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STABLE TORSION RADICALS OVER FBN RINGS

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To Goro Azumaya on his sixtieth birthday

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Gabriel [9] called a localizing subcategory stable if it is closed under injective envelopes. (In this case, the associated torsion radical or torsion theory is also said to be stable.) He showed that over a commutative Noetherian ring R every localizing subcategory of R-Mod is stable. He noted the connections between stability and the Artin–Rees property for ideals, and considered the stability of the localizing subcategory generated by the class of simple modules. Papp [18] studied Noetherian rings for which every localizing subcategory is stable, and Louden [16] utilized stable torsion radicals to define a sheaf over the spectrum of an FBN ring.

This paper gives a characterization of stable torsion radicals over left FBN rings. The equivalent conditions stated in Theorem 1.2 exhibit the connections with the Artin–Rees property and with the notion of ideal invariance introduced by Robson [19]. Over an FBN ring, there is a natural correspondence between torsion radicals in R-Mod and those in Mod-R. In Theorem 1.6 it is shown that corresponding torsion radicals are both stable if and only if they are both ideal invariant, and that such torsion radicals correspond to the biradicals defined by Jategaonkar [12], and thus to the link-closed hereditary sets of prime ideals in the sense of Müller [17]. Finally, it is shown that every torsion radical over an FBN ring is stable if and only if every prime ideal is localizable.

In the second section of the paper, applications are given to the study of certain specific torsion radicals. For a left FBN ring, necessary and sufficient conditions are given under which any finitely generated essential extension of a module of Krull dimension at most α again has Krull dimension at most α. These conditions can be checked easily for an FBN ring or for a left Noetherian ring integral over its center, yielding a unified approach to certain results of Jategaonkar [13] and Chamarie and Hudry [5]. Finally, the results in the first section are applied to determine several conditions under which finitely generated essential extensions of Artinian modules are again Artinian, extending a result of Ginn and Moss [10].

Throughout the paper, R will be assumed to be an associative ring with identity element, and all modules will be assumed to be unital R-modules. The categories of unital left R-modules and unital right R-modules will be denoted by R-Mod and Mod-R, respectively. The injective envelope of a module ${}_RM$ will be denoted by $E(M)$.

1. Stable torsion radicals

The ring R is said to be left FBN if it is left Noetherian and fully left bounded (modulo any prime ideal every essential left ideal contains a nonzero two-sided ideal). Cauchon [4] has shown that a left Noetherian ring R is left FBN if and only if for each finitely generated module ${}_RM$ there exists a finite set of elements $x_1, x_2, \dots, x_n \in M$ such that $\mathrm{Ann}(x_1, \dots, x_n) = \mathrm{Ann}(M)$. A module ${}_RM$ will be called finitely annihilated if this property holds, that is, if there exist $x_1, x_2, \dots, x_n \in M$ with $\mathrm{Ann}(x_1, \dots, x_n) = \mathrm{Ann}(M)$. It is clear that M is finitely annihilated if and only if for some integer $n > 0$ there exists an embedding $0 \to R/\mathrm{Ann}(M) \to M^n$, where M^n denotes the direct sum of n isomorphic copies of M.

The book by Stenström [20] will be used as a basic reference for facts regarding torsion radicals. If σ is a torsion radical (equivalently, σ is the torsion functor defined by an hereditary torsion theory), then a module ${}_RM$ is said to be σ-torsion if $\sigma(M) = M$ and σ-torsionfree if $\sigma(M) = (0)$; a submodule $N \subseteq M$ is said to be σ-dense is M/N is σ-torsion, and σ-closed if M/N is σ-torsionfree. The associated filter of σ-dense left ideals determines σ, since $\sigma(M) = \{m \in M \mid Dm = (0)$ for some σ-dense left ideal $D\}$. Using this characterization of σ, it is easy to see that a prime ideal of R must be either σ-closed or σ-dense. The following facts will also be used: any left ideal which contains a σ-dense left ideal must be σ-dense; the product of two σ-dense left ideals is σ-dense.

An essential extension of a σ-torsionfree module is σ-torsionfree; the torsion radical σ is said to be stable if the corresponding statement holds for σ-torsion modules. A submodule $N \subseteq M$ is said to be essentially closed in M if N has no proper essential extension in M. Equivalently, N is a complement submodule of M. Then the torsion radical σ is stable if and only if for each module ${}_RM$ the submodule $\sigma(M)$ is essentially closed in M [20, Chapter VI, Proposition 7.1].

