PIECEWISE NOETHERIAN RINGS

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0. Introduction

In proving the Principal Ideal Theorem, W. Krull [10] made essential use of the fact that in a commutative Noetherian ring \( R \), for each \( P \)-primary ideal \( Q \) there is a maximal chain of \( P \)-primary ideals between \( Q \) and \( R \). This chain, say \( Q = Q_1 \subset \ldots \subset Q_n \subset P \subset R \), corresponds to a composition series for the Artinian ring \( R_P/QR_P \). Krull [11] called the length of such a composition series the length of \( Q \). We extend this notion of length to arbitrary ideals, and use the terminology ideal-length suggested by Zariski and Samuel [17,p.233]. We show that a ring is Noetherian if and only if each ideal has finite ideal-length and a primary decomposition. Following Krull's original proof, we give a weak version of the Principal Ideal Theorem for rings in which each ideal has finite ideal-length.
We say that the commutative ring $R$ is piecewise Noetherian if each ideal has finite ideal-length and the set of prime ideals of $R$ satisfies the ascending chain condition. Then $R$ is piecewise Noetherian if and only if (i) $R$ has a.e.c. on prime ideals; (ii) $R$ has a.e.c. on $P$-primary ideals for each prime ideal $P$; and (iii) each ideal has only finitely many prime ideals minimal over it. Thus a piecewise Noetherian ring has a Noetherian spectrum, since by Theorem 88 and Exercise 25, page 65 of [9] this is equivalent to conditions (i) and (iii) above.

A valuation domain is piecewise Noetherian if and only if it has Krull dimension as a module in the sense of Gabriel and Rentschler [4,7]. (More generally, each ideal of a valuation domain has finite ideal-length if and only if each primary ideal is a power of a prime ideal, that is, if and only if it is a discrete valuation domain in the sense of Gilmer [5].) This shows immediately that the class of piecewise Noetherian rings is much larger than the class of Noetherian rings, although still retaining some desirable properties. For example, a maximal ideal in a piecewise Noetherian ring must be finitely generated. A one-dimensional piecewise Noetherian ring with finitely generated nilradical is in fact Noetherian. The polynomial ring $R[X]$ is piecewise Noetherian if and only if $R$ is piecewise Noetherian. Factor rings, localizations, and finite integral extensions of piecewise Noetherian rings are again piecewise Noetherian. It follows immediately from the work of Arnold in [1] that if $D$ is a
Prüfer domain for which \( D[[X]] \) has finite dimension, then \( D \) is piecewise Noetherian.

Since the class of piecewise Noetherian rings is relatively large, certain extensions of rings with chain conditions may be piecewise Noetherian even though the original chain conditions are not preserved. For example, if \( R \) has Krull dimension as a module but is not Noetherian, then \( R[X] \) fails to have Krull dimension as a module although it is still piecewise Noetherian. It is well known that the integral closure \( \overline{D} \) of a Noetherian domain \( D \) need not be Noetherian. Heinzer has asked (in [8]) if the maximal ideals of \( \overline{D} \) must always be finitely generated. We show that a positive answer to this question is equivalent to the assertion that the integral closure of any Noetherian domain is piecewise Noetherian.

In the first section of the paper we establish the basic properties of rings in which each ideal has finite ideal-length. In the second, we study piecewise Noetherian rings, in particular the connection with Krull dimension in the sense of [7]. Finally we investigate polynomial and integral extensions of piecewise Noetherian rings.

Throughout the paper, \( R \) will denote an associative, commutative ring with unity. All modules are understood to be unital modules. It may interest the reader to note the corresponding terminology for noncommutative rings. The ideal-length of an ideal \( I \) corresponds to the reduced rank of the ring \( R/I \). This notion has
been used in giving a noncommutative version of the Principal Ideal Theorem for Noetherian rings [3]. The general notion has been studied by the first author in connection with Artinian rings of quotients in [2].

1. Rings with Finite Ideal-lengths

For any ideal I of R, comp(I) is the set of prime ideals of R which are minimal over I. Following [15], we say that R has FC (finite components) if comp(I) is finite for every ideal I of R.

