Ann. Sc. Math. Québec, 1977, Vol. I, No 1, pp. 59-62

ON LEFT FBN RINGS[1]

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Let R be a left Noetherian ring with identity. (All modules considered are unital left modules.) The ring R is said to be left FBN if for each prime ideal P of R, each left ideal of R/P that is essential in R/P contains a nonzero two-sided ideal. It is well known ([6], Proposition VII 2.4) that R is left FBN if for each finitely generated R-module M there exist $m_1, \ldots, m_n \in M$ such that $\mathrm{Ann}(M) = \mathrm{Ann}(m_1, \ldots, m_n)$, and Cauchon has shown in [3] that the converse is true. In this note we give a local version of the above result, and we show that R is left FBN if and only if this local condition holds for each minimal prime torsion theory.

For each R-module M, a torsion theory τ_M is defined in the following way: $\tau_M(X) = \{x \in X \mid f(x) = 0 \text{ for each } f \in \mathrm{Hom}_R(X, \mathrm{E}(M))\}$, where $\mathrm{E}(M)$ denotes the injective envelope of M. Let σ be a torsion theory of R–Mod. Then an R-module X is called σ-torsion (σ-torsionfree) if $\sigma(X) = X$ ($\sigma(X) = 0$), and a submodule Y of X is called σ-dense (σ-closed) if X/Y is σ-torsion (σ-torsionfree). A torsion theory π is said to be prime if there exists a uniform R-module U such that $\pi = \tau_U$. In this case, if $\mathrm{Ann}(x)$ is maximal in the set $\{\mathrm{Ann}(u) \mid u \in U\}$, then each nonzero submodule of Rx is π-dense, and as a consequence the localization $(Rx)_\pi$ is a minimal subobject of U_π. If the ideal P is maximal in the set of annihilators of submodules of U, then P is a prime ideal of R, and $\tau_{R/P}$ is a prime in R–Mod satisfying $\tau_{R/P} \geq \pi$ (there exists a uniform submodule A of R/P such that $\tau_{R/P} = \tau_A$ by Theorem 3.9 of [5]). The ideal P is called an associated prime ideal of U. We denote by $\mathrm{ass}(M)$ the set of prime torsion theories π such that there exists a uniform submodule U of M such that $\pi = \tau_U$.

[1] This is a translation of the original paper, which appeared in French.

[2] I would like to thank Professor J. Lambek and the Department of Mathematics of McGill University for their hospitality.

Lemma 1 *Let π be a prime torsion theory of R–Mod, and let M be a finitely generated R-module such that* $\mathrm{ass}(M) = \{\pi\}$. *If $\{M_\alpha\}_{\alpha\in\Lambda}$ is a family of π-closed submodules of M such that $\cap_{\alpha\in\Lambda} M_\alpha = 0$, then there exists a finite subset Φ of Λ such that $\cap_{\alpha\in\Phi} M_\alpha = 0$.*

Proof. Since M is a Noetherian module and $\mathrm{ass}(M) = \{\pi\}$, M contains an essential submodule $\oplus_{i=1}^n U_i$, such that each module U_i is uniform and defines π. Therefore $\pi M = 0$ and M_π contains an essential subobject $\oplus_{i=1}^n M_i$, such that each subobject M_i is minimal in M_π. The one-to-one correspondence ([6], Corollary IX 4.4) between the set of π-closed submodules of M and the set of subobjects of M_π (which preserves intersections) shows that $\cap_{\alpha\in\Lambda}(M_\alpha)_\pi = 0$, and so there exists a finite subset Φ of Λ such that $\cap_{\alpha\in\Phi}(M_\alpha)_\pi = 0$, and thus $\cap_{\alpha\in\Phi} M_\alpha = 0$. □

Theorem 2 *If σ is a torsion theory of R–Mod, then the following conditions are equivalent.*

(1) *For each σ-closed prime ideal P of R, each essential σ-closed left ideal of R/P contains a nonzero two-sided ideal.*

(2) *For each finitely generated σ-torsionfree R-module M, there exist elements $m_1, \ldots, m_n \in M$ such that* $\mathrm{Ann}(M) = \mathrm{Ann}(m_1, \ldots, m_n)$.

Proof. (1) $\Rightarrow$ (2). Let M be a finitely generated σ-torsionfree R-module. It is sufficient to establish condition (2) when M is a uniform R-module, since each finitely generated R-module X has a decomposition $\cap_{i=1}^n X_i = 0$ such that X/X_i is uniform. Let $\pi = \tau_M$, $A = \mathrm{Ann}(M)$, and $\rho \in \mathrm{ass}(R/A)$. Then there exists a prime ideal P associated to ρ, and it is easy to verify that there exists a submodule N of M such that $P = \mathrm{Ann}(N)$, so P is σ-closed in R. If we show that $\tau_{R/P} = \pi$, then $\rho = \pi$ and $\mathrm{ass}(R/A) = \{\pi\}$, and it follows, from Lemma 1, that there exists a finite subset of the set $\{\mathrm{Ann}(m) \mid m \in M\}$ such that $A = \cap_{i=1}^n \mathrm{Ann}(m_i)$.