Theorem 1.1. *Let R be a left Noetherian ring, let σ be a torsion radical, and let ${}_RM$ be a module for which every finitely generated submodule is finitely annihilated. Then $\sigma(M)$ is essentially closed in M if the following condition is satisfied: if P is a σ-closed prime ideal and Q is a σ-dense prime ideal, then there exists a σ-dense ideal D with $PD \subseteq QP$.*

Proof. Without loss of generality, assume that M is cyclic and that there exists an essential extension N of $\sigma(M)$ in M such that N is not σ-torsion. Then $\mathrm{Ann}(N)$ is not

σ-dense, and so there exists an ideal I maximal in the set $\{\mathrm{Ann}(X) \mid {}_RX \subseteq N$ and $\mathrm{Ann}(X)$ is not σ-dense$\}$, say $I = \mathrm{Ann}(X)$.

Suppose that I is not a prime ideal. Then since R satisfies the ascending chain condition on ideals, there must exist a maximal right annihilator ideal P of R/I, say $P = r(A/I)$, where A is an ideal strictly containing I. A standard argument shows that P must be a prime ideal. Now $APX \subseteq IX = (0)$, with $PX \neq (0)$ since $P \supsetneq I$, so it can be assumed that $A = \mathrm{Ann}(PX)$. By the choice of I, A must be σ-dense, and so PX is σ-torsion. If P is also σ-dense, then AP is σ-dense, and hence $I \supseteq AP$ is σ-dense, a contradiction. Thus P is not σ-dense, and so since it is a prime ideal it must be σ-closed.

By assumption, the submodule PX is finitely annihilated, so there exists an embedding $0 \to R/\mathrm{Ann}(PX) \to (PX)^n$, which shows that $\mathrm{Ann}(PX)$ is σ-dense since PX is σ-torsion. Since $\mathrm{Ann}(PX)$ is an ideal of a left Noetherian ring, it contains a product of prime ideals, each of which contains $\mathrm{Ann}(PX)$. Thus there exist σ-dense prime ideals $Q_1, Q_2, \dots, Q_n$ with $Q_1Q_2 \cdots Q_nPX = (0)$, and so by assumption there exist σ-dense ideals D_i with $PD_i \subseteq Q_iP$, for $1 \leq i \leq n$. Therefore

$$P(D_1D_2 \cdots D_n) \subseteq Q_1P(D_2D_3 \cdots D_n) \subseteq \cdots \subseteq Q_1Q_2 \cdots Q_nP \subseteq I.$$

Thus $PD \subseteq AP$ for the σ-dense ideal $D = D_1D_2 \cdots D_n$. If $DX = (0)$, then $D \subseteq I$ and I must be σ-dense, a contradiction. Thus $C = \mathrm{Ann}(DX)$ is a proper ideal with $C \supseteq P \supsetneq I$, since $PDX \subseteq APX = (0)$, and so C must be σ-dense. But then CD is σ-dense, which implies that $I \supseteq CD$ is σ-dense. This contradiction shows that I is a prime ideal.

Since every submodule of M is finitely annihilated, there exists an embedding $f: R/I \to X^n$, for some positive integer n. Since $\sigma(M)$ is essential in N, $\sigma(X) = \sigma(M) \cap X$ is essential in X, so $\sigma(X^n)$ is essential in X^n, which shows that $f(R/I)$ has a nonzero σ-torsion submodule. Thus I is not σ-closed, and this contradicts the choice of I, since I is a prime ideal. □

Theorem 1.2. *Let R be a left* FBN *ring and let σ be a torsion radical of R-*Mod. *The following conditions are equivalent:*

(1) *The torsion radical σ is stable;*

(2) *For any finitely generated module ${}_RM$, any submodule ${}_RN \subseteq M$, and any σ-dense ideal I, there exists a σ-dense ideal D with $DM \cap N \subseteq IN$;*

(3) *For any ideal T and any σ-dense ideal I, there exists a σ-dense ideal D with $TD \subseteq IT$;*

(4) *For any σ-closed prime ideal P and any σ-dense prime ideal Q, there exists a σ-dense ideal D with $PD \subseteq QP$.*

Proof. (2) ⇒ (3) ⇒ (4) ⇒ (1). Proofs of the first two implications are obvious, and the last follows immediately from Theorem 1.1.