Definition 1.1 Let I be an ideal of R, and let S(I) denote the set complement of $\bigcup\{P: P \in \text{comp}(I)\}$ in R. Then I is said to have finite ideal-length if the ring $R_{S(I)}/IR_{S(I)}$ is an Artinian ring. In this case the length of a composition series for $R_{S(I)}/IR_{S(I)}$ is denoted by $\lambda(I)$ and is called the ideal-length of I. If each ideal of R has finite ideal-length, then the ring R is said to have finite ideal-lengths.

Proposition 1.2 Let I be an ideal of R. Then I has finite ideal-length if and only if comp(I) is finite and the ring $R_P/IR_P$ is Artinian for each $P \in \text{comp}(I)$. In this case $\lambda(I) = \sum_{j=1}^{n} \lambda_j(I)$, where $\lambda_j(I)$ is the length of $R/I$ localized at $P_j$, for $\{P_1, \ldots, P_n\} = \text{comp}(I)$.

Proof. If I has finite ideal-length, then comp(I) is finite since each prime ideal $P \in \text{comp}(I)$ corresponds to a prime ideal of the Artinian ring $R_{S(I)}/IR_{S(I)}$. 

Conversely, to show that $R_{S(I)}/IR_{S(I)}$ is Artinian, let $J_1 \supseteq J_2 \supseteq \ldots \supseteq I$ be a sequence of ideals in $R$. Since $\text{comp}(I)$ is finite, $R_p/IR_p$ is Artinian for each $P \in \text{comp}(I)$ and we may choose $n$ so that $(J_n)_P = (J_{n+1})_P = \ldots$ for each $P \in \text{comp}(I)$. This implies that $(J_n)_{S(I)} = (J_{n+1})_{S(I)} = \ldots$ and so $R_{S(I)}/IR_{S(I)}$ is an Artinian ring.

The last statement of the proposition follows from the fact that $R_{S(I)}/IR_{S(I)}$ splits into a direct sum. □

Proposition 1.3 The following conditions on a ring $R$ are equivalent:

1. Each ideal of $R$ has finite ideal-length;
2. $R$ has FC, and for each ideal $I$ of $R$ and each $P \in \text{comp}(I)$, the ring $R_p/IR_p$ is Artinian;
3. $R$ has FC, and for each prime ideal $P$ of $R$, the set of $P$-primary ideals satisfies the ascending chain condition.

Proof. (1) if and only if (2). This follows from Proposition 1.2.

(2) if and only if (3). If condition (2) holds, let $I$ be an ideal of $R$ and let $I \subseteq I_1 \subseteq \ldots$ be an ascending chain of $P$-primary ideals of $R$. Since $R_p/IR_p$ is Artinian, the sequence $\{(I_j/I)_P\}$ stops. Since the ideals $I_j$ are $P$-primary, this implies that the sequence $\{I_j\}$ stops, so $R$ has a.c.c. on $P$-primary ideals.

If condition (3) holds, then for any ideal $I$ and any $P \in \text{comp}(I)$, the ring $R_p/IR_p$ is Noetherian since each ideal is the
image of a $P$-primary ideal of $R$. Since $R_p/IR_p$ has a unique prime ideal, it must be Artinian. \qed

**Proposition 1.4** If $R$ is a ring with finite ideal-lengths, then the same is true for each factor ring and localization of $R$.

Proof. For factor rings this follows immediately from Proposition 1.2.

Let $S$ be a multiplicatively closed subset of $R$, and let $J$ be an ideal of $R_S$. Then $J = I_S$ for some ideal $I$ of $R$, and $\text{comp}(J)$ is a subset of $\{P_S : P \in \text{comp}(I)\}$ and is therefore finite, so $R_S$ has FC. Since every primary ideal of $R_S$ is the localization of a primary ideal of $R$, for each prime ideal $P$ of $R_S$ the a.c.c. on $P$-primary ideals is inherited. Thus each ideal of $R_S$ has finite ideal-length. \qed

In this paper we will use two types of Krull dimension. When we say $R$ has Krull dimension, we mean the dimension studied in [4, 7], which will be defined in the next section. If $R$ satisfies a.c.c. on prime ideals, the ordinal number which is the supremum of lengths of chains of prime ideals of $R$ is the classical Krull dimension of $R$. To avoid confusion we will refer to the latter as the dimension of $R$, or $\text{dim}(R)$.

**Proposition 1.5** Let $R$ be a ring with finite ideal-lengths.

(a) If $\text{dim}(R) = 0$, then $R$ is Artinian.

