On Universal Localization of Noetherian Rings

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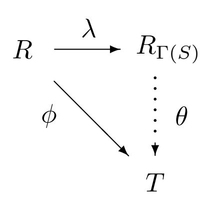
Abstract. We determine the universal localization $R_{\Gamma(S)}$ at certain semiprime ideals S of a Noetherian ring R that is a finite extension of its center. For this class of semiprime ideals, $R_{\Gamma(S)}$ is Noetherian, which need not be true in the general case.

Throughout this note, R will denote an associative ring with identity, and S will denote a semiprime right Goldie ideal of R. (That is, R/S is a semiprime right Goldie ring.) The study of the universal localization $R_{\Gamma(S)}$ of R at S was initiated by P. M. Cohn in [**6**] in the Noetherian case, and continued by the first author (in [**1**], [**2**], and [**3**]) and by others. We note that Goldie's localization, defined and studied in [**9**] and [**10**] for Noetherian rings, is a homomorphic image of the universal localization (see [**2**]). The major results in this paper are based on results in [**13**], the second author's dissertation. The description of $R_{\Gamma(S)}$ when R is a finite extension of its center depends on the seminal work of B. J. Mueller in [**15**].

For any ideal I of R, the set of elements $c \in R$ that are regular modulo I will be denoted by $\mathcal{C}(I)$. For any positive integer n, let $\Gamma_n(S)$ denote the set of all matrices C such that C belongs to the $n \times n$ matrix ring $M_n(R)$ and the image of C in $M_n(R/S)$ is a regular element. This will be abbreviated by saying that C is regular modulo S. The union over all $n > 0$ of $\Gamma_n(S)$ will be denoted by $\Gamma(S)$.

Following Cohn, the universal localization $R_{\Gamma(S)}$ of R at S is defined to be the universal $\Gamma(S)$-inverting ring. That is, $R_{\Gamma(S)}$ satisfies the following conditions: there is a ring homomorphism $\lambda : R \to R_{\Gamma(S)}$ under which each element of $\Gamma(S)$ is invertible, and for any $\Gamma(S)$-inverting ring homomorphism $\phi : R \to T$ there exists a unique ring homomorphism $\theta : R_{\Gamma(S)} \to T$ with $\phi = \theta\lambda$, as in the following diagram.



The ring $R_{\Gamma(S)}$ can be constructed as follows (see [**6**] and [**7**] for details). For each n and each $n \times n$ matrix $[c_{ij}]$ in $\Gamma(S)$, take a set of n^2 symbols $[d_{ij}]$, and take

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Dedicated to the memory of Bruno J. Mueller.

a ring presentation of $R_{\Gamma(S)}$ consisting of all of the elements of R, as well as all of the elements d_{ij} as generators; as defining relations take all of the relations holding in R, together with all of the relations $[c_{ij}][d_{ij}] = I$ and $[d_{ij}][c_{ij}] = I$ which define all of the inverses of the matrices in $\Gamma(S)$.

The universal localization was shown by Cohn in [**6**] to have a number of desirable properties: $\lambda : R \to R_{\Gamma(S)}$ is an epimorphism in the category of rings, and if S is right localizable then $R_{\Gamma(S)} \cong R_S$, where R_S is the right ring of fractions defined by $\mathcal{C}(S)$. The ring $R_{\Gamma(S)}$ modulo its Jacobson radical is naturally isomorphic to $Q_{cl}(R/S)$, the classical ring of right quotients of R/S, and Theorem 1.1 of [**1**] shows that $R_{\Gamma(S)}$ can be defined as the ring universal with respect to this property.

On the other hand, the universal localization also presents some difficulties. In general, it is quite hard to determine $\ker(\lambda)$. If $\mathcal{C}(S)$ is not a right denominator set, then $R_{\Gamma(S)}$ is not flat as a left R-module (Corollary 3.2 of [**1**]). Even when R is finitely generated as a module over its Noetherian center, $R_{\Gamma(S)}$ may not be Noetherian, as shown by the first of the following examples.

