Piecewise Noetherian Rings

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Goldie (1964): if $R$ is a semiprime Noetherian ring, let $Q_{cl}(R)$ be the semisimple Artinian classical ring of quotients of $R$. The \textit{rank} of a finitely generated module $_RM$ is the length of the module $Q_{cl}(R) \otimes_R M$, denoted by $\text{rank}(M)$.

More generally, if $_RM$ is a finitely generated module over a left Noetherian ring $R$ with prime radical $N$, whose index of nilpotence is $k$, then the the \textit{reduced rank} of $M$ is defined to be

$$\rho(_RM) = \sum_{i=1}^{k} \text{rank}(N^{i-1}M/N^iM).$$

A 1979 paper by Chatters, Goldie, Hajarnavis, and Lenagan publicized the notion by using it to give several new proofs of results on Noetherian rings.
Warfield (1979) observed that in order to define the reduced rank all that is needed is that $R/N$ is a semiprime left Goldie ring and $N$ is nilpotent. The definition of $\rho$-rank given by Lam in Lectures on Modules and Rings is equivalent, but allows computation via other descending chains of submodules.

Warfield proved that the following conditions are equivalent, generalizing Small’s theorem:

1. the ring $R$ is a left order in a left Artinian ring
2. $R$ has reduced rank in this general sense and satisfies the regularity condition (any element that is not a zero divisor modulo $N$ is not a zero divisor in $R$).
Definition (1982)

Let $R$ be a ring with prime radical $N$, and let $\gamma$ denote the torsion radical cogenerated by $E(R/N)$. The module $\gamma M$ is said to have finite reduced rank if the module of quotients $Q_\gamma(M)$ has finite length in the quotient category $R-Mod/\gamma$.

This length will be denoted by $\rho(M)$, and $\rho(R)$ is the reduced rank of the module $R R$.

**Note.** If $R$ is a semiprime left Goldie ring, then $\rho(M)$ is the length of $Q_{cl}(R) \otimes_R M$.

For an ideal $I$, let $C(I)$ be the set of elements regular modulo $I$. 
Let $N$ be the prime radical of the ring $R$. The following conditions are equivalent:

(1) $R$ has finite reduced rank on the left;

(2) $R/N$ is a left Goldie ring, $N^k$ is $C(N)$-torsion for some integer $k > 0$, and $N^i/N^{i+1}$ has finite reduced rank as an $R/N$-module, for $i = 1, \ldots, k - 1$;

(3) there exists an ideal $I \subseteq N$ such that $R/I$ has finite reduced rank on the left, $I^t$ is $C(N)$-torsion for some integer $t > 0$, and $I^j/I^{j+1}$ has finite reduced rank as an $R/I$-module, for $u = 1, \ldots, t - 1$.

**Note:** in this case $\rho(RM) = \sum_{i=1}^{k} \rho(N^{i-1}M/N^iM)$, provided each of the terms is finite.

Let $N$ be the prime radical of the ring $R$. The following conditions are equivalent:

(1) $R$ has finite reduced rank on the left;

(2) the set of left annihilators of subsets of $E(R/N)$ satisfies ACC;

(3) $R$ has finitely many minimal prime ideals $\{P_i\}_{i=1}^n$, and for each $i$ the set of left annihilators of $E(R/P_i)$ satisfies DCC.

Theorem

The notion of reduced rank is Morita invariant.
Theorem (1982)

If $R$ has finite reduced rank on the left, then so does the polynomial ring $R[x]$. Moreover, the reduced rank of $R[x]$ is equal to the reduced rank of $R$.

I think that the torsion-theoretic notion of reduced rank is worth studying because of the following generalization of the Small-Warfield theorem on orders in Artinian rings.

Theorem (1982)

The ring $R$ is a left order in a left Artinian ring $\iff$ $R$ has finite reduced rank on the left and satisfies the regularity condition. i.e. $C(N) \subseteq C(0)$. 
A question

If the notion of reduced rank plays such an important role for Noetherian rings, what can be proved if it is the only tool available?

Doug Weakley and I looked at the commutative case. We were able to relate the notion of reduced rank to that of the “ideal length” used by Zariski and Samuel as a tool before localization techniques came to the forefront.

It became apparent that to get useful results we needed to add a chain condition on the prime ideals of the ring.
Main definitions

**Definition**

The ring \( R \) is called *left piecewise Noetherian* if

(i) for each ideal \( I \) of \( R \) the factor ring \( R/I \) has finite reduced rank on the left and

(ii) \( R \) has ACC on prime ideals.

We give two definitions, depending on the choice of the definition of reduced rank.

**Definition**

The ring \( R \) is called *strongly* piecewise Noetherian on the left if in addition the prime radical of each factor ring is nilpotent.