If $\mathrm{Hom}_R(N', R/P) = 0$ for each submodule N' of N, then N is a singular R/P-module, and if $N = Rx_1 + \ldots + Rx_k$, then $\cap_{i=1}^k \mathrm{Ann}(x_i)/P$ is an essential σ-closed left ideal of R/P; therefore it contains a nonzero two-sided ideal I/P, and $IN = 0$, which is impossible since $P = \mathrm{Ann}(N)$. We can therefore conclude that there exists a submodule N' of N such that $\mathrm{Hom}_R(N', R/P) \neq 0$, and if $f(N') \neq 0$, then there exist $y_1, \ldots, y_m \in N'$ such that $P = \mathrm{Ann}(f(y_1), \ldots, f(y_m)) = \mathrm{Ann}(y_1, \ldots, y_m)$, since P is a prime ideal and there does not exist an infinite descending chain of annihilators of R/P. This shows that $\mathrm{E}(R/P)$ is contained in a finite direct sum of copies of $\mathrm{E}(M)$, and Azumaya's theorem ([6], Proposition V 5.4) implies that there exists a submodule A of R/P such that $\mathrm{E}(M) \simeq \mathrm{E}(A)$. Therefore $\tau_{R/P} = \rho$.

(2) $\Rightarrow$ (1). If P is a prime ideal of R and C/P is an essential σ-closed left ideal of R/P, then $R/\operatorname{Ann}(R/C)$ is contained in a finite direct sum of copies of R/C. This shows that $P \subset \operatorname{Ann}(R/C) \subseteq C$, since R/C is a singular R/P module but R/P is a nonsingular R/P-module. □

If R satisfies the conditions of Theorem 2, then it is easy to check that the correspondence which associates to each indecomposable injective R-module E the unique prime ideal maximal in the set of annihilators of submodules of E induces a one-to-one correspondence between the set of isomorphism classes of σ-torsionfree indecomposable injective R-modules and the set of σ-closed prime ideals of R.

Corollary 3 *If σ is a maximal torsion theory of R–Mod, then for each finitely generated σ-torsionfree R-module there exist elements $m_1, \ldots, m_n \in M$ such that* $\operatorname{Ann}(M) = \operatorname{Ann}(m_1, \ldots, m_n)$.

Proof. If σ is maximal, then there exists a minimal prime ideal P such that $\sigma = \tau_{R/P}$ ([2], Theorem 4.6) and P is the only σ-closed prime ideal in R ([1], Proposition 1.2). If A/P is an essential left ideal of R/P, then $\operatorname{Hom}_R(B/A, R/P) = 0$ for each left ideal B such that $A \subseteq B \subseteq R$, since B/A is a singular R/P-module, but R/P is a nonsingular R/P-module. Therefore A/P is not σ-closed in R/P, and thus condition (1) of Theorem 2 is trivially satisfied. □

Theorem 4 *The ring R is left FBN if and only if each torsion theory minimal in the set of prime torsion theories of R–Mod satisfies the conditions of Theorem 2.*

Proof. If R is left FBN, then the conditions of Theorem 2 are satisfied for every torsion theory.

Conversely, it is sufficient to show that for each finitely generated uniform R-module M there exist $m_1, \ldots, m_n \in M$ with $\operatorname{Ann}(M) = \operatorname{Ann}(m_1, \ldots, m_n)$. Let C be a left ideal of R maximal in the set of τ_M-closed left ideals of R. If π is prime in R–Mod and π is strictly contained in τ_M, then C is strictly contained in a left ideal B maximal in the set of π-closed left ideals of R. Since R is left Noetherian, this construction eventually yields a torsion theory σ minimal in the set of prime torsion theories, and $\sigma \leq \tau_M$. Therefore $\sigma M \subseteq \tau_M(M) = 0$

and condition (2) of Theorem 2 show that there exist $m_1, \ldots, m_n \in M$ such that $\mathrm{Ann}(M) = \mathrm{Ann}(m_1, \ldots, m_n)$. $\square$

We note that if each prime torsion theory of R–Mod is maximal, then it follows from Theorem 4 and Corollary 3 that R is left FBN. Therefore R is left Artinian, since each prime ideal of R is maximal ([6], Proposition VIII 1.14). This gives a new proof of Theorem 5.10 of Goldman [4].

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