1. Examples

In Example 4 of [**1**], R_1 is the ring of lower triangular 2×2 matrices over the ring of integers $\mathbb{Z}$ and S is the semiprime ideal $\begin{bmatrix} p\mathbb{Z} & 0 \\ \mathbb{Z} & q\mathbb{Z} \end{bmatrix}$, where p and q are distinct prime numbers. Then $(R_1)_{\Gamma(S)} = \begin{bmatrix} \mathbb{Z}_{(p)} & 0 \\ \mathbb{Q} & \mathbb{Z}_{(q)} \end{bmatrix}$ is neither right nor left Noetherian, where $\mathbb{Q}$ is the field of rational numbers and $\mathbb{Z}_{(p)}$ and $\mathbb{Z}_{(q)}$ are the standard localizations of $\mathbb{Z}$.

To construct a similar example in case S is a prime ideal proved to be more difficult. We note such an example from [**4**]. A well-known example of Nagata (details can be found in [**17**] and [**18**]) provides two commutative Noetherian domains $A \subset B$ such that A is a local ring with maximal ideal M, and B contains two maximal ideals, J_1 and J_2, such that $J_1 \cap J_2 = M$. Let

$$R_2 = \left\{ \begin{bmatrix} b & 0 & 0 \\ x & a & 0 \\ y & z & c \end{bmatrix} \middle| \, b, c \in B; a \in A; x, y, z \in J_1; b - a, c - a \in J_1 \right\}$$

and let

$$P = \left\{ \begin{bmatrix} b & 0 & 0 \\ x & a & 0 \\ y & z & c \end{bmatrix} \middle| \, b, x, y, z, c \in J_1; a \in M \right\}.$$

Then it can be shown that the universal localization $(R_2)_{\Gamma(P)}$ of R_2 at the maximal ideal P is not Noetherian.

We note that $P \subset M_3(J_1)$, which is an ideal of the ring $M_3(B)$. Since B is a commutative Noetherian domain, $\cap_{n=1}^{\infty} (J_1)^n = (0)$ by Theorem 77 of [**12**], so $\cap_{n=1}^{\infty} P^n = (0)$ and thus $\cap_{n=1}^{\infty} P^n \subseteq Q$ for all prime ideals Q of R_2.

The relevant properties of these examples illustrate some of the obstructions to having a Noetherian universal localization. To provide an explanation, we need to utilize the notion of a link between two prime ideals of R. Following [**11**], we say that there is a link from a prime ideal $P \subset R$ to a prime ideal $Q \subset R$, denoted by $P \rightsquigarrow Q$, if there is an ideal A with $PQ \subseteq A \subset P \cap Q$ such that $(P \cap Q)/A$ is nonzero and torsionfree both as a right (R/Q)-module and as a left (R/P)-module. For a prime ideal $P \subset R$, we say that the prime ideal Q belongs to the clique of

P, denoted by $\mathrm{Clq}(P)$, if there exist prime ideals $\{P = P_1, \ldots, P_m\}$ such that for $i = 1, \ldots, m-1$, either $P_i \rightsquigarrow P_{i+1}$ or $P_{i+1} \rightsquigarrow P_i$. When R is a Noetherian ring that is finitely generated as a module over its center Z, it has been shown by Mueller in [**14**] that

$$\mathrm{Clq}(P) = \{Q \in \mathrm{Spec}(R) \mid Q \cap Z = P \cap Z\} .$$

In the ring R_1 of the first example given above, we note that the semiprime ideal S is the intersection of two prime ideals from different cliques. In the second example, in the ring R_2 the prime ideal P belongs to a clique that contains at least one other prime, since P is not localizable. Therefore the intersection of the powers of P is contained in another prime ideal in $\mathrm{Clq}(P)$.

In Theorem 2.3, the main result of this paper, we show that if R is finitely generated as a module over its Noetherian center, and the semiprime ideal S is the intersection of prime ideals in the same clique of R, then $R_{\Gamma(S)}$ is a homomorphic image of a central localization, provided that the intersection of the symbolic powers of S is not contained in any other prime ideal of the given clique. We conclude that under these hypotheses $R_{\Gamma(S)}$ is Noetherian.