(1) ⇒ (2). Let ${}_RX$ be a submodule maximal with respect to the property that $X \cap N = IN$. Then $(N+X)/X$ is σ-torsion since $(N+X)/X \cong N/(X \cap N) = N/IN$ and

I is σ-dense. Since M/X is an essential extension of $(N+X)/X$, it follows by assumption that M/X is σ-torsion, and then $D=\mathrm{Ann}(M/X)$ is σ-dense since M/X is finitely annihilated. Thus $DM\cap N\subseteq X\cap N=IN$. $\square$

Note that since R is a left FBN ring, an ideal D is σ-dense if and only if D contains a product $Q_1Q_2\cdots Q_n$ of σ-dense prime ideals Q_i, $1\leq i\leq n$ [20, Chapter VII, Theorem 3.4]. Thus condition (4) of the previous theorem can be stated in the following form, using only prime ideals: for any σ-closed prime ideal P and any σ-dense prime ideal Q, there exist σ-dense prime ideals $Q_1, Q_2, \ldots, Q_n$ with $PQ_1Q_2\cdots Q_n\subseteq QP$. Condition (3) can be restated as follows: for any ideal T and any σ-dense ideal I, $T^{-1}(IT)=\{r\in R \mid Tr\subseteq IT\}$ is σ-dense.

An ideal I is said to have the Artin–Rees property if for each left ideal A there exists a positive integer n such that $I^n\cap A\subseteq IA$. If R is left Noetherian, then the powers of I generate a torsion radical σ by defining $\sigma(M)=\{m\in M \mid I^n m=(0)$ for some $n>0\}$, for any module ${}_RM$. As observed by Gabriel [9], the ideal I has the Artin–Rees property if and only if the associated torsion radical σ is stable. Condition (2) of Theorem 1.2 shows the similarity between the Artin–Rees property and stability in general. The theorem can be applied to show that an ideal I of a left FBN ring has the Artin–Rees property if and only if for each prime ideal P with $I\not\subseteq P$ there exists a positive integer n such that $PI^n\subseteq IP$.

Example 1. Let R be a left FBN ring, let P be a prime ideal of R, and let σ be the torsion radical cogenerated by $E(R/P)$. Then an ideal D is σ-dense if and only if $D\not\subseteq P$, and so it follows from Theorem 1.2 that σ is stable if and only if for each pair of prime ideals $P_0\subseteq P$ and $Q\not\subseteq P$ there exists an element $r\notin P$ with $P_0 r\subseteq QP_0$.

Let R be the ring of lower triangular matrices of the form

$$\begin{bmatrix} a & 0 \\ b & c \end{bmatrix},$$

where $a\in\mathbb{Z}$ and $b,c\in\mathbb{Q}$. Let

$$P=\begin{bmatrix} p\mathbb{Z} & 0 \\ \mathbb{Q} & \mathbb{Q} \end{bmatrix},$$

for a prime number p,

$$P_0=\begin{bmatrix} 0 & 0 \\ \mathbb{Q} & \mathbb{Q} \end{bmatrix}, \quad \text{and} \quad Q=\begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Q} & 0 \end{bmatrix}.$$

Then $P_0 s\not\subseteq QP_0$ for any element $s\notin P$ and so the torsion radicals of R-Mod determined by P and P_0 are not stable. On the other hand, for any prime number p, $Qr\subseteq P_0Q\subseteq PQ$ for

$$r=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

and so the torsion radical determined by Q is stable.

Definition 1.3. A torsion radical σ of R-Mod is called a *biradical* if there exists a torsion radical τ of Mod-R such that for all ideals $I \subseteq J \subsetneq R$, $\sigma(J/I) = \tau(J/I)$.

This definition was introduced by Jategaonkar [12], who noted that if Σ is a multiplicatively closed subset of a left and right Noetherian ring, and Σ satisfies both the left and right Ore conditions, then Σ defines a biradical. (This follows from the fact that if Σ is an Ore set, then modulo any ideal it is still an Ore set, and hence a denominator set [20, Chapter II, Proposition 1.5].) In particular, any multiplicatively closed set of central elements defines a biradical.

