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ON UNIVERSAL LOCALIZATION[1]

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P. M. Cohn introduced in [4] the universal localization at a semiprime ideal N of a left Noetherian ring R. He gave a construction for a ring of quotients $R_{\Gamma(N)}$ universal with respect to the property that every matrix regular modulo N is invertible over $R_{\Gamma(N)}$. That is, in each ring $\mathrm{M}_n(R_{\Gamma(N)})$ of $n \times n$ matrices over $R_{\Gamma(N)}$, every element of $\mathrm{M}_n(R)$ which is regular modulo $\mathrm{M}_n(N)$ becomes invertible. The ring $R_{\Gamma(N)}$ always exists, but it can be very difficult to determine. In fact, it is hard to compute even the kernel of the mapping $R \to R_{\Gamma(N)}$. On the other hand, $R_{\Gamma(N)}$ has some very desirable properties which are lacking in the torsion theoretic localization $R_{C(N)}$, and so it appears to be worthy of further study. This paper contains the announcement of some preliminary results in studying universal localization. It also contains some explicit computations, since one of the first tasks must be to build a collection of examples. Included is the computation of $\Lambda_{\Gamma(\Pi)}$ for every prime ideal Π of the ring of formal matrices $\Lambda = \begin{bmatrix} R & M \\ N & R \end{bmatrix}$, where R is a commutative ring and M and N are modules over R which have the pairings necessary to define matrix multiplication in Λ. This includes as special cases several examples given by Cohn in [4].

1 Some properties of the universal localization

The ring R is assumed to be an associative ring with identity, and all modules are assumed to be unital. If R is left Noetherian and N is a semiprime ideal of R, then the ring $R_{\Gamma(N)}$ is constructed as follows (see [4] and [5, p.255] for details). Let $\Gamma(N)$ be the set of all square matrices over R which are regular modulo N. For each $n \times n$ matrix $\gamma = (a_{ij}) \in \Gamma(N)$ take a set of n^2 symbols $(a'_{ij}) = \gamma'$, and take a ring presentation of $R_{\Gamma(N)}$ consisting of all of the elements of R, as well as all of the elements a'_{ij} as generators; as defining relations take all of the relations holding in R, together with the relations, in

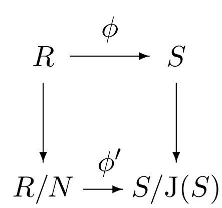
[1] 11/2001: The original term "inversive localization" has been changed throughout to the current term "universal localization"; the notation for the ring of $n \times n$ matrices has also been changed.

matrix form, $\gamma\gamma' = \gamma'\gamma = 1$, for each $\gamma \in \Gamma(N)$. The mapping $\lambda : R \to R_{\Gamma(N)}$ is an epimorphism in the category of rings, and $R_{\Gamma(N)}/\mathrm{J}(R_{\Gamma(N)})$ is the classical ring of quotients of R/N, under the embedding $\lambda' : R/N \to R_{\Gamma(N)}/\mathrm{J}(R_{\Gamma(N)})$ induced by λ. (The Jacobson radical of the ring R will be denoted by $\mathrm{J}(R)$.) The latter property will be used in Theorem 1.1 to characterize $R_{\Gamma(N)}$.

The ring $R_{\Gamma(N)}$ can be constructed in certain cases even when R is not left Noetherian. In fact, Cohn's proofs remain valid when N is any semiprime ideal such that the factor ring R/N is a left Goldie ring (this ensures the existence of the classical ring of quotients $Q_{\mathrm{cl}}(R/N)$). A semiprime (prime) ideal which satisfies this condition will be called a semiprime (prime) Goldie ideal. Working in this generality means that the inversive localization can be defined, for example, at any prime ideal of a ring with polynomial identity.

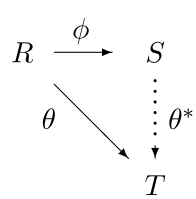
If N is a semiprime Goldie ideal of the ring R, consider the following conditions on a ring S and ring homomorphism $\phi : R \to S$. Note that any ring which satisfies these conditions must be unique (up to isomorphism).