To illustrate this procedure, consider the ring $R_1 = \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$ and the prime ideal $P = \begin{bmatrix} p\mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$, where p is a prime number. Then P is not right localizable, but $N = \begin{bmatrix} p\mathbb{Z} & 0 \\ \mathbb{Z} & p\mathbb{Z} \end{bmatrix}$, the intersection of the prime ideals in the clique of P, *is* right localizable, yielding $\begin{bmatrix} \mathbb{Z}_{(p)} & 0 \\ \mathbb{Z}_{(p)} & \mathbb{Z}_{(p)} \end{bmatrix}$ as the localization at N. Then $(R_1)_{\Gamma(P)}$ is obtained by factoring out the intersection of powers of the extension $\begin{bmatrix} p\mathbb{Z}_{(p)} & 0 \\ \mathbb{Z}_{(p)} & \mathbb{Z}_{(p)} \end{bmatrix}$ of P, and so $(R_1)_{\Gamma(P)} \cong \mathbb{Z}_{(p)}$. (In this case, the construction can be confirmed by noting that P is left localizable.)

2. Main Theorems

In commutative algebra, the Krull Intersection Theorem has been an extremely useful result and there are several interesting results that follow from it. See Kaplansky [**12**] or Eisenbud [**8**] for more information. We begin by developing a noncommutative extension of the Krull Intersection Theorem that leads to information regarding the structure of the universal localization in a particular case.

THEOREM 2.1. *Let R be a right fully bounded Noetherian ring, and let S be a semimaximal ideal of R. Suppose that $S = \cap_{i=1}^m P_i$ for maximal ideals $\{P_i\}_{i=1}^m$ in* $\mathrm{Clq}(P)$*, where $P = P_1$, and let $K = \cap_{n=1}^\infty S^n$. If $K \nsubseteq Q$ for any $Q \in \mathrm{Clq}(P)$ such that $Q \neq P_i$ for $i = 1, \ldots, m$, then $KS = K$.*

PROOF. Let I be a right ideal of R that is maximal with respect to the property that $I \cap K = PK$. (Note that $KS \cap K = KS$, so the given set is nonempty, and then I exists by Zorn's lemma.) It is well-known that $(K+I)/I$ is an essential submodule of R/I. Define $f : K \to (K+I)/I$ by $f(a) = a + I$ for every $a \in K$. We note that f is an onto R-homomorphism, and that $\ker(f) = \{a \in K : a \in I\} = I \cap K = KS$, and thus $(K/KS)_R \cong ((K+I)/I)_R$.

By assumption, R/S is a semisimple Artinian ring, and since $(K+I)/I$ is a finitely generated right R/S-module, it is isomorphic to a direct sum $\oplus_{j=1}^t X_j$ of simple modules X_j. Since R/I is an essential extension of $(K+I)/I$, we can identify R/I with a submodule of the injective envelope of $(K+I)/I$, say $R/I \subseteq \oplus_{j=1}^{t} E(X_j)$. The projection of R/I into each component is a cyclic submodule Rx_j, so in fact we have $R/I \subseteq \oplus_{j=1}^{t} Rx_j$.

By assumption R is a right fully bounded Noetherian ring, and so Theorem 9.12 of [**11**] implies that each submodule Rx_j is Artinian. Furthermore, Corollary 12.14 of [**11**] implies that the annihilator of each composition factor of Rx_j belongs to the clique of the annihilator of X_j. It follows that each annihilator of a composition factor of R/I belongs to the clique of P, and so the same is true for the factor module $R/(K+I)$.

Thus there exists a set of maximal ideals $Q_1, \dots Q_k$ whose product annihilates $R/(K+I)$, and so $\prod_{i=1}^{k} Q_i \subseteq K+I$. Because K annihilates each composition factor of $R/(K+I)$, we have $K \subseteq Q_i$ for $i = 1, \dots, k$. By assumption this implies that each ideal Q_i is one of the original set $\{P_1, \dots, P_m\}$ of maximal ideals. We conclude that $S^n \subseteq K + I$ for some positive integer n. Since S annihilates $(K+I)/I \cong K/KS$, we have $S^{n+1} \subseteq I$, and therefore $K \subseteq I$, so $KS = I \cap K = K$. □