(b) If $\text{dim}(R) = 1$ and the nilradical $N$ of $R$ is finitely
generated, then $R$ is Noetherian.

Proof. (a) If $\dim(R) = 0$, then $R = R_S(0)$ is Artinian.
(b) Suppose that $\dim(R) = 1$ and the nilradical $N$ is finitely generated. We may assume that $N = 0$, since if $R/N$ is Noetherian then for each prime ideal $P$ of $R$, the ideal $P/N$ is finitely generated. Hence $P$ is finitely generated and $R$ is Noetherian by Cohen's Theorem (Theorem 8 of [9]).

Since $R$ has finite ideal-lengths, $\text{comp}(O) = \{P_1, \ldots, P_k\}$ is a finite set. We have $0 = N = \bigcap_{i=1}^k P_i$, so $R$ is isomorphic to a subdirect product of the rings $R/P_i$, $1 \leq i \leq k$. It suffices to prove that $R/P_i$ is Noetherian for each $i$, so we may assume that $R$ is an integral domain with finite ideal-lengths and $\dim(R) \leq 1$.

Let $I$ be a nonzero ideal of $R$. Then $R/I$ has finite ideal-lengths and $\dim(R) = 0$, so $R/I$ is Artinian by the first part of this proposition. Thus each proper factor ring of $R$ is Noetherian, which implies that $R$ is Noetherian. $\square$

**Theorem 1.6** Let $R$ be a ring with finite ideal-lengths and let $M$ be a maximal ideal of $R$. Then either $M$ is finitely generated or $M$ is the union of a strictly ascending chain of prime ideals.

Proof. The ring $R/M^2$ has finite ideal-lengths and has exactly one prime ideal. By Proposition 1.5, $R/M^2$ is an Artinian ring; thus there is a finitely generated ideal $J$ of $R$ such that $M = J + M^2$. It will suffice to establish our conclusion for the
maximal ideal $M/J$ of the ring $R/J$, so we will assume that $J = 0$, which implies $M = M^2$. There are two cases to consider:

Case 1. The ideal $M$ is a minimal prime ideal of $R$. Since $R$ has finite ideal-lengths, $\operatorname{comp}(0) = \{M, Q_2, \ldots, Q_k\}$ is a finite set. We may choose $t \in M - \bigcup_{i=2}^k Q_i$ such that $Rt + \bigcap_{i=2}^k Q_i = R$. Then $M/Rt$ is the unique prime ideal of the ring $R/Rt$, which has finite ideal-lengths, so $R/Rt$ is an Artinian ring and $M/Rt$ is finitely generated, which implies that $M$ is finitely generated.

Case 2. There is a prime ideal $P_1$ of $R$ satisfying $P_1 \supsetneq M$. If there does not exist a prime ideal $P_2$ strictly between $P_1$ and $M$, then the ring $(R/P_1)_M$ is a one-dimensional domain with finite ideal-lengths and thus is Noetherian by Proposition 1.5. Since $M^2 = M$, Nakayama's Lemma would then imply $M = P_1$, a contradiction. A transfinite induction then gives a tower $\{P_\alpha\}_{\alpha \in A}$ of prime ideals such that $\bigcup_{\alpha \in A} P_\alpha = M$. □

**Theorem 1.7** Let $R$ be a ring with finite ideal-lengths and let $P$ be a prime ideal minimal over a principal ideal $Ra$. Then there does not exist a chain of prime ideals $P \supsetneq P_1 \supsetneq P_2$ such that infinitely many of the symbolic powers $P_1^{(k)}$ are finitely generated.

**Proof.** Without loss of generality we may assume that $R$ is a local ring in which $P$ is the unique maximal ideal and $P_1$ is not a minimal prime. Since $R$ has finite ideal-lengths and the ring $R/Ra$ has only one prime ideal, it follows that $R/Ra$ is Artinian. Thus the sequence of ideals $Ra + P_1 \supsetneq Ra + P_1^{(2)} \supsetneq \ldots$ terminates,
with $Ra + P_1(n) = Ra + P_1(n+1) = \ldots$ for some integer $n$.

If the conclusion of the proposition does not hold, then there is an integer $k \geq n$ such that $P_1(k)$ is finitely generated. Applying Nakayama's Lemma to the ideal $P_1^k R P_1$ of $R P_1$ shows that $P_1^{(k+1)} \subseteq P_1(k)$. Since $Ra + P_1(k) = Ra + P_1^{(k+1)}$, each element $y \in P_1(k)$ has the form $y = z + ra$ for some $z \in P_1^{(k+1)}$ and $r \in R$.

