WE SHALL assume that all rings under consideration are associative rings with identity, and that all modules are unital. We show that a ring $R$ is a prime ring if and only if every nonzero torsionless left $R$-module is faithful. This result can be extended to characterize the prime ideals of a ring $R$ in terms of its left $R$-modules.

With each left $R$-module $RM$ is associated an ideal $\text{Ann}(M) = \{r \in R : rm = 0 \text{ for all } m \in M\}$, the annihilator of $M$. The module $RM$ is called faithful if $\text{Ann}(M) = (0)$.

We may also define a left ideal $tr(M)$ associated with $RM$, the sum of the left ideals $f(m)$, for all $R$-homomorphisms $f \in \text{Hom}_R(M, R)$. For each $f \in \text{Hom}_R(M, R)$ and each $r \in R$, the function $g$ defined by $g(m) = f(m)r$, for all $m \in M$, is also a member of $\text{Hom}_R(M, R)$. This can be used to show that in fact $tr(M)$ is an ideal of $R$.

The module $RM$ is said to be torsionless if for each $0 \neq m \in M$ there exists $f \in \text{Hom}_R(M, R)$ such that $f(m) \neq 0$. It is clear that each left ideal of $R$ is torsionless when considered as a left $R$-module.

**Lemma.** Let $RM$ be torsionless. Then $M$ is faithful $\iff \text{Ann}(tr(M)) = (0)$.

**Proof.** $\Rightarrow$. Suppose that $RM$ is torsionless and faithful. Then $\text{Ann}(M) = (0)$, and for each $0 \neq r \in R$ there exists $m \in M$ such that $rm \neq 0$. Since $M$ is torsionless there exists $f \in \text{Hom}_R(M, R)$ such that $f(rm) \neq 0$, and thus $f(m) \in tr(M)$ and $rf(m) \neq 0$. This shows that $\text{Ann}(tr(M)) = (0)$.

$\Leftarrow$. If $r \in \text{Ann}(M)$, then for each $m \in M$ and $f \in \text{Hom}_R(M, R)$ we must have $rf(m) = f(rm) = 0$. Thus $\text{Ann}(M) \subseteq \text{Ann}(tr(M))$, so $\text{Ann}(tr(M)) = (0)$ implies $\text{Ann}(M) = (0)$. Q.E.D.
A ring $R$ is called a prime ring if for all ideals $A, B$ of $R$, $A \cdot B = (0)$ implies $A = (0)$ or $B = (0)$. An ideal $A$ of $R$, $A \neq R$, is called a prime ideal if the quotient ring $R/A$ is a prime ring.

**Theorem 1.** The ring $R$ is a prime ring $\iff$ every non-zero torsionless left $R$-module is faithful.

**Proof.** $\Rightarrow$. Assume that $R$ is a prime ring and let $R^M$ be a non-zero torsionless left $R$-module. This implies that $tr(M) \neq 0$, and then $(\text{Ann}(tr(M))) \cdot (tr(M)) = (0)$. Because we have assumed that $R$ is a prime ring, we must have $\text{Ann}(tr(M)) = (0)$, and it follows from the previous lemma that $R^M$ is faithful.

$\Leftarrow$. Assume that every non-zero torsionless left $R$-module is faithful. If $A, B$ are ideals of $R$ and $B \neq (0)$, then $R^B$ is a non-zero torsionless left $R$-module. By assumption, $R^B$ is faithful so $A \cdot B = (0)$ implies $A \subseteq \text{Ann}(B) = (0)$. Thus for ideals $A, B$ of $R$, $A \cdot B = (0)$ implies $A = (0)$ or $B = (0)$, and $R$ is a prime ring. Q.E.D.

In order to generalize this result, it is convenient to adopt the following terminology.

**Definition.** Let $R^M$ and $R^N$ be non-zero left $R$-modules. If for each $0 \neq m \in M$ there exists $f \in \text{Hom}_R(M, N)$ such that $f(m) \neq 0$, then we will write $R^M > R^N$. If $R^M > R^N$ and $R^N > R^M$, then we will write $R^M \sim R^N$.

With this definition we note that $R^M$ is a non-zero torsionless left $R$-module $\iff R^M > R^M$. If $R^M$ is any module, then for each $m \in M$ there exists $f \in \text{Hom}_R(R, M)$ defined by $f(r) = rm$, for all $r \in R$, and since $R$ has an identity, all $R$-homomorphisms from $R$ to $M$ are of this form. It is clear that $R(R/\text{Ann}(M)) > R^M$, and that $R^R > R^M \iff R^M$ is faithful. Thus Theorem 1 can be restated in the following form: The ring $R$ is a prime ring if and only if $R^M > R^R \Rightarrow R^M \sim R^R$.