J_1. The homomorphism ϕ induces a ring homomorphism $\phi' : R/N \to S/\mathrm{J}(S)$ such that the following diagram commutes. (The mappings $R \to R/N$ and $S \to S/\mathrm{J}(S)$ are the natural projections.)



J_2. The ring $S/\mathrm{J}(S)$ is a classical ring of quotients of R/N, under the embedding $\phi' : R/N \to S/\mathrm{J}(S)$.

J_3. If $\theta : R \to T$ is a ring homomorphism which satisfies conditions J_1 and J_2, then there exists a unique ring homomorphism $\theta^* : S \to T$ such that the following diagram commutes.



Theorem 1.1 *Let N be a semiprime Goldie ideal of R. Then the universal localization $\lambda : R \to R_{\Gamma(N)}$ of R at N can be defined, and it satisfies conditions* J_1, J_2, *and* J_3.

The next theorem was proved by Cohn for Noetherian rings. Let $C(N)$ denote the set of elements of R which are regular modulo N. The torsion theoretic localization at N, which is determined by $C(N)$ when N is a semiprime Goldie ideal, will be denoted by $R_{C(N)}$. (See [3] and [6] for details.) Recall that $C(N)$ is said to be a left denominator set if (i) for each $a \in R$ and $c \in C(N)$ there exist $a_1 \in R$ and $c_1 \in C(N)$ such that $c_1 a = a_1 c$ and (ii) if $ac = 0$ for $a \in R$ and $c \in C(N)$, then there exists $c_1 \in C(N)$ such that $c_1 a = 0$. If $C(N)$ is a left denominator set, then $R_{C(N)}$ is a classical ring of left fractions of R, obtained by inverting the elements of $C(N)$, and in this case it will be denoted by R_N.

Theorem 1.2 *Let N be a semiprime Goldie ideal of R. Then the ring $R_{\Gamma(N)}$ is naturally isomorphic to the ring $R_{C(N)}$ if and only if $C(N)$ is a left denominator set.*

Since the construction of $R_{\Gamma(N)}$ is left-right symmetric, it can sometimes be constructed in this manner even when it differs from $R_{C(N)}$. For example, let R be the ring $\begin{bmatrix} \mathbf{Z} & 0 \\ \mathbf{Z} & \mathbf{Z} \end{bmatrix}$ of lower triangular matrices over the ring of integers $\mathbf{Z}$, with the prime ideal $P = \begin{bmatrix} \mathbf{Z} & 0 \\ \mathbf{Z} & p\mathbf{Z} \end{bmatrix}$, where $p \in \mathbf{Z}$ is prime. It is not difficult to check that $C(P)$ is a right denominator set but not a left denominator set. (This also follows from Proposition 2.2.) Since $\begin{bmatrix} m & 0 \\ n & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0$ for any $m, n \in \mathbf{Z}$, the ideal $\begin{bmatrix} \mathbf{Z} & 0 \\ \mathbf{Z} & 0 \end{bmatrix}$ must be the kernel of $\lambda : R \to R_{\Gamma(P)}$, and computing the classical ring of right fractions shows that $R_{\Gamma(P)}$ is just the localization $\mathbf{Z}_{(p)}$ of $\mathbf{Z}$ at $p\mathbf{Z}$. A computation of the torsion theoretic localization $R_{C(P)}$ (on the left) shows it to be the full ring of 2×2 matrices over $\mathbf{Z}_{(p)}$.

For left Artinian rings it has been possible to explicitly compute $R_{\Gamma(N)}$. It turns out to be just a homomorphic image of R, and is in fact the largest homomorphic image in which $C(N)$ becomes a left denominator set.

Theorem 1.3 *Let N be a semiprime ideal of the left Artinian ring R. Then $R_{\Gamma(N)} = R/N^k$, where $N^k = N^{k+1} = \ldots$.*

Theorem 1.3 follows from part (c) of the next proposition, which has been helpful in computing $R_{\Gamma(N)}$ in a number of examples. The first two parts follow immediately from Theorem 3.2 of [4]. The proof of part (c) has been included since it illustrates some of the techniques which must be used.