Proposition 1.4. *Let R be a left Noetherian ring, and let (σ, τ) be a biradical. Then for any ideal T and any σ-dense ideal I, there exists a σ-dense ideal D with $TD \subsetneq IT$.*

Proof. Since I is σ-dense, the left R/I-module T/IT is σ-torsion, and hence $\tau(T/IT) = \sigma(T/IT) = T/IT$. Since R is left Noetherian, $(T/IT)_R$ is finitely annihilated, so the existence of an embedding $0 \to R/T^{-1}(IT) \to (T/IT)^n$ of right R-modules shows that $T^{-1}(IT)$ is τ-dense. By assumption $D = T^{-1}(IT)$ is σ-dense, and $TD \subsetneq IT$. $\square$

The previous proposition shows that over a left FBN ring, any biradical is stable. In particular, if R is a left FBN ring and C is a subring of the center of R, then for any torsion radical τ of C-Mod, $\sigma = \eta\tau$ is a biradical, where $\eta : R\text{-Mod} \to C\text{-Mod}$ is the forgetful functor. This yields Proposition 1.6 of Chamarie and Hudry [5].

Definition 1.5. Let I be an ideal of R. The torsion radical σ of R-Mod is said to be *invariant under I* if for each σ-dense left ideal D, the ideal ID is σ-dense in I. If σ is invariant under every ideal of R, then σ is said to be *ideal invariant*.

It can be shown that a torsion radical σ is invariant under the ideal I if and only if I deflates σ, in the terminology of Jategaonkar [14]. In [19], Robson gave the definition of ideal invariance in the following (equivalent) form: σ is ideal invariant if for each σ-torsion module ${}_RM$ and each ideal I, the module $I \otimes_R M$ is σ-torsion. If R is a left FBN ring, then since the filter of σ-dense left ideals has a basis of two-sided ideals, σ is invariant under an ideal I if and only if ID is σ-dense in I for all two-sided σ-dense ideals D. With this terminology, Theorem 2.3 of [7] implies that a semiprime ideal S is left localizable if and only if the torsion radical σ cogenerated by $E(R/S)$ is invariant under S, provided R is left Noetherian.

If R is an FBN ring, then any torsion radical σ of R-Mod is completely determined by the set of σ-dense prime ideals, since a left ideal is σ-dense if and only if it contains a product of σ-dense prime ideals. This implies that there is a natural correspondence between the torsion radicals of R-Mod and those of Mod-R. Furthermore, if (σ, τ) is a biradical of R, then σ and τ must correspond to each other in this correspondence, since for any prime ideal P of R, $\sigma(R/P) = \tau(R/P)$.

Theorem 1.6. *Let R be an* FBN *ring, and let σ be a torsion radical of R*-Mod, *with corresponding torsion radical τ of* Mod-R. *Then the following conditions are equivalent:*

(1) *σ and τ are stable;*

(2) *σ and τ are ideal invariant;*

(3) *(σ, τ) is a biradical.*

Proof. (1)⇒(2). This follows from Theorem 1.2, since an ideal of R is σ-dense if and only if it is τ-dense.

(2)⇒(3). Let $I \subsetneq J$ be ideals of R. It suffices to show that ${}_R(J/I)$ is σ-torsion if and only if $(J/I)_R$ is τ-torsion. If J/I is σ-torsion, then $DJ \subsetneq I$ for the ideal $D = \mathrm{Ann}({}_R J/I)$, which is σ-dense since ${}_R(J/I)$ is finitely annihilated. Since σ and τ are corresponding torsion radicals, D is τ-dense, and it follows that DJ is τ-dense in J, since τ is ideal invariant. Thus I is τ-dense in J, that is, $(J/I)_R$ is τ-torsion. Similarly, if $(J/I)_R$ is τ-torsion, then it must be σ-torsion.

(3)⇒(1). This follows from Proposition 1.4 and Theorem 1.2. □

Corollary 1.7. *Let R be an* FBN *ring. Then the following conditions are equivalent:*

(1) *Every torsion radical of R*-Mod *and* Mod-R *is stable;*

(2) *Every torsion radical of R*-Mod *and* Mod-R *is ideal invariant;*

(3) *Every prime ideal of R is localizable.*

Proof. (1)⇔(2). This is clear.

(2)⇒(3). This follows from [7, Theorem 2.3], since the torsion radicals determined by any prime ideal P are by assumption ideal invariant, and hence invariant under P.

