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INJECTIVE MODULES WITH BOTH ASCENDING AND DESCENDING CHAIN CONDITIONS ON ANNIHILATORS

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Jategaonkar [**9**, Proposition 1.9] has shown that if P is a minimal prime ideal of a left Noetherian ring R, then R has finite length with respect to the torsion radical defined at P. On the other hand, Goel, Jain and Singh [**5**] have shown that if a semiprime ring R has finite length with respect to a torsion radical τ and $\tau(R) = 0$, then R is a Goldie ring and τ is defined by the injective envelope $E(R)$ of R. In this case τ is defined by a finite set of minimal prime Goldie ideals of R.

This paper gives necessary and sufficient conditions under which an injective module with the ascending chain condition on annihilators must also have the descending chain condition on annihilators. The results will be stated in terms of the torsion radical defined by the injective module under consideration, as in [**9**] and [**5**]. In particular, Theorem 7 states that R has finite length with respect to the torsion radical τ if and only if R is τ-Noetherian and τ is defined by a finite set of prime ideals, each of which is minimal in the set of prime Goldie ideals of R. In addition, Proposition 6 shows that Jategaonkar's result [**9**, Proposition 1.9] can be extended to rings with Krull dimension.

Throughout the paper, R will denote an associative ring with identity, and R–Mod will denote the category of unital left R-modules. The reader is referred to [**7**] and [**10**] for background material and for any undefined terms. Note that a torsion radical is called an idempotent kernel functor by Goldman and a left exact radical by Stenström.

If ${}_RW$ is an injective R-module, then for a module ${}_RM$ let $\text{rad}_W(M)$ be the intersection of all kernels of R-homomorphisms from M to W. This defines a torsion radical rad_W of R–Mod, and, moreover, every torsion radical of R–Mod can be defined in this way. If τ is a torsion radical, then ${}_RM$ is called τ-torsion if $\tau(M) = M$ and τ-torsionfree if $\tau(M) = 0$; a submodule $N \subseteq M$ is said to be τ-dense if M/N is τ-torsion and τ-closed if M/N is τ-torsionfree. Note that a left ideal $A \subseteq R$ is rad_W-closed if and only if A is the left annihilator of a subset of W. If τ and σ are torsion radicals, then $\tau \leq \rho$ if $\tau(M) \subseteq \rho(M)$ for all modules ${}_RM$. With this notation, $\text{rad}_{E(M)}$ is the smallest torsion radical for which M is torsionfree. For a prime ideal $P \subseteq R$, $\text{rad}_{E(R/P)}$ is called the torsion radical defined by P.

To each torsion radical τ of $R\text{–}Mod$ corresponds a quotient category $R\text{–}Mod/\tau$, where a module ${}_RM$ belongs to $R\text{–}Mod/\tau$ if and only if M is τ-torsionfree and τ-closed in its injective envelope. The quotient functor (or localization functor) Q_τ : $R\text{–}Mod \to R\text{–}Mod/\tau$ assigns to M the intersection in $E(M/\tau(M))$ of all τ-closed submodules which contain $M/\tau(M)$. Usually the notation M_τ will be used in place of $Q_\tau(M)$. Note that there is a one-one correspondence between subobjects of M_τ in $R\text{–}Mod/\tau$ and τ-closed submodules of M.

A nonzero torsion radical π is said to be prime if there exists a module M such that $\pi = \text{rad}_{E(M)}$ and M_π is simple in $R\text{–}Mod/\pi$. Any nonzero module which satisfies this condition is called monoform. Equivalently, ${}_RM$ is monoform if each nonzero endomorphism of M is a monomorphism. A prime torsion radical π is said to be an associated prime of a module M if π is defined by a monoform submodule of M; π is said to belong to the support $\text{supp}(M)$ of M if $M_\pi \neq 0$.

Following Goldman [**7**], a module ${}_RM$ is said to have finite length with respect to the torsion radical τ if it satisfies both the ascending and descending chain conditions on τ-closed submodules. Equivalently, M_τ has finite length in $R\text{–}Mod/\tau$. In particular, R has finite length with respect to τ if and only if the injective module which defines τ satisfies both chain conditions on annihilators. If M satisfies the ascending chain condition on τ-closed submodules it is called τ-Noetherian.