Theorem 1.4 *Let N be a semiprime Goldie ideal of R, let K be the kernel of the homomorphism $\lambda : R \to R_{\Gamma(N)}$, and let I be an ideal of R which is contained in N.*

(a). *If $I \subseteq K$, then $(R/I)_{\Gamma(N/I)} = R_{\Gamma(N)}$.*

(b). *If $I \subseteq K$ and $C(N)$ is a left denominator set modulo I, then $R_{\Gamma(N)} = (R/I)_{N/I}$.*

(c). *If $I = I^2$ and I is finitely generated either as a left or as a right ideal, then $I \subseteq K$.*

Proof. (c) Assume that $I = \sum_{i=1}^{n} Rx_i$, for $x_1, \ldots, x_n \in I$. If $I = I^2$, then $I = \sum_{i=1}^{n} Ix_i$, and so $x_i = \sum_{j=1}^{n} a_{ij}x_j$, for $a_{ij} \in I$. In matrix form, this shows that $(1-\gamma)u = 0$ for the matrix $\gamma = (a_{ij})$ and the vector u which has entries $x_1, \ldots, x_n$. Since $1-\gamma \equiv 1 \pmod{N}$, $1-\gamma \in \Gamma(N)$, and so $1-\gamma$ must be invertible over $R_{\Gamma(N)}$. Therefore $x_i \in K$, for each i, and $I \subseteq K$. □

2 Examples

Let R and S be associative rings with identity, and let ${}_RM_S$ and ${}_SN_R$ be unital bimodules. Let Λ be the ring of 2×2 matrices $\begin{bmatrix} R & M \\ N & S \end{bmatrix}$. To define a multiplication for the ring Λ it is necessary to have a Morita context (see [1]). That is, it is necessary to have bilinear mappings $(\, , \,) : M \otimes_S N \to R$ and $[\, , \,] : N \otimes_R M \to S$, together with associative laws $m_1[n,m] = (m_1,n)m$ and $[n,m]n_1 = n(m,n_1)$, which must hold for all $m, m_1 \in M$ and $n, n_1 \in N$. Some elementary facts about Λ must be given, at the risk of writing down results which are in the folklore of the subject.

If I is an ideal of Λ, then I must have the form $I = \begin{bmatrix} A & X \\ Y & B \end{bmatrix}$, where A and B are ideals of R and S, respectively, and ${}_R X_S$, ${}_S Y_R$ are submodules of M and N, respectively. Furthermore, the following conditions must hold.

$$\begin{array}{llll} (M,Y) \subseteq A & AM \subseteq X & NA \subseteq Y & [Y,M] \subseteq B \\ (X,N) \subseteq A & MB \subseteq X & BN \subseteq Y & [N,X] \subseteq B \end{array}$$

From this point on, it seems to be much the easiest to suppress all mention of the bilinear mappings (,) and [,], except in the statements of theorems.

The above characterization of ideals can be used to show that if I is any ideal with $I \cap R = A$, then

$$\begin{bmatrix} A & AM \\ NA & NAM \end{bmatrix} \subseteq I \subseteq \begin{bmatrix} A & AN^{-1} \\ M^{-1}A & M^{-1}AN^{-1} \end{bmatrix}$$

where $AN^{-1} = \{x \in M \mid xN \subseteq A\}$, $M^{-1}A = \{y \in N \mid My \subseteq A\}$ and $M^{-1}AN^{-1} = \{b \in S \mid MbN \subseteq A\}$. Similarly, if $I \cap M = X$, then

$$\begin{bmatrix} XN & X \\ NXN & NX \end{bmatrix} \subseteq I \subseteq \begin{bmatrix} XM^{-1} & X \\ M^{-1}XM^{-1} & M^{-1}X \end{bmatrix}$$

where $XM^{-1} = \{a \in R \mid aM \subseteq X\}$, $M^{-1}X = \{b \in S \mid Mb \subseteq X\}$ and $M^{-1}XM^{-1} = \{y \in N \mid MyM \subseteq X\}$. Similar conditions can be given in the other two cases.