(3)⇒(1). If every prime ideal P of R is left and right localizable, then the torsion radicals of R-Mod and Mod-R cogenerated by $E({}_R(R/P))$ and $E((R/P)_R)$, respectively, define a biradical and hence are stable. Since R is an FBN ring, [18, Theorem 1] then implies that every torsion radical over R is stable. □

2. Krull dimension of essential extensions

The Krull dimension of a module ${}_R M$ (see [11]) will be denoted by $|M|$, and is defined by transfinite recursion, as follows: if M is Artinian, then $|M| = 0$; if α is an ordinal and $|M| \not< \alpha$, then $|M| = \alpha$ if there is no infinite descending chain $M = M_0 \supsetneq M_1 \supsetneq \cdots$ of submodules M_i such that $|M_{i-1}/M_i| \not< \alpha$ for $i = 1, 2, \ldots$. It can be shown that any Noetherian module has Krull dimension. For a given ordinal α, the set of left ideals $D \subsetneq R$ such that $|R/D| < \alpha$ defines a filter, and the associated torsion radical of R-Mod will be denoted by τ_α. Applying Theorem 1.2 to τ_α gives the following result.

Theorem 2.1. *Let R be a left* FBN *ring. Then the following conditions are equivalent for the ordinal α:*

(1) *If* ${}_RM$ *is a finitely generated, essential extension of* ${}_RN$ *and* $|N| < \alpha$, *then* $|M| < \alpha$;

(2) *For any ideals* I, T *with* $|{}_RR/I| < \alpha$, *there exists an ideal* D *with* $|{}_RR/D| < \alpha$ *and* $TD \subseteq IT$;

(3) *For any prime ideals* P, Q *with* $|{}_RR/P| \geq \alpha$ *and* $|{}_RR/Q| < \alpha$, *there exists an ideal* D *with* $|{}_RR/D| < \alpha$ *and* $PD \subseteq QP$.

Let R be an FBN ring, and let $I \subseteq J$ be ideals of R. It is a direct consequence of [13, Lemma 2.2] that $|{}_RJ/I|$ can be measured as the Krull dimension of the set of ideals between I and J. Thus $|{}_RJ/I| = |J/I_R|$, which shows that for each ordinal α, the torsion radical τ_α is a biradical. It then follows from Proposition 1.4 that the equivalent conditions of Theorem 2.1 hold for any FBN ring. This in turn can be used to give another proof of Jategaonkar's result that over an FBN ring an essential extension of an α-smooth module is α-smooth (see [13] for the relevant definitions).

The ring R is said to be integral over its center if each element $r \in R$ satisfies a monic polynomial equation whose coefficients are elements of the center of R. It is well known that a left Noetherian ring integral over its center is a left FBN ring. Theorem 2.1 can be used to simplify the proof of the following result of Chamarie and Hudry.

Corollary 2.2 [5]. *Let R be a left Noetherian ring integral over its center. If* ${}_RM$ *is a finitely generated, essential extension of* ${}_RN$ *and* $|N| < \alpha$, *then* $|M| < \alpha$.

Proof. Let T be any ideal of R, and let Q be a prime ideal of R with $|{}_RR/Q| < \alpha$. By [5, Corollary 1.8], $|{}_RR/Q| = |{}_RR/R(Q \cap C)|$, where C is the center of R. Thus $T(R(Q \cap C)) = T(Q \cap C) = (Q \cap C)T \subseteq QT$ and condition (3) of Theorem 2.1 is satisfied. $\square$

Example 2. If R is a left FBN ring which satisfies the conditions of Theorem 2.1, then the ring

$$\Lambda = \begin{bmatrix} R & 0 \\ R & R \end{bmatrix}$$

of lower triangular 2×2 matrices over R also satisfies the same conditions. The prime ideals of Λ are of two types:

$$\Pi = \begin{bmatrix} P & 0 \\ R & R \end{bmatrix} \quad \text{or} \quad \Omega = \begin{bmatrix} R & 0 \\ R & Q \end{bmatrix},$$

where P and Q are prime ideals of R. Condition (3) of Theorem 2.2 is easy to check if both prime ideals are of the same type. If Π and Ω are as given above and

$|\Lambda/\Pi| \geq \alpha$ and $|\Lambda/\Omega| < \alpha$, then $|\Lambda/\Delta| < \alpha$ for the ideal

$$\Delta = \Pi^{-1}(\Omega\Pi) = \begin{bmatrix} P+Q & 0 \\ P+Q & Q \end{bmatrix}.$$

If $|\Lambda/\Omega| \geq \alpha$ and $|\Lambda/\Pi| < \alpha$, then the desired conclusion follows from the fact that $\Omega^{-1}(\Pi\Omega) = \Pi$.

