ON NONCOMMUTATIVE PIECEWISE NOETHERIAN RINGS

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Abstract: We extend the definition of a piecewise Noetherian ring to the noncommutative case, and investigate various properties of such rings. In particular, we show that a ring with Krull dimension is piecewise Noetherian. Certain fully bounded piecewise Noetherian rings have Gabriel dimension and exhibit the Gabriel correspondence between prime ideals and indecomposable injective modules.

Commutative piecewise Noetherian rings were introduced and studied by the second author and W. D. Weakley in [6] and [7]. It follows from Proposition 1.3 of [6] that a commutative ring $R$ is piecewise Noetherian if and only if $R$ has a Noetherian spectrum and for all prime ideals $P$, the ascending chain condition is satisfied for $P$-primary ideals. In this paper we extend the definition of a piecewise Noetherian ring to the noncommutative setting by using the general notion of reduced rank defined in [2], and show that it reduces to the original definition in the commutative case. The goal of our paper is to investigate the properties of a ring that depend only on having reduced rank on factor rings and the ascending chain condition on prime ideals.

In Section 1 we state the definition of a left piecewise Noetherian ring, and show that it is a generalization of the definition of a ring with Krull dimension on the left. We show that the notion of piecewise Noetherian is Morita invariant, and is passed on to triangular matrix rings. We also investigate the existence of associated prime ideals.

In Sections 2 and 3 we consider left fully bounded rings that are left piecewise Noetherian. We show that under certain conditions such rings have enough associated prime ideals, and have Gabriel dimension. We also investigate some other properties that hold for left fully bounded Noetherian rings.
In Section 4 we give the following result: if $R$ is a left fully bounded ring such that for any prime ideal $P$ of $R$ the factor ring $R/P$ is left Goldie, and each uniform left $R$-module has an associated prime ideal, then there is a bijection between the set of prime ideas of $R$ and the set of isomorphism classes of indecomposable injective left $R$-modules. This implies immediately that Gabriel's correspondence holds for left fully bounded rings strongly piecewise Noetherian on the left, and for left piecewise Noetherian rings that satisfy Gabriel's condition H. The majority of the results in the paper are derived from the first author's dissertation [18].

Throughout the paper, $R$ will be assumed to be an associative ring with identity element, and all modules will be assumed to be unital $R$-modules. The injective envelope of a module $_RM$ will be denoted by $E(M)$, and the direct sum of $n$ isomorphic copies of $M$ will be denoted by $M^n$. The module $M$ is said to be finitely annihilated if there exist $m_1, \ldots, m_n \in M$ such that $\text{Ann}(M) = \text{Ann}\{m_1, \ldots, m_n\}$, or, equivalently, if $R/\text{Ann}(M)$ can be embedded in $M^n$, for some positive integer $n$.

Any injective module $_RX$ defines a torsion radical $\sigma = \text{rad}_X$, where $\sigma(_RM)$ is the intersection of all kernels of homomorphisms $f \in \text{Hom}_R(M,X)$. The corresponding quotient functor will be denoted by $Q_{\sigma} : R\text{-Mod} \to R\text{-Mod}/\sigma$. The module $_RM$ is called $\sigma$-torsionfree if $\sigma(M) = (0)$, and $\sigma$-torsion if $\sigma(M) = M$; a submodule $M' \subseteq M$ is called $\sigma$-closed if $M/M'$ is $\sigma$-torsionfree. Note that the subobjects of $Q_{\sigma}(M)$ in $R\text{-Mod}/\sigma$ correspond to the $\sigma$-closed submodules of $M$.

If $I$ is an ideal of $R$, then the set of elements of $R$ which are regular modulo $I$ will be denoted by $C(I)$. If the ring $R/I$ is a semiprime left Goldie ring, then the torsion radical $\text{rad}_{C(I)}$ defined by the elements $C(I)$ coincides with $\text{rad}_{E(R/I)}$ by Proposition 1 of [4]. In this case, if a submodule $M'$ of a module $M$ is $\text{rad}_{C(I)}$-closed, then we will call $M'$ a $C(I)$-closed submodule of $M$. The reader is referred to [20] for other definitions and results on quotient categories and torsion radicals, and to [9], [11], and [16] for results on Noetherian rings and modules.
1 Definition and elementary properties

We will use a torsion theoretic definition of reduced rank to give the definition of a left piecewise Noetherian ring. Let $R$ be a ring with prime radical $N$, and let $\gamma = \text{rad}_{E(R/N)}$. The module $_RM$ is said to have finite reduced rank if the module of quotients $Q_\gamma(M)$ has finite length in the quotient category $R\text{-Mod}/\gamma$. In this case the length will be denoted by $\rho_R(M)$ (or simply $\rho(M)$, when the context is clear). The left reduced rank of the ring $R$ is defined to be the reduced rank of the module $_RR$. Proposition 2 of [2] shows that if $N$ is nilpotent, then this definition coincides with the one given by Warfield [21] (see p. 346 of [16] for the definition of $\rho$-rank). We also investigate the situation in which all factor rings of $R$ have finite reduced rank in Warfield’s more stringent sense.

**DEFINITION 1.1.** The ring $R$ is called **left piecewise Noetherian** if it satisfies the ascending chain condition on its prime ideals and the factor ring $R/I$ has finite reduced rank on the left, for every ideal $I$.

The ring $R$ is said to be **strongly left piecewise Noetherian** if in addition the prime radical of each factor ring is nilpotent.

Our first result shows that the class of piecewise Noetherian rings can be viewed as a generalization of the class of rings with (Rentschler-Gabriel) Krull dimension. The reader is referred to [11] for the definition and elementary properties of Krull dimension.