Proposition 2.1 *If* $\Pi = \begin{bmatrix} P & X \\ Y & Q \end{bmatrix}$ *is a prime ideal of* Λ*, then* P *and* Q *are prime ideals (if proper). Furthermore,* Π *must be one of the following types.*

Type 1. If $(M,N) \subseteq P$*, then* $[N,M] \subseteq Q$*,* $X = M$*,* $Y = N$*, and either* $Q = S$ *or* $P = R$*.*

Type 2. If $(M,N) \not\subseteq P$*, then* $[N,M] \not\subseteq Q$*,* $X \neq M$ *and* $Y \neq N$*.*

Proof. If A and B are ideals of R with $AB \subseteq P$, then for the left ideals generated by A and B, $\begin{bmatrix} A & 0 \\ NA & 0 \end{bmatrix} \begin{bmatrix} B & 0 \\ NB & 0 \end{bmatrix} = \begin{bmatrix} AB & 0 \\ NAB & 0 \end{bmatrix} \subseteq \Pi$, and so $A \subseteq P$ or $B \subseteq P$. Similarly, if Q is proper, then it is a prime ideal.

If $MN \subseteq P$, then $I^2 \subseteq \Pi$ for the left ideal $I = \begin{bmatrix} MN & M \\ N & NM \end{bmatrix}$, and so $I \subseteq \Pi$. Then $\begin{bmatrix} R & 0 \\ N & 0 \end{bmatrix} \begin{bmatrix} 0 & M \\ 0 & S \end{bmatrix} \subseteq \Pi$, which shows that either $Q = S$ or $P = R$.

On the other hand, if $MN \not\subseteq P$, then by the above argument $NM \not\subseteq Q$. The conditions which Π satisfies (as an ideal) force $X \neq M$ and $Y \neq N$. $\square$

As the proof of Proposition 2.1 shows, semiprime ideals can be treated in a similar manner. The next proposition determines the universal localization at a prime ideal of Type 1. It also shows that the set of elements regular modulo such a prime ideal need not be a left denominator set, even when R is commutative and $R = S$.

Proposition 2.2 *Let $(M, N) = I$, let P be a prime Goldie ideal of R with $P \supseteq I$, and let Π be the prime Goldie ideal $\begin{bmatrix} P & M \\ N & S \end{bmatrix}$ of Λ.*

(a). $\Lambda_{\Gamma(\Pi)} = (R/I)_{\Gamma(P/I)}$.

(b). *$C(\Pi)$ is a left denominator set if and only if $C(P)$ is a left denominator set and ${}_RM$ is $C(P)$-torsion.*

Proof. (a) Since $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in C(\Pi)$, it follows that the kernel of $\lambda : \Lambda \to \Lambda_{\Gamma(\Pi)}$ must contain $\begin{bmatrix} MN & M \\ N & S \end{bmatrix}$, since $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ N & S \end{bmatrix} = 0$ and $\begin{bmatrix} 0 & M \\ 0 & S \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0$. The universal localization can then be computed by using Proposition 1.4 (a).

(b) $\Rightarrow$) If $m \in M$, then $\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0$, so there must exist an element $\begin{bmatrix} c & x \\ y & b \end{bmatrix} \in C(\Pi)$ with $\begin{bmatrix} c & x \\ y & b \end{bmatrix}\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} = 0$. Thus $cm = 0$ for some $c \in C(P)$, and M is $C(P)$-torsion. It is just as easy to show that $C(P)$ must be a left denominator set.

