Hom}_R(M/X) \neq 0$. If ${}_RW$ is a nonzero M-injective module, then by assumption we have $\text{Hom}_R(M/\text{Rad}(M), W) \neq 0$. Thus W contains a nonzero submodule belonging to $\sigma[M/\text{Rad}(M)]$, so by assumption there is a submodule of W whose left annihilator in $M/\text{Rad}(M)$ is a minimal prime $M/\text{Rad}(M)$-ideal. But then the annihilator is in fact a minimal prime M-ideal, and so the proof can be completed by applying Lemma 2.4. □

Our final results concern a characterization of maximal torsion radicals via the quotient categories that they define. The theory of quotient categories of

$\sigma[M]$ is developed in Section 9 of [12]. If τ is a torsion radical in $\sigma[M]$, a module ${}_RW$ in $\sigma[M]$ is said to be (M,τ)-injective if it is injective relative to all exact sequences $0 \to X \to Y$ in $\sigma[M]$ such that the image of X is τ-dense in Y. That is, if $i : X \to Y$ is one-to-one in $\sigma[M]$, and $i(X)$ is σ-dense in Y, then any R-homomorphism $f : X \to W$ can be extended to $\widehat{f} : Y \to W$ with $\widehat{f}i = f$. We note that there is a version of Baer's criterion for (M,τ)-injective modules. It is shown in [1] that W is (M,τ)-injective if and only if for any R-homomorphism $f : A \to W$ such that A is a τ-dense left ideal of R and $R/\ker(f)$ belongs to $\sigma[M]$ there exists an element $w \in W$ with $f(a) = aw$ for all $a \in A$.

The quotient category of $\sigma[M]$ determined by τ is the full subcategory of all τ-torsionfree (M,τ)-injective modules, denoted by $\sigma[M]/\tau$. There is an associated quotient functor $Q_\tau : \sigma[M] \to \sigma[M]/\tau$ defined on the module ${}_RX$ in $\sigma[M]$ by letting $Q_\tau(X)$ be the (M,τ)-injective envelope of the factor module $X/\tau(X)$.

We say that an M-ideal $K \subseteq M$ is a *torsion M-ideal* if $K = \tau(M)$ for some torsion radical τ of $\sigma[M]$. This is equivalent to the condition that $K = \mathrm{Ann}_M(W)$ for a module ${}_RW$ injective in $\sigma[M]$, and also equivalent to the condition that $f(K) = 0$ for all R-homomorphisms $f : M \to \widehat{M/K}$ (see [8]),

Proposition 2.8. *Let M be any left R-module. The following conditions are equivalent for a torsion radical τ of $\sigma[M]$ with $\tau(M) = K$:*

(1) the torsion radical τ is a maximal torsion radical of $\sigma[M]$;

(2) every injective object in the quotient category $\sigma[M]/\tau$ is a cogenerator;

(3) the torsion radical τ is defined by $\widehat{M/K}$ and K is maximal in the set of proper τ-closed torsion M-ideals.

Proof. (1) $\Longrightarrow$ (2): In general, if W is any injective object in $\sigma[M]/\tau$, then $\tau \leq \mathrm{rad}_W$ since W is τ-torsionfree. It follows that if τ is a maximal torsion radical, then for any injective object W we must have $\mathrm{rad}_W = \tau$, and hence W is a cogenerator in $\sigma[M]/\tau$.

(2) $\Longrightarrow$ (3): Since $W = \widehat{M/K}$ is injective in $\sigma[M]/\tau$, it is a cogenerator and hence $\mathrm{rad}_W = \tau$. If L is any torsion M-ideal with $K \subseteq L$ and $Z = \widehat{M/L}$, then as before we must have $\tau = \mathrm{rad}_Z$, and it follows that $L = \mathrm{rad}_Z(M) = \tau(M) = K$.

(3) $\Longrightarrow$ (1): If γ is a torsion radical of $\sigma[M]$ with $\tau \leq \gamma$, then $M/\gamma(M)$ is τ-torsionfree since it is γ-torsionfree, and thus $\gamma(M)$ is a τ-closed torsion M-ideal

with $K \subseteq \gamma(M)$. By hypothesis we must have $K = \gamma(M)$, and this implies that $\tau = \gamma$, because τ is defined by $\widehat{M/K}$. □

Corollary 2.9. *If M is Noetherian and τ is a maximal torsion radical of $\sigma[M]$, then $\sigma[M]/\tau$ has an injective cogenerator that is an essential extension of a simple object.*

Proof. If M is Noetherian, then there exists a maximal τ-closed submodule N of M. It follows that the quotient module $Q_\tau(M/N)$ is simple in $\sigma[M]/\tau$. The desired conclusion is a consequence of Proposition 2.8, since $\widehat{M/N}$ is a cogenerator for $\sigma[M]/\tau$. □

Corollary 2.10. *Let ${}_RM$ be quasi-projective, such that each prime M-ideal is completely prime. Let τ be a torsion radical of $\sigma[M]$, let $\tau(M) = K$, and assume further that τ is defined by $\widehat{M/K}$. If K is a prime M-ideal, then τ is a maximal torsion radical of $\sigma[M]$.*

Proof. It follows from Corollary 2.2 that K is a maximal τ-closed M-ideal. The desired conclusion is therefore a consequence of condition (3) of Proposition 2.8. □

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