**THEOREM 1.2.** If $R$ has Krull dimension on the left, then it is strongly piecewise Noetherian on the left.

**Proof.** Assume that $R$ has Krull dimension on the left, with prime radical $N$. Then Lemma 5.6 of [12] shows that $N$ is nilpotent, and for each positive integer $i$ the factor module $_R(N^i/N^{i+1})$ has finite uniform dimension since $R$ has Krull dimension on the left. It follows that $R$ has finite reduced rank on the left, and Proposition 1.2 (i) of [12] shows that this argument applies to the factor ring $R/I$, for each proper ideal $I$ of $R$. Furthermore, Theorem 7.1 of [12] shows that any ring with Krull dimension on the left satisfies the ascending chain condition on prime ideals, and so $R$ is left piecewise Noetherian. Finally, as noted above,
the prime radical of each factor ring is nilpotent, and so \( R \) is strongly piecewise Noetherian on the left. \( \square \)

**EXAMPLE 1.** A well-known ring that is strongly piecewise Noetherian on the left but does not have Krull dimension on the left.

Consider the following ring, constructed by Small to show that a right Noetherian ring need not be left Noetherian. We let \( R = \begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Z} \end{bmatrix} \) and \( N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \), where \( N \) is the prime radical of \( R \). The ring \( R \) is right Noetherian, since \( R/N \) is commutative and Noetherian, and on the right \( N \) has the structure of \( \mathbb{Q}_\mathbb{Q} \), making it a minimal right ideal. On the left, however, \( N \) has the structure of \( \mathbb{Z}_\mathbb{Q} \), which is not a Noetherian module. Furthermore, \( R \) does not have Krull dimension (as a module) on the left, since the cyclic left \( R \)-module \( M = \begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Z} \end{bmatrix}/\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) does not have finite uniform dimension.

Since \( R/N \) is Noetherian, it is clear that \( R \) satisfies the ascending chain condition on prime ideals. To show that each factor ring has finite reduced rank, we first prove that any nonzero ideal \( I \) of \( R \) must contain \( N \). If \( I \) is an ideal that contains a nonzero element \( \begin{bmatrix} p & 0 \\ q & n \end{bmatrix} \) and \( p \neq 0 \), then multiplying on the left by \( \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \) produces a nonzero element of \( N \). If \( n \neq 0 \), then multiplying on the right by \( \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \) yields a nonzero element of \( N \). If \( p = 0 \) and \( n = 0 \), then \( q \neq 0 \). In each of these cases \( I \) contains a nonzero element of \( N \), which implies that \( I \supseteq N \), since \( N \) is a minimal right ideal. Thus if \( I \neq (0) \), then \( R/I \) is Noetherian, and therefore has finite reduced rank. Finally, to complete the argument when \( I = (0) \), we note that \( R \) is an order in the ring of \( 2 \times 2 \) lower triangular matrices over \( \mathbb{Q} \), and therefore has finite reduced rank on the left by Theorem 4 of [2]. \( \square \)

Theorem 1.5 below will show that Definition 1.1 reduces to the original definition in the commutative case. We first need to define what we mean by a \( P \)-primary left ideal, and we need to give a new characterization of rings with finite reduced rank.

**DEFINITION 1.3.** Let \( P \) be a prime ideal of the ring \( R \). A left ideal \( A \subseteq R \) is said to be \( P \)-primary if \( A \) is \( C(P) \)-closed and there exists an ideal \( I \subseteq A \) such that \( I \subseteq P \) and \( P/I \) is a minimal prime ideal of \( R/I \).
PROPOSITION 1.4. Let $N$ be the prime radical of the ring $R$. The following conditions are equivalent:

1. $R$ has finite reduced rank on the left;
2. $R$ has finitely many minimal prime ideals $\{P_i\}_{i=1}^n$ and $E(R/P_i)$ satisfies the descending chain condition on annihilators, for $1 \leq i \leq n$;
3. $R$ has finitely many minimal prime ideals $\{P_i\}_{i=1}^n$ and $E(R/P_i)$ satisfies the ascending chain condition on annihilators, for $1 \leq i \leq n$.

Proof. (1) $\Rightarrow$ (2): If $R$ has finite reduced rank on the left, then the proof of Theorem 1 of [2] shows that $E(R/N) \cong \bigoplus_{i=1}^n E(R/P_i)$ for minimal prime ideals $P_i$, with the conclusion that each module $E(R/P_i)$ satisfies the descending chain condition on annihilators.

(2) $\Rightarrow$ (3): By Theorem 1.4 of [17], the descending chain condition on annihilators of subsets of $E(R/P_i)$ forces the ascending chain condition.

(3) $\Rightarrow$ (1): Assume that $E(R/P_i)$ satisfies the ascending chain condition on annihilators for $1 \leq i \leq n$, and let $\sigma_i = \text{rad}_E(R/P_i)$, for $1 \leq i \leq n$. Since these left annihilators correspond to the subobjects of the localization $Q_{\sigma_i}(R)$ in the quotient category $R\text{-Mod}/\sigma_i$, it follows that $Q_{\sigma_i}(R)$ is a Noetherian object in the quotient category, and therefore its factor $Q_{\sigma_i}(R/P_i)$ has finite uniform dimension. We conclude that $R/P_i$ has finite uniform dimension on the left, and, moreover, that $R/P_i$ satisfies the ascending chain condition on left annihilators. Thus each factor ring $R/P_i$ is left Goldie, and so $R/N$ is left Goldie. It follows from Proposition 7.6 of [11] that $E(R/N) \cong \bigoplus_{i=1}^n E(R/P_i)$, and therefore $E(R/N)$ satisfies the ascending chain condition on left annihilators. By Theorem 1 of [2], this is equivalent to the condition that $R$ has finite reduced rank on the left.

A ring is said to have Noetherian spectrum if it satisfies the ascending chain condition on its prime ideals and each of its semiprime ideals is a finite intersection of prime ideals.