In commutative ring theory, one approach to determining the kernel of a localization map is to utilize symbolic powers. In [**3**] this notion was extended to the setting of universal localization. Let R be a left Noetherian ring, with semiprime ideal S, and let $\lambda : R \to R_{\Gamma(S)}$ be a universal localization of R at S. The n^{th} *symbolic power of* S, denoted by $S^{(n)}$, is defined as $S^{(n)} = \lambda^{-1}(R_{\Gamma(S)}\lambda(S^n)R_{\Gamma(S)})$. We note that Proposition 1.3 of [**2**] shows that this definition is equivalent to the one given by B. J. Mueller in [**14**], where $S^{(n)}$ is defined to be the intersection of all ideals I containing S^n for which $\mathcal{C}(S) \subseteq \mathcal{C}(I)$.

For our purposes, the importance of the symbolic powers is related to their connection to the kernel of the universal localization map. In [**3**] it is shown that if R is a right Noetherian ring, with semiprime ideal S, and $\lambda : R \to R_{\Gamma(S)}$ is the universal localization of R at S, then $S^{(n)} = \lambda^{-1}(J(R_{\Gamma(S)})^n)$.

Corollary 2.2. *Let R be a right fully bounded Noetherian ring, and let S be a semimaximal ideal of R. Suppose that $S = \cap_{i=1}^{m} P_i$ for maximal ideals $\{P_i\}_{i=1}^{m}$ in* $\mathrm{Clq}(P)$*, where $P = P_1$, and let $K = \cap_{n=1}^{\infty} S^n$. Let $\lambda : R \to R_{\Gamma(S)}$ be the canonical mapping of R into the universal localization $R_{\Gamma(S)}$. Then* $\ker(\lambda) = K$.

Proof. Since R is a right fully bounded Noetherian ring and S is a semimaximal ideal, it follows that R/S is semisimple Artinian by Proposition 9.4 of [**11**]. Therefore each factor S^i/S^{i+1} is torsionfree over R/S since it is a module over a semisimple Artinian ring, and so $S^n = S^{(n)}$ for each n. It follows from Proposition 1.9 of [**3**] that $\ker(\lambda) \subseteq \cap_{i=1}^{\infty} S^{(n)} = K$.

Since K is finitely generated as a right ideal and $KS = K$ by Theorem 2.1, it follows directly from Lemma 1.5 of [**1**] that $K \subseteq \ker(\lambda)$. □

When R is a Noetherian ring that is finitely generated as a right module over its center we will construct the universal localization in two steps, first using an Ore localization, and then following it by a universal localization. This procedure uses an extension of Proposition 1.7 of [**1**], which shows that if P is a prime ideal minimal over the semiprime Goldie ideal N, then $R_{\Gamma(P)}$ is the universal localization of $R_{\Gamma(N)}$ at a maximal ideal. For the ideal I of R, we denote the ideal of $R_{\Gamma(S)}$ generated by $\lambda(I)$ by I^e.

THEOREM 2.3. *Let R be a ring that is finitely generated as a module over its Noetherian center, and let S be a semiprime ideal of R, with $S = \cap_{i=1}^{m} P_i$ for prime ideals $\{P_i\}_{i=1}^{m}$ minimal over S. Suppose that each prime ideal P_i belongs to $\mathrm{Clq}(P)$, where $P = P_1$, and let $N = \cap Q_i$ be the intersection of all prime ideals Q_i in $\mathrm{Clq}(P)$.*

(a) *The following conditions are equivalent:*

(1) *$R_{\Gamma(S)}$ is a homomorphic image of R_N;*

(2) *$\cap_{n=1}^{\infty} S^{(n)} \nsubseteq Q$ for any $Q \in \mathrm{Clq}(P)$ such that $Q \neq P_i$ for $i = 1, \ldots, m$.*

(b) *If the conditions given in part (a) hold, then $R_{\Gamma(S)} = R_N / \cap_{n=1}^{\infty} (S^e)^n$.*

PROOF. Since R is assumed to be module-finite over its Noetherian center, and N is the intersection of a clique of prime ideals, it follows that N satisfies the hypotheses of Theorem 14.21 of [**11**], and so N is classically right localizable. Since R is a right fully bounded Noetherian ring, the same is true of R_N, by Lemma 2.1 of [**5**]. Furthermore, it follows from Section 4.3 of [**14**] that S^e is a semi-maximal ideal of R_N. Then Lemma 14.18 of [**11**] gives us that $NR_N = J(R_N)$, and that $R_N/J(R_N) \cong Q_{cl}(R/N)$, so $R_N/(NR_N) \cong Q_{cl}(R/N)$.