$\Leftarrow$) Let $C^\Delta(\Pi)$ denote the elements of $C(\Pi)$ of the form $\begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}$ for $c \in C(\Pi)$. If $\gamma = \begin{bmatrix} c & x \\ y & d \end{bmatrix} \in C(\Pi)$, then $c_1 x = 0$ for some $c_1 \in C(P)$, since M is $C(P)$-torsion, and then $\gamma^*\gamma \in C^\Delta(\Pi)$, for $\gamma^* = \begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix} \in C^\Delta(\Pi)$. Given $\alpha = \begin{bmatrix} a & w \\ z & b \end{bmatrix} \in \Lambda$, by assumption it is possible to find $a_1 \in R$ and $c_2 \in C(P)$ with $c_2 a = a_1 c_1 c$. In addition there exists $c_3 \in C(P)$ with $c_3 w = 0$, so setting $\gamma_1 = \begin{bmatrix} c_3c_2 & 0 \\ 0 & 0 \end{bmatrix}$, $\alpha_1 = \begin{bmatrix} c_3a_1 & 0 \\ 0 & 0 \end{bmatrix}$ gives $\gamma_1\alpha = (\alpha_1\gamma^*)\gamma$. If $\alpha\gamma = 0$, then there exist $\alpha_1 \in \Lambda$ and $\gamma_1 \in C^\Delta(\Pi)$ with $\gamma_1\alpha = \alpha_1\gamma^*$. Thus $\alpha_1(\gamma^*\gamma) = \gamma_1(\alpha\gamma) = 0$. Since $C(P)$ is a left denominator set, it is easy to find $\gamma_2 \in C^\Delta(\Pi)$ with $\gamma_2\alpha_1 = 0$, and then $(\gamma_2\gamma_1)\alpha = \gamma_2(\alpha_1\gamma^*) = 0$. □

A module ${}_RM$ is said to be prime if $AX \neq 0$ for all nonzero ideals $A \subseteq R$ and all nonzero submodules $X \subseteq M$. Similarly, M is said to be semiprime if $AX = 0$ implies $AM \cap X = 0$, for all ideals A and submodules X. Semiprime ideals of Type 2 can be characterized in a manner similar to that of the following proposition, with semiprime ideals and modules instead of prime ideals and modules. The third condition must be replaced by the condition that for submodules $W \subseteq M$ and $Z \subseteq N$,

$WNW \subseteq X$ implies $W \subseteq X$ and $ZMZ \subseteq Y$ implies $Z \subseteq Y$. Note that the conditions of the proposition are symmetric, in the sense that the conditions could have been given in terms of $(M/X)_S$, ${}_S(N/Y)$ and $(\ ,\ )$.

Proposition 2.3 *Let* $\Pi = \begin{bmatrix} P & X \\ Y & Q \end{bmatrix}$ *be an ideal of* Λ*, with prime ideals* $P \not\supseteq (M,N)$ *and* $Q \not\supseteq [N,M]$ *and submodules* $X \neq M$ *and* $Y \neq N$*. The following conditions are equivalent.*

(1). Π *is a prime ideal.*

(2). ${}_R(M/X)$ *and* $(N/Y)_R$ *are prime modules over* R/P*.*

(3). *For* $m \in M$ *and* $n \in N$*,* $[N,m] \subseteq Q$ *implies* $m \in X$ *and* $[n,M] \subseteq Q$ *implies* $n \in Y$*.*

Proof. (1)$\Rightarrow$(2) If ${}_RA \subseteq R$ and ${}_RW \subseteq M$ with $AW \subseteq X$, then $\begin{bmatrix} A & 0 \\ NA & 0 \end{bmatrix}\begin{bmatrix} 0 & W \\ 0 & NW \end{bmatrix} = \begin{bmatrix} 0 & AW \\ 0 & NAW \end{bmatrix} \subseteq \begin{bmatrix} P & X \\ Y & Q \end{bmatrix}$. Thus either $A \subseteq P$ or $W \subseteq X$, and so M/X is a prime R/P-module. Similarly, $(N/Y)_R$ is a prime R/P-module.

(2)$\Rightarrow$(3) If $m \in M$ and $Nm \subseteq Q$, then $MN(Rm) \subseteq MQ \subseteq X$, and so $Rm \subseteq X$. Similarly, $nM \subseteq Q$ implies $n \in Y$.