THEOREM 1.5. The ring $R$ is left piecewise Noetherian if and only if $R$ has Noetherian spectrum and for each prime ideal $P$ it satisfies the ascending chain condition on $P$-primary left ideals.
Proof. First suppose that $R$ is left piecewise Noetherian. By definition, $R$ satisfies the ascending chain condition on its prime ideals, and for any semiprime ideal $S$, the factor ring $R/S$ has finite reduced rank on the left. Proposition 1.4 shows that $S$ is a finite intersection of prime ideals.

Let $P$ be a prime ideal of $R$, and suppose that $A_1 \subseteq A_2 \subseteq \cdots$ is an ascending chain of $P$-primary left ideals of $R$. By definition there exists an ideal $I \subseteq R$ such that $I \subseteq A_1 \subseteq A_2 \subseteq \cdots$. Since each $P$-primary ideal $A_i$ is $C(P)$-closed, it is a left annihilator of $E(R/P)$. But $P/I$ is a minimal prime ideal of $R/I$, and since $R/I$ has finite reduced rank on the left, by Proposition 1.4 the annihilators of $E(R/I R/P) = \{x \in E(R/P) \mid Ix = 0\}$ satisfy the ascending chain condition. It follows that the ascending chain $A_1 \subseteq A_2 \subseteq \cdots$ must terminate.

Conversely, suppose that $R$ has Noetherian spectrum and satisfies the ascending chain condition on $P$-primary left ideals for each prime ideal $P$. Suppose that $I$ is an ideal of $R$, and consider a minimal prime ideal $P/I$ of $R/I$. Any ascending chain of annihilators of $E(R/I R/P)$ in $R/I$ corresponds to an ascending chain $I \subseteq A_1 \subseteq A_2 \subseteq \cdots$ of $P$-primary ideals of $R$, and therefore must terminate. It follows from Proposition 1.4 that $R/I$ has finite reduced rank on the left. Thus $R$ is left piecewise Noetherian.

\begin{proposition} Let $R$ be a left piecewise Noetherian ring. Then the ring of $2 \times 2$ lower triangular matrices over $R$ is also left piecewise Noetherian. The same statement holds if $R$ is strongly piecewise Noetherian on the left. \end{proposition}

\begin{proof} Suppose that $R$ is left piecewise Noetherian. Theorem 3 of \cite{1} states that if every factor ring of $R$ has finite reduced rank, then the same is true of the ring $T$ of $2 \times 2$ lower triangular matrices over $R$. By Proposition 1.17 (3) of \cite{15}, any ideal $I$ of $T$ has the form $\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$, where $A, B, C$ are ideals of $R$ and $A + C \subseteq B$. Since $K^2 = (0)$ for the ideal $K = \begin{bmatrix} 0 & 0 \\ R & 0 \end{bmatrix}$, it follows that $K$ is contained in each prime ideal of $T$. Therefore the prime ideals of $T$ have the form $\begin{bmatrix} P_1 & 0 \\ R & R \end{bmatrix}$ or $\begin{bmatrix} R & 0 \\ R & P_2 \end{bmatrix}$ for prime ideals $P_1, P_2$ of $R$. Thus any ascending chain of prime ideals in $T$ is in one-to-one correspondence with an ascending chain of prime ideals in $R$, and it follows immediately that $T$ satisfies the ascending chain condition on prime ideals. Therefore $T$ is left piecewise Noetherian. \end{proof}
Let $R$ be strongly piecewise Noetherian on the left, with $I = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ an ideal of $T$. By the above remarks about prime ideals, it follows that the inverse image of the prime radical of $R/I$ is $\begin{bmatrix} N_1 & 0 \\ R & N_2 \end{bmatrix}$, where $N_1/A$ and $N_2/C$ are the prime radicals of $R/A$ and $R/C$, respectively. By assumption there exist positive integers $n, m$ such that $(N_1)^n \subseteq A$ and $(N_2)^m \subseteq C$. A short calculation shows that $\begin{bmatrix} N_1 & 0 \\ R & N_2 \end{bmatrix}^{n+m} \subseteq \begin{bmatrix} N_1^{n+m} & 0 \\ N_1^n + N_2^n & N_2^m \end{bmatrix} \subseteq \begin{bmatrix} A & 0 \\ A + C & C \end{bmatrix}$, and so the prime radical of $T/I$ is nilpotent, verifying that $T$ is strongly piecewise Noetherian on the left.

We recall that a property of a ring $R$ is said to be Morita invariant if it is preserved by any equivalence between $R$–Mod and $T$–Mod, for any ring $T$.

**THEOREM 1.7.** The following properties are Morita invariant.

(i) “Left piecewise Noetherian”

(ii) “Strongly piecewise Noetherian on the left”

*Proof.* (i) Suppose that the ring $R$ is Morita equivalent to the ring $T$, with mutually inverse category equivalences $F : R$–Mod $\rightarrow T$–Mod and $G : T$–Mod $\rightarrow R$–Mod.

Proposition 18.44 of [16] shows that the lattice of ideals in $R$ is isomorphic to the lattice of ideals of $T$. Furthermore, under this isomorphism prime ideals of $R$ correspond to prime ideals of $T$. Thus the ascending chain condition on prime ideals of $R$ yields the same condition on the prime ideals of $T$.

Let $J$ be any ideal of $T$. Then Corollary 18.49 of [16] shows that $T/J$ is Morita equivalent to $R/I$, where $I$ is the ideal of $R$ corresponding to $J$ in the above correspondence. Theorem 4 of [3] shows that having finite reduced rank on the left is a Morita invariant property. Thus $T/J$ has finite reduced rank on the left since by assumption each factor ring $R/I$ has finite reduced rank on the left.