Thus $ra = y - z \in P_1(k)$. Since $a \not\in P_1$, it follows that $r \in P_1(k)$. Thus $P_1(k) \subseteq P_1^{(k+1)} + P_1(k)a$, and so $P_1(k) = P_1^{(k+1)} + P_1(k)a$.

Let $M = P_1(k)/P_1^{(k+1)}$. Then $M$ is a finitely generated $R$-module and we have just shown that $a M = M$; by Nakayama's Lemma, there exists $b \in R$ such that $(1 - ba)M = 0$. Since $a \in P$ and $R$ is local, $1 - ba$ is a unit, which implies $M = 0$, contradicting $P_1^{(k+1)} \subseteq P_1(k)$.

Thus for each $k \geq n$, $P_1(k)$ is not finitely generated. \qed

The ring $R$ is said to be Laskerian if every ideal of $R$ can be expressed as a finite intersection of primary ideals.

**Theorem 1.8** The following conditions on the ring $R$ are equivalent:

1. $R$ is Noetherian;
2. $R$ is Laskerian and every primary ideal of $R$ has finite ideal length;
3. $R$ is Laskerian and has finite ideal lengths.

Proof. (1) implies (2). This is well known.
(2) implies (3). This follows from the fact that a Laskerian ring has FC.

(3) implies (1). Theorem 4.3 of [18] states that a ring is Noetherian if and only if each maximal ideal is finitely generated and each finitely generated ideal has a primary decomposition. Thus it suffices to show that each maximal ideal of \( R \) is finitely generated. This follows from Theorem 1.6, since by Theorem 4 of [6], a Laskerian ring has the ascending chain condition on prime ideals. \( \Box \)

Shock has shown in [16] that a ring \( R \) is Noetherian if the following conditions hold: (i) if \( I \) and \( J \) are ideals of \( R \) with \( I \not\subseteq J \subseteq R \), then there exists an ideal \( J' \) with \( I \subseteq J' \not\subseteq J \) such that \( J/J' \) is simple; (ii) for each ideal \( I \subseteq R \) the socle of \( R/I \) is finitely generated. A proof of Theorem 1.8 can be given using Shock's theorem and the following fact, which can be shown without difficulty: if \( R \) is a subdirectly irreducible ring in which the ideal \( (0) \) has finite ideal-length and a primary decomposition, then \( R \) is Artinian.

2. Piecewise Noetherian Rings

Definition 2.1 The ring \( R \) will be called piecewise Noetherian if it has finite ideal-lengths and has a.c.c. on prime ideals.

It follows immediately from Proposition 1.4 that any factor
ring or localization of a piecewise Noetherian ring is again piecewise Noetherian. As noted in the introduction, a piecewise Noetherian ring has Noetherian spectrum. We often make use of the fact that a zero-dimensional piecewise Noetherian ring is Artinian, as shown by Proposition 1.5.

**Proposition 2.2** Each maximal ideal of a piecewise Noetherian ring is finitely generated.

**Proof.** This follows from Proposition 1.6. □

The Krull dimension of an R-module M will be denoted by $K\dim(M)$, and is defined as follows: if M is Artinian, then $K\dim(M) = 0$; if $\alpha$ is an ordinal and $K\dim(M) < \alpha$, then $K\dim(M) = \alpha$ if there is no infinite descending chain $M_i$ such that $K\dim(M_{i+1}/M_i) < \alpha$ for $i = 1, 2, \ldots$. The Krull dimension of a ring R is its Krull dimension as an R-module.

If the ring R has finite Krull dimension as a module, then by Proposition 7.8 of [7] we have $K\dim(R) = \dim(R)$.

It is easy to see that a Noetherian ring is piecewise Noetherian. Since a Noetherian ring has Krull dimension (Proposition 1.3 of [7]), the following known proposition is stronger. We give a simple proof in the commutative case.