More generally, Jategaonkar has introduced in [**9**] a relative

Krull dimension, defined with respect to a torsion radical τ, by considering the deviation of the lattice of τ-closed submodules of a module. Equivalently, one can consider the Krull dimension in $R\text{–}Mod/\tau$ of the localization of a module. Thus for a module ${}_RM$ the relative Krull dimension $K_\tau \dim(M)$ is defined inductively as follows: $K_\tau \dim(M) = -1$ if $M_\tau = 0$ (i.e. M is τ-torsion); $K_\tau \dim(M) = 0$ if M_τ is Artinian in $R\text{–}Mod/\tau$; if α is an ordinal and $K_\tau \dim(M) \not< \alpha$, then $K_\tau \dim(M) = \alpha$ provided there is no infinite descending chain $M_\tau = X_0 \supseteq X_1 \supseteq \ldots$ of subobjects X_i of M_τ such that for $i = 1, 2, \ldots$, $K_\tau \dim(X_{i-1}/X_i) \not< \alpha$. The relative Krull dimension of the ring R is defined to be the relative Krull dimension of the module ${}_RR$. A module M which is not τ-torsion is said to be α-critical with respect to τ if $K_\tau \dim(M) = \alpha$ and $K_\tau \dim(M/N) < \alpha$ for each τ-closed submodule N which properly contains $\tau(M)$. The proofs of Theorem 2.1 and Proposition 1.3 of [**3**] remain valid in the above setting. Thus every module with relative Krull dimension contains an α-critical submodule, and any τ-Noetherian module has relative Krull dimension (with respect to τ).

Throughout the paper τ will denote a fixed torsion radical of $R\text{–}Mod$.

LEMMA (1). Let ${}_RM$ be a τ-torsionfree module which has relative Krull dimension with respect to τ.

(a) M has finite uniform dimension.

(b) Every nonzero submodule of M contains a monoform submodule.

Proof. (a) If M contains an infinite durect sum $\oplus_{i=1}^{\infty} M_i$ of nonzero submodules, then M_τ contains an infinite coproduct $\coprod_{i=1}^{\infty}(M_i)_\tau = Q_\tau(\oplus_{i=1}^{\infty} M_i)$ of nonzero objects in R–Mod/τ. This is a contradiction, since by assumption M_τ has Krull dimension in R–Mod/τ, and the proof of [**8**, Proposition 1.4] remains valid in R–Mod/τ.

(b) If N is a nonzero submodule of M, then N_τ has Krull dimension in R–Mod/τ, so N_τ contains a subobject X which is α-critical with respect to τ, for some ordinal α. If $0 \neq f : N \cap X \to N \cap X$, then f extends uniquely to $Q_\tau : X \to X$, since $N \cap X$ is τ-dense in X. Now $Q_\tau(f) \neq 0$, so it must be a monomorphism since X is α-critical. Thus f is a monomorphism and $N \cap X$ is the required monoform submodule of N. □

PROPOSITION (2). If R has relative Krull dimension with respect to τ and I is a τ-closed semiprime ideal of R, then R/I is a semiprime Goldie ring.

Proof. By assumptiion R/I is τ-torsionfree and has relative Krull dimension with respect to τ, so by Lemma 1, R/I has finite uniform dimension and enough monoform left ideals. The desired conclusion then follows from [**8**, Theorem 3.3]. □

PROPOSITION (3). Assume that R has relative Krull dimension with respect to τ. If $R \supset P_0 \supset P_1 \supset \ldots \supset P_n$ is a proper chain of τ-closed prime ideals of R, then $K_\tau \dim(R) \geq n$.

Proof. By Proposition 2, each of the rings R/P_i is a prime Goldie ring, and thus each nonzero ideal of R/P_i contains a regular element. Jategaonkar's proof [**9**, Lemma 1.10] of the corresponding result for left Noetherian rings can then be used. □

COROLLARY (4). If $K_\tau \dim(R) = 0$*, then any* τ*-closed prime ideal of* R *is minimal in the set of prime Goldie ideals of* R*.*

Proof. Assume that P_0 and P_1 are prime Goldie ideals of R such that P_0 is τ-closed and $R \supset P_0 \supseteq P_1$. Since R/P_1 is a prime Goldie ring, it satisfies the descending chain condition on left annihilators. If $\tau(R/P_1) \neq 0$, then there must exist elements $x_1, \ldots, x_n \in \tau(R/P_1)$ with $\mathrm{Ann}(x_1, \ldots, x_n) = P_1$. These elements can be used to define an embedding $R/P_1 \to \tau(R/P_1)^n$, which shows that R/P_1 must be τ-torsion. This is a contradiction since P_0 is τ-closed and therefore $\tau(R/P_1) = P_0/P_1$. Thus P_1 is τ-closed, and so by Proposition 3, $P_0 = P_1$. □

Goldman [**7**, Theorem 4.1(3)] has shown that if R is left Noetherian, ${}_RM$ is finitely generated, and π is a prime torsion radical in $\mathrm{supp}(M)$, then M has finite length with respect to π if and only if π is maximal in $\mathrm{supp}(M)$. Proposition 5 gives a more general result, using methods which are quite different from those used by Goldman.