Let $\eta : R \to R_N$ and $\lambda : R_N \to (R_N)_{\Gamma(S^e)}$ be the standard ring homomorphisms. A modification of the proof of Proposition 1.7 of [**1**] shows that $\lambda\eta$ is the canonical ring homomorphism from R into $R_{\Gamma(S)}$. (That $(R_N)_{\Gamma(S^e)}$ corresponds to $R_{\Gamma(S)}$ also follows from our Proposition 3.2 given below.) Since R_N is a right fully bounded Noetherian ring and S^e is a semi-maximal ideal, it follows that R_N/S^e is semisimple Artinian by Proposition 9.4 of [**11**]. Therefore each factor $(S^e)^i/(S^e)^{i+1}$ is torsionfree over R_N/S^e since it is a module over a semisimple Artinian ring, and so $(S^e)^n = (S^e)^{(n)}$ for each n. Thus by Theorem 1.10 of [**3**], we have $\ker(\lambda) \subseteq \cap(S^e)^n$. Letting $K = \cap_{i=1}^{\infty}(S^e)^n$, it follows that $\ker(\lambda) \subseteq K$.

(1) $\Rightarrow$ (2) Suppose that $R_{\Gamma(S)}$ is a homomorphic image of R_N. We know that $R_{\Gamma(S)}$ modulo its Jacobson radical is isomorphic to $Q_{cl}(R/S)$. so its Jacobson radical must be S^e. Since R_N is a right fully bounded Noetherian ring, by Theorem 9.13 of [**11**] we have $\cap_{i=1}^{\infty}(S^e)^n \subseteq \ker(\lambda)$, since $R_{\Gamma(S)}$ is also a right fully bounded Noetherian ring. Thus $\ker(\lambda) = K$. Because S^e is the Jacobson radical of $R_{\Gamma(S)}$, the maximal ideals of $R_{\Gamma(S)}$ correspond to the prime ideals of R minimal over S, and so we cannot have $K \subseteq Q^e$ for any other prime ideal Q minimal over N. It follows that $K \nsubseteq Q$ for any $Q \in \mathrm{Clq}(P)$ such that $Q \neq P_i$ for $i = 1, \ldots, m$.

(2) $\Rightarrow$ (1) Suppose that the given condition holds. Then it follows from Corollary 2.2 that $K = \ker(\lambda)$. Hence if we verify that $(R_N)_{\Gamma(S^e)} = R_N/K$, we will obtain the desired result.

Since $N = \cap Q_i$ where the prime ideals Q_i are the elements of the clique of P, all maximal ideals of R_N are of the form Q^e, where $Q \in \mathrm{Clq}(P)$. However, since $K \nsubseteq Q^e$ for any $P \neq Q \in \mathrm{Clp}(P)$, it follows that S^e/K is the intersection of the maximal ideals of R_N/K and so $S^e/K = J(R_N/K)$. It is well-known that two prime ideals P^e and Q^e of R_N are linked in R_N if and only if P and Q are linked in R. (See the remarks on page 12 of [**5**].) It follows that the prime ideals of R_N minimal over S^e form a clique in R_N/K, and hence S^e/K is a localizable semiprime ideal. But then since $S^e/K = J(R_N/K)$ and the classical localization of R_N at S^e coincides with $(R_N)_{\Gamma(S^e)} = R_N/K$, we have $R_{\Gamma(S)} = (R_N)_{\Gamma(S^e)} = R_N/K$. □

COROLLARY 2.4. *If the ring R and semiprime ideal S satisfy the hypotheses of Theorem 2.3, then $R_{\Gamma(S)}$ is a Noetherian ring.*

3. Some general results

We conclude this note with several propositions that shed some light on the general case of universal localization at a semiprime Goldie ideal. If Δ is any set of matrices over R, then there exists a universal Δ-inverting ring R_Δ, which can be constructed in a fashion similar that of $R_{\Gamma(S)}$ (see Theorem 7.2.4 of [**7**]).