(ii) Consider the prime radical $N(T/J)$ of the factor ring $T/J$, which corresponds to the prime radical of the corresponding factor ring $R/I$. Corollary 18.45 of [16] shows that the nilpotence of $N(R/I)$ implies that $N(T/J)$ is nilpotent. Thus if $R$ is strongly piecewise Noetherian on the left, then so is $T$. 

\[ \square \]
A nonzero module \( RX \) is said to be a **prime module** if \( \text{Ann}(Y) = \text{Ann}(X) \) for all nonzero submodules \( Y \subseteq X \). An **associated prime ideal** of the module \( RM \) is an ideal \( P \) such that \( P = \text{Ann}(X) \) for a prime submodule \( X \subseteq M \). Lemma 1 of [7] shows that every nonzero module over a commutative piecewise Noetherian ring has an associated prime ideal. We have been unable to prove this result for the general case of a noncommutative piecewise Noetherian ring. Theorem 1.8, Proposition 2.2, and Theorem 3.3 address this question.

**THEOREM 1.8.** If \( R \) is strongly piecewise Noetherian on the left, then every nonzero left \( R \)-module has an associated prime ideal.

**Proof.** In Lemma 2 of [19], it is shown that every nonzero left \( R \)-module contains a prime submodule if (i) \( R \) satisfies the ascending chain condition on its prime ideals, and (ii) for each ideal \( I \subset R \) there exists a finite collection \( \{P_1, \ldots, P_n\} \) of prime ideals such that \( P_1 \cdots P_n \subseteq I \subseteq \cap_{i=1}^n P_i \). By taking the prime ideals in the prime radical of \( R/I \) (with some multiplicity, if necessary) it follows that if \( R \) is strongly piecewise Noetherian on the left, then it satisfies conditions (i) and (ii).

For the reader’s convenience we outline a proof of the theorem. Let \( RM \) be any nonzero module, let \( I = \text{Ann}(M) \), and let \( P_1, \ldots, P_n \) be the prime ideals given in condition (ii) above. Since \( (P_1 \cdots P_n)M = (0) \), it can be shown that \( P_iM' = (0) \) for some \( i \) and some nonzero submodule \( M' \subseteq M \). Therefore the set of prime ideals of \( R \) that annihilate a nonzero submodule of \( M \) contains a maximal element, say \( P \), with \( PM'' = (0) \) for a nonzero submodule \( M'' \). It is then easily checked that \( M'' \) is a prime submodule, and thus \( P \) is an associated prime ideal of \( M \).

\[ \square \]

2 **Fully bounded piecewise Noetherian rings**

A ring \( R \) is said to be **left fully bounded** if each prime factor ring is left bounded, i.e. if \( A/P \) contains a nonzero ideal \( I/P \) for each prime ideal \( P \) and each left ideal \( A/P \) that is essential in \( R(R/P) \). In case \( R \) is piecewise Noetherian, Proposition 14.1 of [11] shows that \( R \) has classical Krull dimension, which will be denoted by \( k\text{dim}(R) \).
The Gabriel dimension of a module, introduced in [10], is defined by trans-finite recursion: if \( \alpha > 0 \) is an ordinal, then \( G\dim(M) \leq \alpha \) if every nonzero homomorphic image of \( M \) is \( \beta \)-simple for some \( \beta \leq \alpha \). (The module \( R X \) is \( \beta \)-simple if \( G\dim(X) \not< \beta \), \( \beta \) is not a limit ordinal, and \( G\dim(X') < \beta \) for every proper homomorphic image \( X' \) of \( X \).) We refer the reader to [13] for additional information.

**THEOREM 2.1.** Let \( R \) be a left fully bounded ring that is strongly piecewise Noetherian on the left. Then \( R \) has Gabriel dimension on the left, and \( G\dim R = k\dim(R) + 1 \).

*Proof.* The result follows immediately from the proof of Theorem 1.1 of [14]. As noted by the authors of [14], the proof requires the following hypotheses: \( R \) satisfies the ascending chain condition on prime ideals, each prime factor ring of \( R \) is left Goldie and left bounded, and each nonzero ideal of \( R \) contains a finite product of prime ideals that contain it. For the reader’s convenience, we include their proof.

The proof proceeds by induction on the classical Krull dimension of \( R \). The prime radical of \( R \) is nilpotent, and since it is a finite intersection of minimal prime ideals, there are prime ideals \( P_1, P_2, \ldots, P_n \) such that \( P_n \cdots P_2 P_1 = 0 \). In the chain \( R \supset P_1 \supset P_2 P_1 \supset \ldots \supset (0) \) the \( i \)th factor is a left \( R/P_i \)-module. It follows that if each factor ring \( R/P_i \) has Gabriel dimension on the left, then so does \( R \).

If \( k\dim(R) = 0 \), then each prime ideal \( P_i \) is maximal, and therefore simple Artinian since \( R \) is left fully bounded. Thus the theorem is valid in this case.

Now suppose that \( k\dim(R) = \alpha \), and that the theorem is valid for every factor ring \( R/I \) for which \( k\dim(R/I) < \alpha \). As noted above, we may assume without loss of generality that \( R \) is a left fully bounded prime left Goldie ring. Let \( A \) be a cyclic uniform left ideal of \( R \). If \( B \) is a nonzero submodule of \( A \), then \( A/B \) is a cyclic torsion module over \( R \) since \( A \) is uniform, so \( A/B \cong R/C \) for an essential left ideal \( C \subseteq R \). Because \( R \) is left bounded there exists a nonzero ideal \( I \subseteq C \), and \( G\dim(R/I) < \alpha \) by hypothesis. It follows that \( G\dim(R/C) < \alpha + 1 \), since the theorem holds for \( R/I \), and therefore \( G\dim(R/I) = k\dim(R/I) + 1 \). Consequently, \( G\dim(A) \leq \alpha + 1 \). By Goldie’s theorem, \( R \) can be embedded in a
finite direct sum of cyclic uniform left ideals. We conclude that $G\dim(R)$ exists, and that $G\dim(R) \leq \alpha + 1$. 

A nonzero left $R$-module $U$ is said to be \textit{monoform} if for each submodule $U' \subseteq U$ every nonzero $R$-homomorphism $f : U' \to U$ is a monomorphism. Note that any monoform module is uniform, but the converse does not hold. Corollary 2.10 of [12] shows that if $R$ has Gabriel dimension on the left, then every nonzero left $R$-module contains a monoform submodule.