The torsion radical defined at a minimal prime ideal P of a ring with Krull dimension is maximal in the set of proper torsion radicals of R–Mod [**2**, Theorem 4.6]. It follows from either

Proposition 5 or Proposition 6 below that in this case $E(R/P)$ has the descending chain condition on annihilators. This extends Jategaonkar's result [**9**, Proposition 1.9] stated in the introduction, and also sharpens [**3**, Theorem 2.5] which shows that for each submodule $M \subseteq E(R/P)$ there exists a finite set of elements $m_1, \dots, m_n \in M$ such that $\text{Ann}(M) = \text{Ann}(m_1, \dots, m_n)$.

PROPOSITION (5). Let π be a prime torsion radical of R–Mod. If ${}_RM$ has relative Krull dimension with respect to π, then $K_\pi \dim(M) = 0 \Leftrightarrow \pi$ *is maximal in* $\text{supp}(M)$.

Proof. $\Rightarrow$) Let $\tau \in \text{supp}(M)$, with $\tau \geq \pi$. Then $\tau(M) \neq M$ and $\tau(M)$ is π-closed in M, since $\pi(M/\tau(M)) \supseteq \tau(M/\tau(M)) = 0$, so by assumption $M/\tau(M)$ contains a minimal π-closed submodule N. Now since $\tau \geq \pi$, N must also be a minimal τ-closed submodule of $M/\tau(M)$, and thus both N_τ and N_π are simple objects in the respective quotient categories. Since a prime torsion radical is defined by any object which is simple in the corresponding quotient category, this shows that N defines both τ and π, and so $\tau = \pi$.

$\Leftarrow$) Assume that π is maximal in $\text{supp}(M)$. Let $N \subset M$ be any proper π-closed submodule fo M. By Lemma 1, M/N must contain an essential finite direct sum of monoform modules, say $\oplus_{i=1}^{\infty} M_i = X/N$. Each M_i defines a prime torsion radical π_i in $\text{supp}(M)$, and $\pi_i \geq \pi$ since M_i is π-torsionfree. Thus by assumption $\pi_i = \pi$, so each $(M_i)_\pi$ is simple in R–Mod/π. This shows that $\oplus_{i=1}^{n} M_i$ has finite length with respect to π. Now if

$N_1 \supseteq N_2 \supseteq \ldots$ is a descending chain of π-closed submodules of M, let $N = \cap_{i=1}^{\infty} N_i$. Then M/N contains an essential submodule X/N which has finite length with respect to π, and so for some k, $X \cap N_k = N$. Since X/N is essential in M/N, this shows that $N_k = \cap_{i=1}^{\infty} N_i$, and the chain terminates at N_k. □

PROPOSITION (6). Let P be minimal in the set of prime Goldie ideals of R, and let $\pi = \mathrm{rad}_{E(R/P)}$. If R has relative Krull dimension with respect to π and every π-torsionfree module has an associated prime ideal, then $K_\pi \dim(R) = 0$.

Proof. Since P is a prime Goldie ideal, π is a prime torsion radical, and so by Proposition 5 it is sufficient to show that π is a maximal torsion radical of *R–Mod*. If $\tau \geq \pi$ is a proper torsion radical ($\tau \neq 1$), then τ is defined by a nonzero injective module W which is τ-torsionfree, and hence π-torsionfree. By assumption W has an associated prime ideal Q which must be a prime Goldie ideal by Proposition 2. Since Q is the annihilator of a submodule of W, Q must be τ-closed in R. It follows that $\mathrm{rad}_{E(R/Q)} \geq \tau \geq \mathrm{rad}_{E(R/P)}$, and so by [**1**, Lemma 3.5], $Q \subseteq P$. But P is a minimal prime Goldie ideal, so $Q = P$ and therefore $\tau = \pi$, which shows that π is in fact maximal. □

An injective module ${}_R W$ is said to be Σ-injective if every direct sum of copies of W is injective, which occurs if and only if W has the ascending chain condition on annihilators of subsets [**10**, Chapter XIII, Proposition 2.4]. An analogous definition can be given for quasi-injective modules. Cailleau and Renault [**4**, Proposition 5] have shown that a quasi-injective module is Σ-quasi-injective if and only if it can be expressed as a direct sum of indecomposable Σ-quasi-injective modules.

If ${}_RW$ has the descending chain condition on annihilators, then for each submodule $X \subseteq W$ there exist elements $x_1, \ldots, x_n$ in X such that $\mathrm{Ann}(X) = \mathrm{Ann}(x_1, \ldots, x_n)$. A module ${}_RM$ with elements $x_1, \ldots, x_n$ such that $\mathrm{Ann}(M) = \mathrm{Ann}(x_1, \ldots, x_n)$ will be called finitely annihilated. Note that over a commutative ring any finitely generated module is finitely annihilated, although the converse does not hold. (Consider the set of rational numbers as a module over the ring of integers.)