Lemma 3.1. *Let Γ and Δ be sets of matrices over the ring R, with $\Delta \subseteq \Gamma$. Let $\lambda : R \to R_\Gamma$ be the universal Γ-inverting map and let $\eta : R \to R_\Delta$ be the universal Δ-inverting map.*

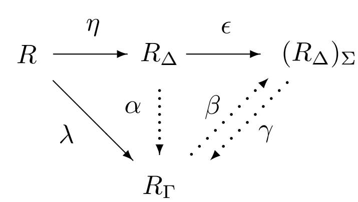
If $\Sigma = \eta(\Gamma)$, and $\epsilon : R_\Delta \to (R_\Delta)_\Sigma$ is the universal Σ-inverting map, then there exists a natural isomorphism $\gamma : (R_\Delta)_\Sigma \to R_\Gamma$ with $\gamma\epsilon\eta = \lambda$.

Proof. Since $\Delta \subseteq \Gamma$, it follows that $\lambda : R \to R_\Gamma$ is a Δ-inverting map. Then since $\eta : R \to R_\Delta$ is a universal Δ-inverting map, there exists a unique homomorphism $\alpha : R_\Delta \to R_\Gamma$ with $\alpha\eta = \lambda$.

Since $\epsilon : R_\Delta \to (R_\Delta)_\Sigma$ inverts $\Sigma = \eta(\Gamma)$, it follows that $\epsilon\eta$ is Γ-inverting. Since $\lambda : R \to R_\Gamma$ is a universal Γ-inverting map, there exists a unique homomorphism $\beta : R_\Gamma \to (R_\Delta)_\Sigma$ with $\beta\lambda = \epsilon\eta$.

Since $\alpha\eta = \lambda$ and λ is Γ-inverting, it follows that $\alpha : R_\Delta \to R_\Gamma$ inverts $\Sigma = \eta(\Gamma)$. Since $\epsilon : R_\Delta \to (R_\Delta)_\Sigma$ is a universal Σ-inverting map, there exists a unique homomorphism $\gamma : (R_\Delta)_\Sigma \to R_\Gamma$ with $\gamma\epsilon = \alpha$.

This leads to the following diagram.



We claim that $\beta = \gamma^{-1}$, which will show that γ is the required isomorphism. We have

$$(\gamma\beta)\lambda = \gamma(\beta\lambda) = \gamma(\epsilon\eta) = (\gamma\epsilon)\eta = \alpha\eta = \lambda$$

and then $\gamma\beta = 1$ since λ is an epimorphism in the category of rings. On the other hand,

$$(\beta\gamma)(\epsilon\eta) = \beta(\gamma\epsilon)\eta = \beta(\alpha\eta) = \beta\lambda = \epsilon\eta$$

and so $\beta\gamma = 1$ since both η and ϵ are epimorphisms in the category of rings. □

Proposition 3.2. *Let S be a semiprime right Goldie ideal of the ring R, and suppose that $\mathcal{C} \subseteq \mathcal{C}(S)$ is a right denominator set. If $R\mathcal{C}^{-1}/SR\mathcal{C}^{-1}$ is the classical ring of right quotients of R/S via the embedding induced by the canonical mapping $\eta : R \to R\mathcal{C}^{-1}$, then the universal localization $R_{\Gamma(S)}$ of R at S can be constructed as the universal localization of $R\mathcal{C}^{-1}$ at the semiprime ideal $SR\mathcal{C}^{-1}$.*

Proof. In order to use Lemma 3.1, let $\overline{\eta} : R/S \to R\mathcal{C}^{-1}/SR\mathcal{C}^{-1}$ be the mapping induced by η, and let $\pi : R \to R/S$ and $\widehat{\pi} : R\mathcal{C}^{-1} \to R\mathcal{C}^{-1}/SR\mathcal{C}^{-1}$ be the canonical projections, as in the following diagram.