**PROPOSITION 2.2.** Let $R$ be a left piecewise Noetherian ring that is fully bounded on the left. Then every left $R$-module has an associated prime ideal if and only if every left $R$-module contains a monoform prime submodule.

\textit{Proof.} Suppose that $X$ is a prime submodule of $R \! M$ that has an associated prime ideal $P = \text{Ann}(X)$. Let $K$ be any cyclic submodule of $X$. Then $\text{Ann}(K) = P$ since $X$ is a prime module, and so $K$ is a faithful and finitely generated $R/P$-module, where $R/P$ is a prime left Goldie ring since $R$ is left piecewise Noetherian. Because $K$ is a finitely annihilated $R/P$ module by Proposition 9.7 of [11], and $R/P$ contains a monoform left ideal, it follows that $X$ contains a monoform submodule.

The converse is clear. 

**LEMMA 2.3.** Let $R$ be a left fully bounded left piecewise Noetherian ring for which every left $R$-module has an associated prime ideal. Let $R \! U$ be uniform, faithful, and Noetherian. If $P$ is an associated prime ideal of $U$, then the right annihilator of $P$ in $R$ is an essential left ideal of $R$.

\textit{Proof.} Let $R \! U$ be a uniform, faithful, Noetherian module, let $P = \text{Ann}(X)$ for a prime submodule of $U$, and let $A$ be the right annihilator of $P$ in $R$. Since by assumption every nonzero left ideal of $R$ has an associated prime ideal, it follows from Proposition 2.2 that every nonzero left ideal of $R$ contains a uniform submodule. Thus to prove that $A$ is an essential left ideal, it suffices to show that it has nonzero intersection with every nonzero uniform left ideal of $R$.

Let $I$ be a uniform left ideal of $R$ and set $P'$ to be the associated prime ideal of $I$. That is, let $P' = \text{Ann}(J)$ for some prime submodule $J \subseteq I$. The module
$U$ is faithful, so $J \cdot U \neq (0)$. Let $B = \text{Ann}(JU)$. Then $(BJ)U = B(JU) = (0)$, so $BJ = (0)$ since $U$ is faithful, and thus $B \subseteq P'$. For the reverse inclusion, we observe that $P'J = (0)$ implies that $P'(JU) = (0)$, so $B = \text{Ann}(J) = P'$. Now we have that $(0) \neq X \subseteq U$ and $(0) \neq JU \subseteq U$, where $U$ is uniform. Thus $JU \cdot X \neq (0)$, and $JU \cdot X \subseteq X$, where $X$ is prime. Therefore $\text{Ann}(X) = \text{Ann}(JU \cdot X)$. We also have $B = \text{Ann}(JU) \subseteq \text{Ann}(JU \cdot X)$ since $JU \cdot X \subseteq JU$.

Thus $P' = B = \text{Ann}(JU) \subseteq \text{Ann}(X) = P$.

Now $JU \cap X$ is nonzero because it contains $JU \cdot X$, which is nonzero. Note that $JU$ is finitely generated since $U$ is Noetherian. Since $R$ is left piecewise Noetherian, $R/P'$ is left Goldie, and $JU$ is a faithful finitely generated uniform left $R/P'$ module, so it is nonsingular by Corollary 9.3 of [11]. But $P \cdot (JU \cap X) = (0)$, so $P/P'$ cannot be nonzero since then it would be an essential left ideal of $R/P'$, contradicting the fact that $JU$ is nonsingular. It follows that $P \subseteq P'$, which shows that $P = P'$. Hence $J \cdot P = (0)$ since $J \cdot P' = (0)$, and so $A \cap I \neq (0)$.

\[ \square \]

**PROPOSITION 2.4.** Let $R$ be a left fully bounded left piecewise Noetherian ring such that every left $R$-module has an associated prime ideal, and let $\mu M$ be a Noetherian module. The following conditions are equivalent.

1. $M$ is finitely annihilated;
2. $R/\text{Ann}(M)$ is a left Noetherian ring;
3. every factor ring of $R/\text{Ann}(M)$ has finite uniform dimension (on the left).

**Proof.** If $M$ is finitely annihilated, then $R/\text{Ann}(M)$ can be embedded in $M^n$ for some $n$, and it follows that $R/\text{Ann}(M)$ is left Noetherian. Thus (1) implies (2), and (2) implies (3) follows immediately.

To show that (3) implies (1), assume that every factor ring of $R/\text{Ann}(M)$ has finite uniform dimension on the left. Since $M$ is Noetherian by hypothesis, it follows from Lemma 7.9 of [9] that there exist submodules $K_1, K_2, \ldots, K_n \subseteq M$ such that each $M/K_i$ is uniform and $K_1 \cap K_2 \cap \ldots \cap K_n = (0)$. Then $\text{Ann}(M) = \cap_{i=1}^n \text{Ann}(M/K_i)$, and so $M$ is finitely annihilated if each factor $M/K_i$ is finitely annihilated. Thus, without loss of generality, we can reduce to the case of a uniform module.
Accordingly, let $U$ be a faithful Noetherian uniform left $R$-module such that every factor ring of $R$ has finite uniform dimension (on the left). By hypothesis every left $R$-module has an associated prime ideal. Let $P = \text{Ann}(X)$, where $X$ is a prime submodule of $U$. Then the right annihilator of $P$ is an essential left ideal of $R$ by Lemma 2.3. Set $L = \{u \in U \mid Pu = (0)\}$, and note that $P = \text{Ann}(L)$. Corollary 4.1.3 of [22] states that a finitely generated uniform faithful module over a left bounded prime left Goldie ring has zero singular submodule. It follows that $Z(L) = (0)$ as a left $R/P$-module, so $L$ is torsionfree as an $R/P$-module. Since $L$ is torsionfree and uniform as an $R/P$-module, we note that any submodule of $L$ has rank equal to 1.