(3)$\Rightarrow$(1) Suppose that $\begin{bmatrix} A & W \\ Z & B \end{bmatrix}\begin{bmatrix} C & U \\ V & D \end{bmatrix} \subseteq \begin{bmatrix} P & X \\ Y & Q \end{bmatrix}$ for two ideals of Λ. Then $BD \subseteq Q$ implies that $B \subseteq Q$ or $D \subseteq Q$, since Q is prime, so suppose that $B \subseteq Q$. Then $NW \subseteq B \subseteq Q$ implies that $W \subseteq X$ and $ZM \subseteq B \subseteq Q$ implies that $Z \subseteq Y$. Finally, $NA \subseteq Z \subseteq Y$ implies that $(MN)A \subseteq MY \subseteq P$, which implies that $A \subseteq P$ since $MN \not\subseteq P$. □

At the beginning of this section it was remarked that if I is any ideal of Λ such that $I \cap R = P$, then $I \subseteq \begin{bmatrix} P & PN^{-1} \\ M^{-1}P & M^{-1}PN^{-1} \end{bmatrix}$. As can be seen from the conditions of the proposition, $X = PN^{-1}$, $Y = M^{-1}P$, and $Q = M^{-1}PN^{-1}$, so $I \subseteq \Pi$. Thus any prime ideal of Type 2 contains all ideals which "lie over" any of its components. It also follows from the proposition that if $cm \in X$ for $c \in C(P)$ and $m \in M$, then $m \in X$. (If $cm \in X$, then $c(mN) \subseteq XN \subseteq P$ implies $mN \subseteq P$, so $x \in X$.)

The final proposition shows that if R and S are commutative, then the universal localization is defined at any prime ideal of Λ. In this situation I have been unable to determine the universal localization at prime ideals of Type 2, unless the denominator set conditions are satisfied, so it remains an open question. If $S = R$, (still commutative) then Q must be equal to P since it is the annihilator of M/X, and it follows immediately that the denominator set conditions must be satisfied. I have been told by Bill Blair that part (a) of Proposition 2.4 has been know to Lance Small for several years, for semiprime ideals. (With some slight modifications the proof given below can be used to obtain the more general result.)

Proposition 2.4 *Let* $\Pi = \begin{bmatrix} P & X \\ Y & Q \end{bmatrix}$ *be a prime ideal of* Λ, *with* $(M, N) \not\subseteq P$. *Assume that* P *and* Q *are prime ideals such that* R/P *and* S/Q *are left Goldie rings.*

(a). Λ/Π *is a left Goldie ring.*

(b). $Q_{\mathrm{cl}}(\Lambda/\Pi) = \begin{bmatrix} Q_{\mathrm{cl}}(R/P) & Q_{\mathrm{cl}}(R/P) \otimes_R (M/X) \\ Q_{\mathrm{cl}}(S/Q) \otimes_S (N/Y) & Q_{\mathrm{cl}}(S/Q) \end{bmatrix}$

(c). $C(\Pi)$ *is a left denominator set if and only if* $C(P)$ *and* $C(Q)$ *are left denominator sets and the following conditions hold.*

(i) For $c \in C(P)$, $d \in C(Q)$, $m \in M$, $n \in N$, *there exist* $c_1 \in C(P)$, $d_1 \in C(Q)$, $m_1 \in M$, $n_1 \in N$ *with* $c_1 m = m_1 d$ *and* $d_1 n = n_1 c$.

(ii) If $m \in M$ *and* $n \in N$ *with* $md = 0$ *and* $nc = 0$ *for* $d \in C(Q)$, $c \in C(P)$, *then there exist* $c_1 \in C(P)$ *and* $d_1 \in C(Q)$ *with* $c_1 m = 0$ *and* $d_1 n = 0$.

(d). *If* $C(\Pi)$ *is a left denominator set, then* $\Lambda_\Pi = \begin{bmatrix} R_P & R_P \otimes_R M \\ S_Q \otimes_S N & S_Q \end{bmatrix}$.