THEOREM (7). If τ is defined by the injective module ${}_RW$, then the following conditions are equivalent.

(1) R has finite length with respect to τ.

(2) R is τ-Noetherian and every submodule of W is finitely annihilated.

(3) $W = \oplus_{\nu \in J} V_\nu$, where each V_ν is an indecomposable Σ-injective module isomorphic to a direct summand of $E(R/P_i)$, $1 \leq i \leq n$, and each P_i is minimal in the set of prime Goldie ideals of R.

In this case, $\{P_1, \ldots, P_n\}$ is the set of all τ-closed prime ideals of R, and τ is defined by $\oplus_{i=1}^n E(R/P_i)$.

Proof. (1) $\Rightarrow$ (2) This follows immediately from the remarks preceding the theorem.

(2) $\Rightarrow$ (3) If R is τ-Noetherian, then W must be Σ-injective, and so W is isomorphic to a direct sum $\oplus_{\nu \in J} V_\nu$ of indecomposable Σ-injective modules. For each $\nu \in J$, among annihilators of nonzero submodules of V_ν there is a maximal one, say P_ν, which must be a prime Goldie ideal by Proposition 2. Since V_ν is finitely annihilated, R/P_ν can be embedded in a finite direct sum $(V_\nu)^n$ of copies of V_ν. Thus $(V_\nu)^n$ contains an isomorphic copy of $E(R/P_\nu)$, and so since V_ν is indecomposable, the Krull-Remak-Schmidt-Azumaya theorem [**10**, Chapter V, Proposition 5.4] shows that V_ν is isomorphic to a direct summand of $E(R/P_\nu)$.

If P_ν contains a prime Goldie ideal P, then as in the proof of Corollary 4, P must be π_ν-closed, where π_ν is the prime torsion radical defined by V_ν (or $E(R/P_\nu)$). Thus P is the annihilator of a submodule $X \subseteq V_\nu$. By assumption, X is finitely annihilated, so R/P can be embedded in a finite direct sum of copies of V_ν, and hence in a finite direct sum of copies of $E(R/P_\nu)$. This implies that $P = P_\nu$, and so P_ν must be minimal in the set of prime Goldie ideals of R.

If $I = \cap_{\nu \in J} P_\nu$, then I is a τ-closed semiprime ideal, and so by Proposition 2, R/I is a semiprime Goldie ring. Therefore R/I has only finitely many minimal prime ideals, and so only finitely many of the ideals P_ν are distinct.

(3) $\Rightarrow$ (1) If V_ν defines the prime torsion radical π_i, then R is π_i-Noetherian since V_ν is Σ-injective, and so π_i satisfies the conditions of Proposition 6. Thus R has finite

length with respect to π_i, and since there are only finitely many π_i's, it is not difficult to show that R has finite length with respect to $\tau = \cap_{i=1}^n \pi_i$. (Construct a composition series for R with respect to π_1, and then refine it by constructing a π_2-composition series for each factor, and so on. The end result will be a τ-composition series for R.)

Finally, if conditions (1) – (3) are satisfied, let P be any τ-closed prime ideal of R. Then P is an annihilator of a submodule X of W, so R/P can be embedded in a finite direct sum of copies of W, since X is finitely annihilated. Thus implies that $R(R/P)$ is isomorphic to $E(R/P_i)$ for some i, and then $P = P_i$. It is clear that τ is defined by $\oplus_{i=1}^n E(R/P_i)$. □

If R is a semiprime Goldie ring which has finite length with respect to τ and $\tau(R) = 0$, then each minimal prime ideal $P_1, \ldots, P_n$ must be τ-closed. It follows from Theorem 7 that τ must be defined by $E(R) = \oplus_{i=1}^n E(R/P_i)$, and so τ is the Goldie torsion radical, yielding the theorem of Goel, Jain and Singh [**5**]. Theorem 7 also implies that if R has Krull dimension, then it has finite length with respect to τ if and only if $\tau \geq E(R/N)$, where N is the prime radical of R.

Theorem 7 can be extended to give conditions under which any finitely generated, finitely annihilated module has finite length with respect to τ. To simplify the statement (which would involve the Σ-quasi-injectivity of an appropriate submodule of W), the ring R will be assumed to be left Noetherian.

Corollary (8). Let R be left Noetherian, $\tau = \mathrm{rad}_W$*, and let* ${}_RM$ *be finitely generated and finitely annihilated, with* $A = \mathrm{Ann}(M)$*. Then M has finite length with respect to* $\tau \Leftrightarrow$ *every associated prime torsion radical of* $\{x \in W \mid Ax = 0\}$ *is defined by a prime ideal minimal over A.*

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