Let $u_1 \in U$ and set $I_1 = L \cap \text{Ann}(u_1)$. If $I_1 = (0)$, then $\text{Ann}(u_1) = (0)$ since $L$ is essential in $R$, so $\text{Ann}(u_1) = \text{Ann}(U)$, and we are done. If $I_1 \neq (0)$, then there exists a nonzero element $u_2 \in U$ such that $I_1 u_2 \neq (0)$. We have $P(I_1 u_2) = (0)$ by the definition of $I_1$, and so $I_1 u_2$ has rank 1 as an $R/P$-module. Set $I_2 = I_1 \cap \text{Ann}(u_2) = L \cap \text{Ann}(u_1, u_2)$, and define $f : I_1 \to I_1 u_2$ by $f(a) = a \cdot u_2$ for all $a \in I_1$. Then $\text{ker}(f) = I_2$, and so $I_1/I_2 \cong I_1 \cdot u_2$. Consequently, $I_1/I_2$ has rank 1 over $R/P$. If $I_2 = (0)$, then $\text{Ann}(u_1, u_2) = \text{Ann}(U)$, and we are done.

Continuing in this manner yields a chain of $R/P$-submodules $I_1 \supset I_2 \supset \cdots$ in which each factor $I_i/I_{i+1}$ has rank one over $R/P$. Since rank is an additive function and $I_1$ has finite rank over $R/P$ because $R$ has finite uniform dimension, this chain must terminate after at most $\rho(I_1)$ steps. It follows that there are finitely many elements $u_1, u_2, \ldots, u_n \in U$ such that $L \cap \text{Ann}(u_1, u_2, \ldots, u_n) = (0)$, and so $\text{Ann}(u_1, u_2, \ldots, u_n) = (0)$ since $L$ is an essential left ideal of $R$. Thus $U$ is finitely annihilated, completing the proof.

**Corollary 2.5.** Let $R$ be a left fully bounded ring that has Krull dimension on the left. Then every Noetherian left $R$-module is finitely annihilated.

**Proof.** If $R$ has Krull dimension on the left, then every cyclic module has finite uniform dimension, and so condition (3) of Proposition 2.4 is satisfied for any Noetherian left $R$-module.
3 Piecewise Noetherian rings with condition H

We can obtain some of the results in the previous section if we drop the condition that $R$ is strongly piecewise Noetherian on the left, while adding the condition that $R$ satisfies Gabriel’s condition H, defined as follows in [10]. The ring $R$ is said to satisfy condition $H$ on the left if every finitely generated left $R$-module is finitely annihilated. We note that left Artinian rings, commutative rings, and all rings finitely generated as a left module over their center satisfy condition H. On the other hand, if $R$ satisfies condition H on the left, then all factor rings of $R$ are left bounded (see Proposition 4.3.1 of [22]). Corollary 3.2 of [5] shows that condition H is a Morita invariant property.

For a left Noetherian ring $R$, it has been shown by Cauchon [8] that $R$ is left fully bounded if and only if $R$ satisfies condition H on the left. As the next example shows, this equivalence fails even for rings that are strongly piecewise Noetherian on the left.

**EXAMPLE 2.** The ring $R = \begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Z} \end{bmatrix}$ of Example 1 is fully bounded and strongly piecewise Noetherian on the left, but does not satisfy condition H.

The ring $R$ is fully bounded since every prime ideal of $R$ has one of the following forms: $\begin{bmatrix} 0 & 0 \\ \mathbb{Q} & \mathbb{Z} \end{bmatrix}$, $\begin{bmatrix} \mathbb{Q} & 0 \\ 0 & \mathbb{Z} \end{bmatrix}$, or $\begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & p\mathbb{Z} \end{bmatrix}$, for a prime number $p$. In each case the factor ring is commutative, and hence bounded.

To show that $R$ does not satisfy condition H on the left, we must exhibit a finitely generated left $R$-module that is not finitely annihilated. As in Example 1, consider the cyclic left $R$-module $M = R/I$, where $I = \begin{bmatrix} 0 & 0 \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$. We claim that $M$ is faithful. To see this, let $a = \begin{bmatrix} x & 0 \\ y & m \end{bmatrix} \in \text{Ann}(M)$. Since $a$ must annihilate the coset of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, we have $x = 0$ and $y \in \mathbb{Z}$. Then we must have $xq_1 + mq_2 \in \mathbb{Z}$ for all $q_1, q_2 \in \mathbb{Q}$, and it follows that $x = m = 0$.

Suppose towards contradiction that $M$ is finitely annihilated. Then there is an embedding from $R$ into a finite direct sum $M^k$ of copies of $M$. It is clear that $R(J/I)$ is essential in $R/I$, for $J = \begin{bmatrix} 0 & 0 \\ \mathbb{Q} & \mathbb{Z} \end{bmatrix}$, so the image of $R$ in $M^k$ must include elements that have finite order in the underlying additive group.
This is a clear contradiction, so the embedding is impossible, and $M$ cannot be finitely annihilated. \(\square\)

We next show that left piecewise Noetherian rings satisfying condition H have enough associated prime ideals and have Gabriel dimension.

**Lemma 3.1.** If the left $R$-module $M_1 \oplus \cdots \oplus M_n$ has a prime submodule with annihilator $P$, then some component $M_i$ has a prime submodule whose annihilator is $P$.

*Proof.* Let $X \subseteq M_1 \oplus \cdots \oplus M_n$ be a prime submodule with annihilator $P$. If $X \cap M_1 \neq (0)$, then $X \cap M_1$ is a prime submodule of $M_1$ with annihilator $P$. If $X \cap M_1 = (0)$, then there is an embedding of $X$ into $M_2 \oplus \cdots \oplus M_n$. We can continue this argument until some component $M_i$ contains a submodule isomorphic to a nonzero submodule of $X$, and this submodule is prime with annihilator $P$. \(\square\)

**Lemma 3.2.** Let $R$ be a ring with finite reduced rank on the left, and let $\gamma = \text{rad}_{E(R/N)}$, where $N$ is the prime radical of $R$. If the module $R M$ is $\gamma$-torsionfree, then it has an associated prime ideal.