$$\begin{array}{ccc} R & \xrightarrow{\eta} & R\mathcal{C}^{-1} \\ \pi \big\downarrow & & \big\downarrow \widehat{\pi} \\ R/S & \xrightarrow{\overline{\eta}} & R\mathcal{C}^{-1}/SR\mathcal{C}^{-1} \end{array}$$

The role of Δ in Lemma 3.1 will be played by the set $\mathcal{C}$, while the role of Σ will be played by $\eta(\Gamma(S))$. It follows from Lemma 3.1 that the universal localization $R_{\Gamma(S)}$ is naturally isomorphic to the universal inverting ring of the set $\eta(\Gamma(S))$ (defined on $R\mathcal{C}^{-1}$). We claim that any ring homomorphism out of $R\mathcal{C}^{-1}$ inverts $\eta(\Gamma(S))$ if and only if it inverts $\Gamma(SR\mathcal{C}^{-1})$, which will complete the proof.

Since $\overline{\eta}$ is the embedding of R/S in its classical ring of left quotients, a matrix C is regular modulo S if and only if $\overline{\eta}\pi(C)$ is invertible, which occurs if and only if $\widehat{\pi}\eta(C)$ is invertible. This shows that $\eta(\Gamma(S)) \subseteq \Gamma(SR\mathcal{C}^{-1})$.

To prove the reverse inclusion, let C be a matrix over $R\mathcal{C}^{-1}$ that is regular modulo $SR\mathcal{C}^{-1}$. By assumption, $R\mathcal{C}^{-1}/SR\mathcal{C}^{-1}$ is a simple Artinian ring, so a matrix is regular modulo $SR\mathcal{C}^{-1}$ if and only if it is invertible modulo $SR\mathcal{C}^{-1}$. In $R\mathcal{C}^{-1}$ we can find a common denominator $d \in \mathcal{C}$ for the entries of C, and hence we can write $C = C^*(dI)^{-1}$, where C^* has entries in $\eta(R)$, say $C^* = \eta(C')$, and I is the identity matrix of the appropriate size. Because dI is invertible over $R\mathcal{C}^{-1}$, it follows that $\widehat{\pi}(C)$ is invertible if and only if $\widehat{\pi}(C^*)$ is invertible. Furthermore, any ring homomorphism out of $R\mathcal{C}^{-1}$ will invert C if and only if it inverts C^*. In particular, since C is regular modulo $SR\mathcal{C}^{-1}$, this implies that $\widehat{\pi}$ inverts C, so $\overline{\eta}\pi(C') = \widehat{\pi}\eta C'$ is invertible, and therefore $C' \in \Gamma(S)$. This shows that any ring homomorphism out of $R\mathcal{C}^{-1}$ will invert $\eta(\Gamma(S))$ if and only if it inverts $\Gamma(SR\mathcal{C}^{-1})$, and allows us to conclude that the universal $\Gamma(SR\mathcal{C}^{-1})$-inverting ring is naturally isomorphic to the universal $\eta(\Gamma(S))$-inverting ring. □

PROPOSITION 3.3. *Let S be a semiprime right Goldie ideal of R, and let Z be the center of R. If $S \cap Z = N$, then $R_{\Gamma(S)}$ is an algebra over Z_N.*

PROOF. Let $\lambda : R \to R_{\Gamma(S)}$ be the natural mapping. Each element $c \in Z \setminus N$ is regular modulo S, so it is inverted by λ. The universal property defining the localization Z_N yields a homomorphism $\mu : Z_N \to R_{\Gamma(S)}$ such that the following diagram is commutative.

$$\begin{array}{ccc} Z & \subseteq & R \\ \big\downarrow & & \big\downarrow \lambda \\ Z_N & \xrightarrow{\mu} & R_{\Gamma(S)} \end{array}$$

To show that μ defines an algebra structure on $R_{\Gamma(S)}$ it is only necessary to show that the image of μ is in the center of $R_{\Gamma(S)}$. For $b, d \in Z$ with $d \in Z \setminus N$, we have $\mu(d^{-1}b) = \lambda(d)^{-1}\lambda(b)$, and it is routine to show that such elements lie in the center of $R_{\Gamma(S)}$, using the fact that each element of $R_{\Gamma(S)}$ has the form $\lambda(a)\lambda(C)^{-1}\lambda(r^t)$, for some $a, r \in R^n$ and $C \in \Gamma_n(S)$. □

The final two results are well-known in case the semiprime ideal S is localizable.