Suppose that $R^M > R^N$. If $r \notin \text{Ann}(M)$, then there exists $m \in M$ such that $rm \neq 0$, so there exists $f \in \text{Hom}_R(M, N)$ such that $f(rm) \neq 0$. Thus $rf(m) \neq 0$, which shows that $r \notin \text{Ann}(N)$, and so $R^M > R^N$ implies $\text{Ann}(N) \subseteq \text{Ann}(M)$.

If $R^M > R^N$ and $R^N > R^P$, then for $0 \neq m \in M$ there exists $f \in \text{Hom}_R(M, N)$ such that $f(m) \neq 0$. Since $R^N > R^P$, there exists
$g \in \text{Hom}_R(N,P)$ such that $g(f(m)) \neq 0$, and thus $RM > RP$. This also shows that $RM \sim RN = RP \Rightarrow RM \sim RP$.

**Theorem 2.** Let $A$ be an ideal of $R$, $A \neq R$. Then the following are equivalent:

(i) $A$ is a prime ideal;

(ii) $RM > R(R/A) \Rightarrow RM \sim R(R/A)$.

**Proof.** If $RM > R(R/A)$, then $A = \text{Ann}(R/A) \subseteq \text{Ann}(M)$, and $M$ is a left $R/A$-module. Every $R$-homomorphism from $M$ to $R/A$ is also an $R/A$-homomorphism, so $R_A M > R_A(R/A)$. Conversely, every left $R/A$-module can be regarded as a left $R$-module, and $R_A M > R_A(R/A) \Rightarrow RM \sim R(R/A)$. The theorem then follows immediately from the restatement of Theorem 1, since, by definition, $A$ is a prime ideal if and only if $R/A$ is a prime ring.

**Theorem 3.** For a module $RP$ the following are equivalent:

(i) $RM > RP \Rightarrow RM \sim RP$;

(ii) $RP \sim R(R/A)$ for a prime ideal $A$ of $R$.

**Proof.** (i) $\Rightarrow$ (ii). Assume that $RM > RP \Rightarrow RM \sim RP$. Let $A = \text{Ann}(P)$. Then $A \neq R$, and $R(R/A) > RP \Rightarrow R(R/A) \sim RP$. We will show that $A$ is a prime ideal, using Theorem 2. If $RM > R(R/A)$, then $RM > RP$ and therefore $RM \sim RP$, from which it follows that $RM \sim R(R/A)$.

(ii) $\Rightarrow$ (i). Assume that $RP \sim R(R/A)$. Then $RM > RP$ implies $RM > R(R/A)$. By Theorem 2, $RM \sim R(R/A)$, and then $RM \sim RP$. Q.E.D.

**References**


Gangadhar Meher College
Sambalpur, Orissa, India
and
Northern Illinois University
DeKalb, Illinois, U.S.A.
The results in this paper were put into the proper context in my paper *On Maximal Torsion Radicals*, Can. J. Math. 25 (1973), 712–726. The relevant part of that paper is included below.

**Maximal radicals and prime ideals**

A subfunctor $\rho$ of the identity on $R$-$\text{Mod}$ is a functor such that for all $M \in R$-$\text{Mod}$, $\rho(M)$ is a submodule of $M$, and if $f \in \text{Hom}_R(M,N)$, then $f(\rho(M)) \subseteq \rho(N)$. Such a functor $\rho$ is called a *radical* of $R$-$\text{Mod}$ if $\rho(M/\rho(M)) = 0$ for all $M \in R$-$\text{Mod}$. A radical is *proper* if it is not the identity functor on $R$-$\text{Mod}$, or, equivalently, if $\rho(R) \neq R$.

If $\rho$ and $\sigma$ are radicals with $\rho(M) \subseteq \sigma(M)$ for all $M \in R$-$\text{Mod}$, we write $\rho \leq \sigma$, and if $\rho$ is a radical then we call $\rho$ a *maximal radical* if $\rho$ is proper and for any other radical $\sigma$ with $\rho \leq \sigma$, either $\rho = \sigma$ or $\sigma$ is the identity on $R$-$\text{Mod}$.

If $\rho$ is a radical, then a module $rM$ is called *$\rho$-torsion* if $\rho(M) = M$, and *$\rho$-torsionfree* if $\rho(M) = 0$. A submodule $M_0 \subseteq M$ is called *$\rho$-dense* if $M/M_0$ is $\rho$-torsion, and *$\rho$-closed* if $M/M_0$ is $\rho$-torsionfree. A (left) ideal $A$ of $R$ is a *maximal* $\rho$-closed (left) ideal if it is maximal in the set of proper $\rho$-closed (left) ideals.

For any module $rN$, we define $\text{rad}_N : R$-$\text{Mod} \to R$-$\text{Mod}$ by setting

$$\text{rad}_N(M) = \bigcap_{f \in \text{Hom}_R(M,N)} \ker(f)$$

for all $M \in R$-$\text{Mod}$. Then it can be shown that $\text{rad}_N$ is a radical, and that $\text{rad}_N(R) = \text{Ann}(N)$.