*Proof.* By assumption the $\gamma$-closed left ideals of $R$ satisfy the ascending and descending chain conditions. Therefore the set of left annihilators of subsets of $M$ satisfies the ascending chain condition since $M$ is $\gamma$-torsionfree. Thus we can choose an ideal $P$ maximal in the set of annihilators of submodules of $M$. Proposition 2.12 of [11] shows that $P$ is an associated prime ideal of $M$. \(\square\)

**Theorem 3.3.** If $R$ is left piecewise Noetherian and satisfies condition H on the left, then every nonzero left $R$-module has an associated prime ideal.

*Proof.* Let $M$ be a nonzero left $R$-module. Then $M$ contains a nonzero cyclic submodule, which we denote by $K$. By assumption $K$ is finitely annihilated, so there exists an embedding $0 \to R/\text{Ann}(K) \to K^n$, for some positive integer $n$. It follows from Lemma 3.1 that if $R(R/\text{Ann}(K))$ has an associated prime ideal, then so does $K$, and hence $M$. Thus it suffices to show that each factor ring $R/I$ has an associated prime ideal (as a left $R$-module).
Suppose that there exists an ideal \( I \subset R \) such that the module \( _RR/I \) has no associated prime ideal. We will use the notation \( N(I) \) for the preimage in \( R \) of the prime radical \( N(R/I) \). Consider the set \( S \) of all semiprime ideals \( S \) such that \( S = N(I) \) for an ideal \( I \) for which the module \( _RR/I \) has no associated prime ideal. By assumption, \( S \) is nonempty, and since by hypothesis \( R \) has Noetherian spectrum, the set \( S \) contains a maximal element \( S = N(A) \), where \( _RR/A \) has no associated prime ideal. By passing to the factor ring \( R/A \), we may assume, without loss of generality, that \( A = (0) \) and \( N(A) = N \), the prime radical of \( R \).

Accordingly, we assume that \( R \) is a left piecewise Noetherian ring over which every finitely generated left \( R \)-module is finitely annihilated such that \( _RR \) has no associated prime ideal, and such that if \( I \) is any ideal of \( R \) for which \( N \subset N(I) \), then \( _RR/I \) does have an associated prime ideal. Let \( \gamma = \text{rad}_{E(R/N)} \). If \( _RR \) is \( \gamma \)-torsionfree, then Lemma 3.2 implies that it has an associated prime ideal and we are done.

Assume further that \( \gamma(R) \neq (0) \), and that \( 0 \neq x \in \gamma(R) \). Then \( Rx \) is finitely annihilated, and so there exists an embedding \( 0 \to R/\text{Ann}(Rx) \to (Rx)^n \) for some positive integer \( n \). It follows that \( R/\text{Ann}(Rx) \) is \( \gamma \)-torsion, and therefore \( R/N(\text{Ann}(Rx)) \) is also \( \gamma \)-torsion. We now consider two cases: (i) \( N(\text{Ann}(Rx)) = N \), and (ii) \( N(\text{Ann}(Rx)) \supset N \). The first case leads to a contradiction, since \( R/N \) is \( \gamma \)-torsionfree. Therefore \( N(\text{Ann}(Rx)) \supset N \), which implies that \( _RR/N(\text{Ann}(Rx)) \) has an associated prime ideal, since \( N \) was chosen to be a maximal element of the set \( S \). It follows from Lemma 3.1 that \( Rx \) has an associated prime ideal, and this completes the proof.

An ideal of \( R \) is said to be right \( T \)-nilpotent if for each sequence \( a_1, a_2, \ldots \) of elements of \( I \) there exists an integer \( n \) such that \( a_n \cdots a_2a_1 = 0 \). It follows from Lemma 2.9 of [20] that if \( I \) is right \( T \)-nilpotent, then for each nonzero module \( _RM \) the submodule \( \{ m \in M \mid Im = 0 \} \) is nonzero.

**COROLLARY 3.4.** If \( R \) is left piecewise Noetherian and satisfies condition \( H \) on the left, then the prime radical of \( R \) is right \( T \)-nilpotent.
Proof. Let $N$ be the prime radical of $R$. Since Theorem 3.3 shows that each nonzero left $R$-module has an associated prime ideal, it follows that each nonzero left $R$-module contains a nonzero $R/N$-submodule. Thus if $_RM$ is nonzero, the submodule $\{m \in M \mid Nm = 0\}$ is nonzero, and so $N$ is right T-nilpotent. \(\square\)

**Theorem 3.5.** If $R$ is left piecewise Noetherian and satisfies condition H on the left, then $R$ has Gabriel dimension on the left, and $G\dim R = k\dim(R) + 1$.

Proof. Let $N$ be the prime radical of $R$. Since $N$ is right T-nilpotent, every nonzero left $R$-module contains a nonzero $R/N$-submodule, and so to show that $R$ has Gabriel dimension it suffices to show that $R/N$ has Gabriel dimension. Since $R$ satisfies condition H on the left, it is left fully bounded, and so the proof of Theorem 2.1 can be carried over to this case. \(\square\)

**Proposition 3.6.** Let $R$ be a left piecewise Noetherian ring that satisfies condition H. Then $R$ is left Artinian if and only if each prime ideal of $R$ is maximal.

Proof. Theorem 9 of [2] shows that these conditions are equivalent for any ring that has finite reduced rank on the left and satisfies condition H. \(\square\)

If $R$ is a Noetherian ring that is left fully bounded, then any finitely generated essential extension of a simple left $R$-module is Artinian (see Theorem 8.1 of [11]). We can give an equivalent condition in certain cases.