Example 3. If R is an integral domain not equal to its quotient field F, then the ring

$$\Lambda = \begin{bmatrix} R & 0 \\ F & F \end{bmatrix}$$

does not satisfy the conditions of Theorem 2.1. Let

$$\Pi = \begin{bmatrix} 0 & 0 \\ F & F \end{bmatrix} \quad \text{and} \quad \Omega = \begin{bmatrix} R & 0 \\ F & 0 \end{bmatrix}.$$

Then Λ/Ω is Artinian, but $\Pi^{-1}(\Omega\Pi) = (0)$, and Λ is not left Artinian.

If R is a commutative, Noetherian ring, it is well known that any finitely generated essential extension of an Artinian module is Artinian. As shown by Jategaonkar [13, Theorem 3.7], this property has among its consequences that the intersection of powers of the Jacobson radical of R must be zero. The commutative result has been extended to FBN rings by Jategaonkar [13], and then to bimodules over Noetherian rings by Ginn and Moss [10]. This condition is equivalent to the stability of the torsion radical τ_1 generated by the class of Artinian modules, and this connection with stability will be exploited in the following results.

Corollary 2.3. *Let R be a left* FBN *ring such that for each maximal ideal Q and each non-maximal prime ideal P there exists an ideal D such that R/D is left Artinian and $PD \subsetneq QP$. Then $\bigcap_{n=1}^{\infty} J^n = (0)$, where J is the Jacobson radical of R.*

Proof. By Theorem 2.1, the torsion radical τ_1 is stable, since the τ_1-dense prime ideals of R are the maximal ideals of R. The result follows from the above remarks. (See [6, Lemma 7.4] for a proof that $\bigcap_{n=1}^{\infty} J^n = (0)$ if finitely generated essential extensions of Artinian modules are Artinian. A short proof can also be given by using condition (2) of Theorem 1.2.) $\square$

Theorem 2.4. *Let R be a left and right Noetherian ring, and let $_RM$ be a finitely generated module which contains an essential Artinian submodule. If every submodule of M is finitely annihilated, then M is Artinian.*

Proof. Lenagan [15] has shown that in a left and right Noetherian ring, an ideal is Artinian on the left if and only if it is Artinian on the right. This is equivalent to the condition that the torsion radical τ_1 of R-Mod is a biradical, and so the theorem follows from Proposition 1.4 and Theorem 1.1. □

Theorem 2.5. *Let R be a left and right Noetherian ring which is K-symmetric. Then for any ordinal α, the ideal $\tau_\alpha(R)$ is a complement left ideal and a complement right ideal, and is both a right annihilator and a left annihilator in R.*

Proof. A ring is said to be K-symmetric if for any ordinal α, the torsion radical τ_α is a biradical. It follows from Theorem 1.1 that $\tau_\alpha(R)$ is essentially closed in ${}_RR$ and in R_R, since every left or right ideal of R is finitely annihilated. Since $R/l(\tau_\alpha(R))$ can be embedded in a finite direct sum of copies of $\tau_\alpha(R)$, it follows that $|R/l(\tau_\alpha(R))|<\alpha$, and therefore $|r(l(\tau_\alpha(R)))|<\alpha$, which implies that $r(l(\tau_\alpha(R)))=\tau_\alpha(R)$. Similarly, $l(r(\tau_\alpha(R)))=\tau_\alpha(R)$. □

Fisher [8] has shown that if M is a Noetherian, injective module over a commutative ring, then M must in fact be Artinian. The following theorem shows that this result can be extended to any left FBN ring which satisfies the conditions of Theorem 2.1 for $\alpha=1$. In [6], Chatters and Hajarnavis have shown that an injective left ideal of a left and right Noetherian ring must be Artinian, and this result is also a corollary of the following theorem.

Theorem 2.6. *Let R be a left Noetherian ring such that for any ideal T and any maximal ideal Q such that R/Q is Artinian, there exists an ideal D with R/D left Artinian and $TD \subsetneq QT$. If ${}_RM$ is a finitely generated injective module such that every submodule of M is finitely annihilated, then M is Artinian.*

Proof. Without loss of generality, M may be assumed to be indecomposable. By Theorem 1.1, to show that $\tau_1(M)=M$ it suffices to show that M has a nonzero Artinian submodule. Since every submodule of M is finitely annihilated, a standard argument shows that M is isomorphic to a direct summand of $E(R/P)$ for a prime ideal P of R. Since $E(R/P)$ is finitely generated, it follows that the classical ring of quotients of R/P coincides with R/P by [6, Lemma 1.29], and thus R/P is Artinian as a left R-module. □

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