Proof. (a) By Theorem 2.2 of [2], a prime ring is a left Goldie ring if and only if it contains a uniform left ideal and each nonzero ideal contains a set of elements $a_1, \ldots, a_n$ for which the left annihilator $\ell(a_1, \ldots, a_n)$ is zero. Assume (without loss of generality) that $\Pi = 0$. If U is a uniform left ideal of R, then $\begin{bmatrix} U & 0 \\ NU & 0 \end{bmatrix}$ is a uniform left ideal of Λ. This follows from the observation that for any left ideal $\begin{bmatrix} A & 0 \\ Z & 0 \end{bmatrix} \subseteq \begin{bmatrix} U & 0 \\ NU & 0 \end{bmatrix}$, $NZ \subseteq A$ is nonzero if Z is nonzero. For any ideal $\begin{bmatrix} A & X \\ Y & B \end{bmatrix}$ which is nonzero, both A and B must be nonzero, so there exist elements $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_k \in B$ with $\ell_R(a_1, \ldots, a_n) = 0$ and $\ell_S(b_1, \ldots, b_k) = 0$. Using these elements it is easy to construct a finite set of matrices in $\begin{bmatrix} A & X \\ Y & B \end{bmatrix}$ with zero left annihilator. (If, for the left annihilator $\begin{bmatrix} C & W \\ Z & D \end{bmatrix}$, it can be shown that $C = 0$ and $D = 0$, then $MZ = 0$ and $NW = 0$ imply that $Z = 0$ and $W = 0$.)

(b) This will follow from (a) and (d).

(c) Before beginning the proof of (c) it is necessary to establish some properties of $C(\Pi)$. If $\gamma \in C(\Pi)$, then there exists $\gamma' \in C(\Pi)$ such that $\gamma'\gamma \in C(\Pi)$ and $\gamma'\gamma$ is a diagonal matrix module Π. Thus $\gamma'\gamma$ must have elements of $C(P)$ and $C(Q)$, respectively, along the diagonal. It suffices to show this when $\Pi = 0$. Since any regular element of a prime Goldie ring generates an essential left ideal, it follows that $\begin{bmatrix} A & 0 \\ Y & 0 \end{bmatrix} = \Lambda\gamma \cap \begin{bmatrix} R & 0 \\ N & 0 \end{bmatrix}$ it essential in $\begin{bmatrix} R & 0 \\ N & 0 \end{bmatrix}$. It can be checked that A must be an essential left ideal of R, so A contains a regular element. A similar argument can be given in S, and so $\Lambda\gamma$ contains an element $\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} = \gamma'\gamma$, where $c \in C(P)$, $d \in C(Q)$. If $\alpha\gamma' = 0$, then $\alpha(\gamma'\gamma) = 0$, so $\alpha = 0$ and γ' is left regular, which implies that γ' is regular.

To prove (c), first assume that $C(\Pi)$ is a left denominator set. It is not difficult to show that $C(P)$ and $C(Q)$ are left denominator sets, by using the preceding remark and the fact that $\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} \in C(\Pi)$ if $c \in C(P)$ and $d \in C(Q)$. Given $m \in M$ and $d \in C(Q)$, there must exist $\begin{bmatrix} a_1 & m_1 \\ n_1 & b_1 \end{bmatrix} \in \Lambda$ and $\begin{bmatrix} c_1 & x_1 \\ y_1 & d_1 \end{bmatrix} \in C(\Pi)$ with $\begin{bmatrix} a_1 & m_1 \\ n_1 & b_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} c_1 & x_1 \\ y_1 & d_1 \end{bmatrix} \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$. This implies that $m_1 d = c_1 m$, and it can be assumed, as above, that $c_1 \in C(P)$. The other condition can be proved similarly.

Conversely, assume that the stated conditions hold. Let $C^{\Delta}(\Pi)$ be the set of elements of $C(\Pi)$ of the form $\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$. If $\gamma \in C(\Pi)$, then there exists $\gamma' \in C(\Pi)$ with $\gamma'\gamma = \begin{bmatrix} c & m \\ n & d \end{bmatrix}$, where $c \in C(P)$ and $d \in C(Q)$. By assumption there exist $c_1 \in C(P)$ and $m_1 \in M$ with $c_1 m = m_1 d$. Thus $\begin{bmatrix} c_1 & -m_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c & m \\ n & d \end{bmatrix} = \begin{bmatrix} c_1 c - m_1 n & 0 \\ n & d \end{bmatrix}$, and it can be checked that $\begin{bmatrix} c_1 & -m_1 \\ 0 & 1 \end{bmatrix} \in C(\Pi)$. The argument can be repeated for the entry n, so that the following result holds. If $\gamma \in C(\Pi)$, then there exists $\gamma^* \in C(\Pi)$ such that $\gamma^*\gamma \in C^{\Delta}(\Pi)$.