If $\rho$ is a radical and $rN$ is $\rho$-torsionfree, then for any module $rM$ and any $f \in \text{Hom}_R(M,N)$ we must have $f(\rho(M)) \subseteq \rho(N) = 0$, and thus $\rho \leq \text{rad}_N$. On the other hand, if $\rho \leq \text{rad}_N$, then $\text{rad}_N(N) = 0$ implies $\rho(N) = 0$. Therefore $\rho \leq \text{rad}_N$ if and only if $\rho(N) = 0$. This result will prove to be useful in the characterization of maximal radicals.

**Lemma.** Let $A$ be an ideal of $R$. Then $A$ is a prime ideal if and only if $\text{rad}_N \geq \text{rad}_{R/A}$ implies $\text{Ann}(N) = A$, for all nonzero $N \in R$-$\text{Mod}$.

**Proof.** Assume that $A$ is a prime ideal and that $\text{rad}_N \geq \text{rad}_{R/A}$. Then $\text{rad}_{R/A}(N) = 0$, so there exists $0 \neq f \in \text{Hom}_R(N,R/A)$ since $N \neq 0$, which implies that $f(N) \neq 0$. Because $R/A$ is a prime ring and $\text{Ann}(N) \cdot f(N) = 0$, it follows that $\text{Ann}(N) \subseteq A$. On the other hand, by assumption $\text{Ann}(N) = \text{rad}_N(R) \geq \text{rad}_{R/A}(R) = \text{Ann}(R/A) = A$.

Conversely, let $B$ and $C$ be ideals of $R$ with $BC \subseteq A$. If $A \subseteq C$, then $C/A \neq 0$ and $\text{rad}_{C/A} \geq \text{rad}_{R/A}$. By assumption $A = \text{Ann}(C/A) \supseteq B$, and this is sufficient to show that $A$ is a prime ideal. Q.E.D.
PROPOSITION. Let $A$ be an ideal of $R$. Then $A$ is a prime ideal if and only if $A$ is a maximal $\rho$-closed ideal for a radical $\rho$.

PROOF. If $A$ is a prime ideal, let $\rho = \text{rad}_{R/A}$. Then $A$ is $\rho$-closed, and if $B$ is any proper $\rho$-closed ideal, then $\text{rad}_{R/A} \geq \text{rad}_{R/A}$, and the lemma implies that $B = \text{Ann}(R/B) \subseteq A$.

Conversely, if $A$ is a maximal $\rho$-closed ideal and $\text{rad}_{N} \geq \text{rad}_{R/A}$ for some $0 \neq N \in R$-Mod, then $\text{Ann}(N) \supseteq A$ and $\text{Ann}(N)$ is $\rho$-closed since $\text{rad}_{N} \geq \rho$. By assumption we must have $\text{Ann}(N) = A$, and then the lemma implies that $A$ is a prime ideal. Q.E.D.

THEOREM. Let $\rho$ be a radical of $R$-Mod, with $\rho(R) = A$. Then $\rho$ is a maximal radical if and only if $\rho = \text{rad}_{R/A}$ and $A$ is a prime ideal.

PROOF. Suppose that $\rho$ is a maximal radical. Then $A \neq R$ and $\rho(R/A) = 0$ implies $\rho \leq \text{rad}_{R/A}$, so $\rho = \text{rad}_{R/A}$ since $\rho$ is maximal. Furthermore, $A$ is a maximal $\rho$-closed ideal since any larger $\rho$-closed ideal would determine a larger radical. The proposition then shows that $A$ is a prime ideal.

Conversely, if $A$ is a prime ideal and $\rho = \text{rad}_{R/A}$, then the proof of the proposition shows that $A$ is a maximal $\rho$-closed ideal. If $\alpha$ is a radical with $\alpha \geq \rho$, then $\alpha(R)$ is $\rho$-closed and contains $A$. Hence either $\alpha(R) = R$ or $\alpha(R) = A$ and $\alpha \leq \text{rad}_{R/A} = \rho$. Q.E.D.

COROLLARY. Every proper radical of $R$-Mod is contained in a maximal radical if and only if for each $0 \neq M \in R$-Mod there exists a submodule $M_0 \subseteq M$ such that $\text{Ann}(M_0)$ is a prime ideal.

PROOF. If $0 \neq M \in R$-Mod and every proper radical is contained in a maximal radical, then the theorem shows that there exists a prime ideal $P$ with $\text{rad}_{R/P} \geq \text{rad}_M$. Let $M_0 = \{m \in M : Pm = 0\}$. It can easily be shown that $\text{Ann}(M_0) = P$.

Conversely, let $\rho$ be a proper radical with $\rho(R) = A$. By assumption there exists a left ideal $A \supseteq B$ for which $\text{Ann}(B/A) = P$ is a prime ideal. Thus $\rho \leq \text{rad}_{R/A} \leq \text{rad}_{R/P}$, and $\text{rad}_{R/P}$ is a maximal radical. Q.E.D.