PROPOSITION 3.4. *Let S be a semiprime ideal of R. If $R_{\Gamma(S)}$ is right Noetherian and I is any ideal of R, then $I^e = \lambda(I)R_{\Gamma(S)}$.*

PROOF. We need to show that $\lambda(I)R_{\Gamma(S)}$ is a left ideal of $R_{\Gamma(S)}$, where $\lambda : R \to R_{\Gamma(S)}$ is the canonical ring homomorphism. Let $x \in \lambda(I)R_{\Gamma(S)}$, with $x = \sum_{i=1}^m \lambda(a_i)q_i$, where $q_i \in R_{\Gamma(S)}$ and $a_i \in I$, for each i. If q is any element of $R_{\Gamma(S)}$, then since $\Gamma(S)$ contains all permutation matrices and is closed under products, we may assume without loss of generality that q is the $(1,1)$–entry of $\lambda(C)^{-1}$ for some $C \in \Gamma_n(S)$. In the matrix ring $M_n(R_{\Gamma(S)})$, we have

$$\lambda(C)M_n(\lambda(I)) \subseteq M_n(\lambda(I))$$

since I is an ideal of R. Thus

$$M_n(\lambda(I)) \subseteq \lambda(C)^{-1}M_n(\lambda(I)) \ .$$

Since $R_{\Gamma(S)}$ is right Noetherian, so is $M_n(R_{\Gamma(S)})$. The ascending chain of right ideals

$$\lambda(C)^{-1}M_n(\lambda(I))M_n(R_{\Gamma(S)}) \subseteq \lambda(C)^{-2}M_n(\lambda(I))M_n(R_{\Gamma(S)}) \subseteq \dots$$

must terminate at some point, say

$$\lambda(C)^{-k}M_n(\lambda(I))M_n(R_{\Gamma(S)}) = \lambda(C)^{-k-1}M_n(\lambda(I))M_n(R_{\Gamma(S)}) \ .$$

This implies that

$$M_n(\lambda(I))M_n(R_{\Gamma(S)}) = \lambda(C)^{-1}M_n(\lambda(I))M_n(R_{\Gamma(S)}) \ .$$

The element $q\left(\sum_{i-1}^m \lambda(a_i)q_i\right)$ is the $(1,1)$–entry of the matrix

$$\lambda(C)^{-1}\left(\sum_{i=1}^m \begin{bmatrix} \lambda(a_i) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q_i & 0 \\ 0 & 0 \end{bmatrix}\right)$$

which belongs to $M_n(\lambda(I))M_n(R_{\Gamma(S)})$, and so it must have the form required to be an element of $\lambda(I)R_{\Gamma(S)}$. □

PROPOSITION 3.5. *Let S be a semiprime ideal of R. If $R_{\Gamma(S)}$ is Noetherian as a right module over R, then $R_{\Gamma(S)} = \lambda(R)$, where $\lambda : R \to R_{\Gamma(S)}$ is the canonical mapping.*

PROOF. Let $q \in R_{\Gamma(S)}$ and suppose that q is an entry in the matrix $\lambda(C)^{-1}$, for $C \in \Gamma_n(S)$. It is sufficient to show that $\lambda(C)^{-1}$ has entries in $\lambda(R)$. By assumption, the matrix ring $T = M_n(R_{\Gamma(S)})$ is Noetherian as a module over R, so we may consider the ascending chain of submodules $\lambda(C)^{-1}T \subseteq \lambda(C)^{-2}T \subseteq \dots$. By assumption this chain must terminate after finitely many steps, say $\lambda(C)^{-k}T = \lambda(C)^{-k-l}T$. Therefore $\lambda(C)^{-1}T = M_n(\lambda(R))$, and so the entries of $\lambda(C)^{-1}$ belong to $\lambda(R)$. □

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