**Proposition 3.7.** Let $R$ be a left piecewise Noetherian ring that satisfies condition H. Then the following conditions are equivalent:

1. A finitely generated module $_RM$ with finite uniform dimension is Artinian if every associated prime ideal of $M$ is maximal;

2. If $_RM$ is a finitely generated essential extension of a simple module, then $M$ is Artinian.

Proof. (1) $\Rightarrow$ (2). Suppose that condition (1) holds and that $_RU$ is a finitely generated essential extension of a simple module $S$, but $U$ is not Artinian. Then there exists an associated prime ideal $P$ of $U$ that is not maximal. Thus $P = \text{Ann}(X)$ for some nonzero prime submodule $X \subseteq U$, and $S \subseteq X$ since $U$ is
an essential extension of $S$. The submodule $X$ is finitely generated, faithful, and uniform as a left $R/P$-module. Since $R$ satisfies condition H, it is left fully bounded by Proposition 7.6 of [9], so it follows from Proposition Corollary 4.1.3 of [22] that the singular submodule of $R/PX$ is $(0)$. Since $S \subseteq X$ is a simple left $R/P$-module, we have $S \cong R/A$ for a maximal left ideal $A \supseteq P$, and $A/P$ cannot be an essential left ideal of $R/P$ since $Z(S) = (0)$. Thus there is a nonzero left ideal $B/P \subseteq R/P$ such that $(A/P) \cap (B/P) = (0)$. It follows that $B/P$ is isomorphic to a submodule of $R/A$, so $B/P$ is a minimal left ideal of $R/P$, and hence $R/P$ is a simple Artinian ring by Theorem 1.24 of [9]. This contradicts our assumption that $P$ is not a maximal ideal of $R$. We conclude that $R$ is Artinian, as required.

(2) $\Rightarrow$ (1). Assume that condition (2) holds and let $rM$ be a finitely generated module with finite uniform dimension such that every associated prime ideal of $M$ is maximal in $R$. Since $M$ has finite uniform dimension, its injective envelope $E(M)$ is a finite direct sum $E(M) = \bigoplus_{i=1}^{n} E(U_i)$ of injective envelopes of uniform submodules $U_i$). The image of $M$ in $E(U_i)$ is finitely generated and uniform, so without loss of generality we may assume that $M$ is uniform. We will show that $M$ contains a minimal submodule, and then the hypothesis will imply that $M$ is Artinian. Then there exists an associated prime ideal $P = \text{Ann}(X)$ of $M$, where $X$ is a prime submodule of $M$, and $P$ is maximal in $R$ (by hypothesis). Since $R$ satisfies condition H, the ring $R/P$ is simple Artinian and can therefore be embedded into a finite direct sum of copies of the finitely generated submodule $X$. It follows that $X$, and therefore $M$, contains a minimal submodule, showing that $M$ is a finitely generated essential extension of a simple module, and therefore Artinian.

4 The Gabriel correspondence

If $R$ is left Noetherian, it is well-known that $R$ is left fully bounded if and only if there is a bijection between the set of prime ideals of $R$ and the set of isomorphism classes of indecomposable injective left $R$-modules. In general, if this bijection exists we say that the Gabriel correspondence holds for $R$.
THEOREM 4.1. Let $R$ be a left fully bounded ring such that for any prime ideal $P$ of $R$ the factor ring $R/P$ is left Goldie, and every uniform left $R$-module has an associated prime ideal. Then the Gabriel correspondence holds for $R$.

Proof. Let $P$ be a prime ideal of $R$. By assumption, $R/P$ is a prime left Goldie ring, so $R(R/P)$ has finite uniform dimension. Thus, by Proposition 5.20 of [11], $R/P$ contains an essential direct sum $U \oplus U_2 \oplus \cdots \oplus U_n$ of $n$ uniform submodules, for some $n$. The proof of Proposition 7.23 of [11] shows that $E(U_i) \cong E(U)$ for $2 \leq i \leq n$, and so our desired bijection $\Phi$ is defined by setting $\Phi(P)$ to be the isomorphism class of $E(U)$, where $U$ is any uniform left ideal of $R/P$. That $\Phi$ is well defined follows easily from the Krull-Schmidt-Azumaya Theorem.

To show that $\Phi$ is an injection, suppose that $P_1$ and $P_2$ are prime ideals of $R$ such that $\Phi(P_1) = \Phi(P_2)$, with nonzero uniform left ideals $U_1 \subseteq R/P_1$ and $U_2 \subseteq R/P_2$. Then there exists an isomorphism $f : E(U_1) \to E(U_2)$, and $f(U_1) \cap U_2 \neq (0)$ since $f(U_1) \neq (0)$ and $U_2$ is essential in $E(U_2)$. We have $\text{Ann}(f(U_1) \cap U_2) = \text{Ann}(f(U_1)) = P_1$ and $\text{Ann}(f(U_1) \cap U_2) = \text{Ann}(U_2) = P_2$, so $P_1 = P_2$.

To show that $\Phi$ is a surjection, let $E$ be an indecomposable injective left $R$-module. Then since $E$ is uniform, by hypothesis it has an associated prime ideal, say $P = \text{Ann}(U)$, where $U$ is a cyclic prime submodule of $E$. Corollary 9.3 of [11] shows that since $R/P$ is left bounded and $U$ is a faithful cyclic uniform module, it must be a torsionfree $R/P$-module. Lemma 7.2.2 of [11] shows that any uniform left ideal of $R/P$ is isomorphic to a submodule of $U$, and therefore $E$ belongs to the isomorphism class of $E(R/P)$, which shows that $\Phi$ is a surjection. \qed

THEOREM 4.2. The Gabriel correspondence holds for any left fully bounded left piecewise Noetherian ring $R$ such that either

(i) $R$ is strongly piecewise Noetherian on the left, or

(ii) $R$ satisfies condition $H$ on the left.

Proof. The result follows from Theorem 4.1 and Theorems 1.8 and 3.3. \qed

COROLLARY 4.3. The Gabriel correspondence holds for any commutative piecewise Noetherian ring.
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