Now suppose that $\alpha \in \Lambda$ and $\gamma \in C(\Pi)$, with $\gamma^*\gamma = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} \in C^{\Delta}(\Pi)$ and $\alpha = \begin{bmatrix} a & x \\ y & b \end{bmatrix}$. Using the given conditions, it is possible to find $c_1 \in C(P)$, $d_1 \in C(Q)$ and $\alpha_1 = \begin{bmatrix} a_1 & x_1 \\ y_1 & b_1 \end{bmatrix} \in \Lambda$ such that $c_1 a = a_1 c$, $c_1 x = x_1 d$, $d_1 y = y_1 c$ and $d_1 b = b_1 d$. Thus $\gamma_1 \alpha = (\alpha_1 \gamma^*)\gamma$, for $\gamma_1 = \begin{bmatrix} c_1 & 0 \\ 0 & d_1 \end{bmatrix} \in C^{\Delta}(\Pi)$. If $\alpha\gamma = 0$, then by the above computations there exist $\alpha_1 = \begin{bmatrix} a_1 & x_1 \\ y_1 & b_1 \end{bmatrix} \in \Lambda$ and $\gamma_1 \in C^{\Delta}(\Pi)$ with $\gamma_1\alpha = \alpha_1\gamma^*$. Thus $\alpha_1(\gamma^*\gamma) = \gamma_1(\alpha\gamma) = 0$, so it follows that $a_1c = 0$, $x_1d = 0$, $y_1c = 0$ and $b_1d = 0$. Using the given conditions it is possible to find $\gamma_2 \in C^{\Delta}(\Pi)$ with $\gamma_2\alpha_1 = 0$. Thus $(\gamma_2\gamma_1)\alpha = (\gamma_2\alpha_1)\gamma^* = 0$.

(d) Assume that $C(\Pi)$ is a left denominator set. If $\alpha \in \Lambda$ and $\gamma\alpha = 0$ for $\gamma \in C(\Pi)$, then $\gamma^*\gamma\alpha = 0$, where $\gamma^*\gamma \in C^{\Delta}(\Pi)$. This shows that the entries of α must belong to the appropriate torsion submodules, which will be denoted by $\tau(\ )$, so with this notation the $C(\Pi)$-torsion idea of Λ is $\begin{bmatrix} \tau_P(R) & \tau_P(M) \\ \tau_Q(N) & \tau_Q(S) \end{bmatrix}$. It suffices to consider the case in which Λ has no $C(\Pi)$-torsion. The localization $M_P \simeq R_P \otimes_R M$ can be given a right S_Q-module structure as follows. For $c^{-1}m \in M_P$ and $d^{-1}b \in S_Q$, let $(c^{-1}m)(d^{-1}b) = (c_1c)^{-1}m_1b$, where $c_1m = m_1d$ for $c_1 \in C(P)$ and $m_1 \in M$. In addition, a pairing can be defined for $c^{-1}m \in M_P$ and $d^{-1}n \in N_Q$ by setting $(c^{-1}m, d^{-1}m) = (c_1c)^{-1}(m_1, n) \in R_P$, where $c_1 \in C(P)$ and $m_1 \in M$ with $c_1m = m_1d$. This can be done formally by showing that $R_P \otimes_R M) \otimes_S Q \simeq R_P \otimes_R M$. In this way the set of matrices $\Omega = \begin{bmatrix} R_P & M_P \\ N_Q & S_Q \end{bmatrix}$ can be given a ring structure, with Λ identified in the natural way with a subring of Ω. If $\gamma \in C(\Pi)$, then there exists $\gamma^* \in C(\Pi)$ with $\gamma^*\gamma \in C^{\Delta}(\Pi)$. It is clear that $\gamma^*\gamma$ is invertible in Ω, so γ is left invertible, and hence invertible (γ is a regular element of Ω). If $\omega \in \Omega$, it is evident that $\gamma \in C^{\Delta}(\Pi)$ can be found with $\gamma\omega = \alpha \in \Lambda$, so $\omega = \gamma^{-1}\alpha$, and this shows that Ω is the left ring of fractions Λ_Π defined by $C(\Pi)$. □

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