**Note:** every semiprime ideal \( S \) of a piecewise Noetherian ring is left Goldie. \((R/S \text{ is a semiprime left Goldie ring.})\)
The Krull dimension of $R^M$, denoted $K\dim(M)$, is defined by a transfinite recursion as follows: if $M = 0$, then $K\dim(M) = -1$, and if $\alpha$ is an ordinal with $K\dim(M) \not\geq \alpha$, then $K\dim(M) = \alpha$ provided there is no infinite descending chain $M = M_0 \supset M_1 \supset \ldots$ of submodules $M_i$, for $i = 1, \ldots$, such that $K\dim(M_{i-1}/M_i) \not\geq \alpha$. If there is no such ordinal $\alpha$, then we say that $M$ does not have Krull dimension.

$K\dim(M) = 0 \iff M$ is Artinian. $K\dim(\mathbb{Z}) = 1$.

Any Noetherian module has Krull dimension. Any module with Krull dimension has finite uniform dimension.

If $F$ is a field, the Krull dimension of the polynomial ring $F[x_1, \ldots, x_n]$ is $n$. $R[x]$ has Krull dimension on the left $\iff R$ is left Noetherian.
“Piecewise Noetherian” generalizes Krull dimension

Theorem

If $R$ has Krull dimension on the left, then it is strongly piecewise Noetherian on the left.

Proof.

Gordon and Robson (1973) showed that a ring with Krull dimension has ACC on prime ideals and a nilpotent prime radical. If $R$ has Krull dimension, so does each factor ring, which then has finite reduced rank on the left by a result of Lenagan (1979). Finally, each factor ring has nilpotent prime radical.
Let \( R = \begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Z} \end{bmatrix} \) and \( N = \begin{bmatrix} 0 & 0 \\ \mathbb{Q} & 0 \end{bmatrix} \). The ring \( R \) is well-known to be right but not left Noetherian. (As a left ideal, \( N \) is not Noetherian.) Note that \( _RR \) does not have Krull dimension since the cyclic left \( R \)-module \( M = \begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Z} \end{bmatrix} / \begin{bmatrix} 0 & 0 \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix} \) does not have finite uniform dimension.

The ring \( R \) is strongly piecewise Noetherian on the left: any nonzero ideal \( I \) of \( R \) must contain \( N \), and \( R/N \) is commutative Noetherian; \( R \) itself has finite reduced rank on the left because it is a left order in the left Artinian ring \( \begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Q} \end{bmatrix} \).
Motivation for the terminology

**Definition (2017)**

Let $P$ be a prime ideal of the ring $R$. A left ideal $A \subseteq R$ is said to be $P$-primary if there exists an ideal $I \subseteq A$ such that $P$ is minimal over $I$ and $R/A$ is $C(P)$-torsionfree.

Recall: A ring has Noetherian spectrum if it has ACC on prime ideals and each semiprime ideal is a finite intersection of primes.

**Theorem (1984; 2017)**

The ring $R$ is left piecewise Noetherian if and only if it has Noetherian spectrum and ACC on $P$-primary left ideals, for each prime ideal $P$. 
The commutative case

Theorem (1984)

A commutative ring \( R \) is piecewise Noetherian \( \iff \) \( R \) has Noetherian spectrum and for each ideal \( I \) and each prime ideal \( P \) minimal over \( I \), the localized ring \( R_P/IR_P \) is Artinian.

Theorem (1984)

Let \( R \) be commutative and piecewise Noetherian.
(a) Each maximal ideal of \( R \) is finitely generated.
(b) \( R \) is Noetherian \( \iff \) every ideal is a finite intersection of primary ideals.

Theorem (1984)

A commutative ring \( R \) is piecewise Noetherian \( \iff \) the polynomial ring \( R[x] \) is piecewise Noetherian.
More commutative results

Theorem (1987)

*Over a commutative piecewise Noetherian ring, every module has an associated prime ideal.*

Recall that a Prüfer domain is a Dedekind domain if and only if it is Noetherian. Popescu (1984) has defined a notion of a *generalized Dedekind domain*.

Theorem (Facchini, 1994)

*A Prüfer domain is a generalized Dedekind domain if and only if it is piecewise Noetherian.*
Now we return to the general case ($R$ is not necessarily commutative).

**Theorem (2017)**

The following are Morita invariant:

(i) “left piecewise Noetherian”;
(ii) “strongly piecewise Noetherian on the left”.

**Theorem (2017)**

Let $R$ be a left piecewise Noetherian ring or be strongly piecewise Noetherian on the left. Then the ring of lower triangular matrices over $R$ has the same property.
### Existence of associated prime ideals

**Theorem (2017)**  
*If $R$ is strongly piecewise Noetherian on the left, then every nonzero left $R$-module has an associated prime ideal.*

A ring $R$ is said to satisfy Gabriel’s condition H (on the left) if every finitely generated left $R$-module is finitely annihilated. (If $_RM$ is finitely generated, then there exist $m_1, \ldots, m_n \in M$ such that $\text{Ann}(M) = \text{Ann}(m_1, \ldots, m_n).$)

**Theorem (2017)**  
*If $R$ is left piecewise Noetherian and has condition H on the left, then every nonzero left $R$-module has an associated prime ideal.*
Cauchon (1976) showed that a left Noetherian ring satisfies Gabriel’s condition $H \iff$ it is left fully bounded. (For each prime ideal $P$, each essential left ideal of $R/P$ contains an ideal essential in $R/P$.) This fails for left piecewise Noetherian rings.

Let $R = \begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Z} \end{bmatrix}$ and $M = \begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Z} \end{bmatrix} / \begin{bmatrix} 0 & 0 \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$. Then $R$ is left fully bounded, but $M$ is a cyclic module that is not finitely annihilated, so $R$ does not satisfy Gabriel’s condition $H$ on the left.

Matlis (1958) showed that over a commutative Noetherian ring there is a one-to-one correspondence between indecomposable injective modules and prime ideals of the ring. For noncommutative rings we call this the *Gabriel correspondence*, first investigated by Gabriel (1962).
Lemma (2017)

Let $R$ be a left fully bounded ring such that for any prime ideal $P$ of $R$ the factor ring $R/P$ is left Goldie, and every uniform left $R$-module has an associated prime ideal. Then there is a bijection between the set of prime ideals of $R$ and the set of isomorphism classes of indecomposable injective left $R$-modules.

Theorem (2017)

The Gabriel correspondence holds for any left piecewise Noetherian ring $R$ such that either

(i) $R$ is left fully bounded and strongly piecewise Noetherian on the left, or

(ii) $R$ satisfies Gabriel’s condition H on the left.
Let \( P \) be a prime ideal of \( R \). Then \( R/P \) is a prime left Goldie ring, with \( \text{udim}(R/P) = n \). Then \( R/P \) contains an essential left ideal that is a direct sum \( U \oplus U_2 \oplus \cdots \oplus U_n \) of \( n \) uniform submodules (Goodearl, Warfield 5.20). Next, \( E(U_i) \cong E(U) \) for \( 1 < i \leq n \) (Goodearl, Warfield 7.23), so define \( \Phi(P) \) to be the isomorphism class of \( E(U) \), where \( U \) is any uniform left ideal of \( R/P \). Since \( E(R/P) \cong E(U) \oplus \cdots \oplus E(U_n) \), the Krull-Schmidt-Azumaya theorem implies that \( \Phi \) is well-defined.

To show that \( \Phi \) is an injection, suppose that \( P_1 \) and \( P_2 \) are prime ideals of \( R \) such that \( \Phi(P_1) = \Phi(P_2) \), \( U_1 \subseteq R/P_1 \) and \( U_2 \subseteq R/P_2 \).
Then there exists an isomorphism $f : E(U_1) \to E(U_2)$, and $f(U_1) \cap U_2 \neq (0)$ since $f(U_1) \neq (0)$ and $U_2$ is essential in $E(U_2)$. We have $\text{Ann}(f(U_1) \cap U_2) = \text{Ann}(f(U_1)) = P_1$ and $\text{Ann}(f(U_1) \cap U_2) = \text{Ann}(U_2) = P_2$, so $P_1 = P_2$.

To show that $\Phi$ is a surjection, let $E$ be an indecomposable injective left $R$-module. Then since $E$ is uniform, by hypothesis it has an associated prime ideal, say $P = \text{Ann}(U)$, where $U$ is a cyclic prime submodule of $E$. Since $R/P$ is left bounded and $U$ is a faithful cyclic uniform module, $U$ must be a torsionfree $R/P$-module (Goodearl, Warfield 9.3). Any uniform left ideal of $R/P$ is isomorphic to a submodule of $U$ (Goodearl, Warfield 7.2.2), and therefore $E$ belongs to the isomorphism class of $E(R/P)$, which shows that $\Phi$ is a surjection.
Gabriel dimension


The *Gabriel dimension* of $R^{}M$ is defined by transfinite recursion: if $\alpha > 0$ is an ordinal, then $G\dim(M) \leq \alpha$ if every nonzero homomorphic image of $M$ is $\beta$-simple for some $\beta \leq \alpha$. (The module $R^{}X$ is $\beta$-simple if $G\dim(X) \not\prec \beta$, $\beta$ is not a limit ordinal, and $G\dim(X') < \beta$ for every proper homomorphic image $X'$ of $X$.)

Any Noetherian module $M$ has Gabriel dimension, and $G\dim(M) = K\dim(M) + 1$.

A module with Gabriel dimension has Krull dimension $\iff$ each factor module has finite uniform dimension.

Gordon and Small (1984) showed that an affine polynomial identity ring has Gabriel dimension.
Theorem (2017)

Let $R$ be a left piecewise Noetherian ring $R$. Then $R$ has Gabriel dimension if either

(i) $R$ is left fully bounded and strongly piecewise Noetherian on the left, or

(ii) $R$ satisfies Gabriel’s condition $H$ on the left.

Corollary

A commutative piecewise Noetherian ring has Gabriel dimension.
Future directions

Conditions equivalent to piecewise Noetherian that can be more easily checked are needed.

Can the piecewise Noetherian condition be translated into reasonable conditions for group rings and/or enveloping algebras? For other broad classes of rings?

The localization of a commutative piecewise Noetherian ring at a prime ideal is again piecewise Noetherian. Is the same statement true at a localizable prime ideal of a noncommutative ring? Is the universal localization of a piecewise Noetherian ring again piecewise Noetherian?