**Proposition 2.3** If R has Krull dimension as an R-module, then R is piecewise Noetherian.
Proof. Let $R$ be a ring with Krull dimension. By Theorem 7.1 of [7], $R$ has the ascending chain condition on prime ideals. We finish the proof by showing that $R$ satisfies condition (2) of Proposition 1.1. By Proposition 7.3 of [7], a ring with Krull dimension has FC. Let $I$ be an ideal of $R$ and let $P \in \text{comp}(I)$. The ring $(R/I)_P$ has Krull dimension and is zero-dimensional, thus is Artinian. □

Corollary 2.4 Let $R$ be a ring with Krull dimension as a module over itself.

(a) The ring $R$ is Noetherian if and only if each ideal of $R$ has a primary decomposition.

(b) Every maximal ideal of $R$ is finitely generated.

Proof. These follow from Theorem 1.8 and Proposition 2.2. □

It is a classical result that the nilradical of a Noetherian ring is nilpotent. This is also true for rings with Krull dimension. We have been able to establish this result for piecewise Noetherian rings in certain special cases, but it is false in general, as the following example shows.

Example. There exists a one-dimensional piecewise Noetherian ring whose nilradical is not nilpotent.

Proof. Let $k$ be a field and let $R_0 = k[X_1, Y_1, X_2, Y_2, \ldots]$. For each $i > 0$ let $M_i$ be the ideal of $R_0$ generated by $X_1$ and all the
Let $I$ be the complement in $R_0$ of $\bigcup \{M_i : i > 0\}$ and let $I$ be the ideal of $R_0$ generated by $\{X_i : i > 0\}$. Define $R = (R_0/I)_S$.

Let $X_i, Y_i$ denote the images in $R$ of $X_i, Y_i$ respectively, and let $m_i$ denote $(M_i/I)_S$. For any $f \in R - M_i$ we have $X_i + f \in S$, so for each $i > 0$, $m_i$ is a maximal ideal of $R$. The localization $R_{m_i}$ has exactly two prime ideals: a maximal ideal, generated by the images of $X_i$ and $Y_i$, and the nilradical, generated by the image of $Y_i$.

Thus $R$ is one-dimensional and by Cohen's Theorem [Theorem 8 of 9], $R_{m_i}$ is a Noetherian ring, which shows that $R$ has a.c.c. on $P$-primary ideals for each prime ideal $P$. If $f \in R_0 \setminus (Y_1, Y_2, \ldots)$, then $f$ involves only finitely many terms $X_i$. Thus $R$ has FC; it has nilradical $N = (y_1, y_2, \ldots)$. Since $(N_{m_i})^i$ is a nonzero ideal of $R_{m_i}$ for each $i$, $N$ is not nilpotent in $R$.

**Lemma 2.5** Let $R$ be a one-dimensional piecewise Noetherian ring with nilradical $N$, and let $\{M_\alpha : \alpha \in A\}$ be the set of maximal ideals of $R$. If $X$ is any $S(N)$-torsion module, then $X = \oplus_{\alpha \in A} X_\alpha$, where $X_\alpha = \{x \in X : (M_\alpha)^nx = 0 \text{ for some integer } n > 0\}$.

Proof. For each element $x \in X$ there exists $c \in S(N)$ with $cx = 0$. Since $c$ is not contained in any minimal prime ideal of $R$, the ring $R/cR$ is Artinian. Thus $Rx$ is a module over an Artinian ring, and so it has a decomposition of the desired type. This shows that $x$ can be written uniquely as a sum of elements in $\oplus_{\alpha \in A} X_\alpha$. $\square$
An $R$-module is said to have finite uniform dimension if it does not contain an infinite direct sum of nonzero submodules.

**Proposition 2.6** Let $R$ be a one-dimensional piecewise Noetherian ring with nilradical $N$. Then $N$ is nilpotent if either (i) $R$ has only finitely many maximal ideals, or (ii) $R$ has finite uniform dimension.

Proof. The ring $R_{S(N)}$ is Artinian, so the ideal $NR_{S(N)}$ is nilpotent. This implies that for some $k$, $N^k$ is $S(N)$-torsion.

First, assume that $R$ has only finitely many maximal ideals. It suffices to show that for each maximal ideal $M$ there is an integer $h$ such that the localization of $N^h$ at $M$ is zero. Therefore we may assume that $R$ has a unique maximal ideal $M$.

If $M$ is also a minimal prime ideal then $R$ is Artinian and we are done. If not, choose $t \in M - \bigcup \{P : P \in \text{comp}(0)\}$. Then $R/rt$ is Artinian so $(N + Rt)/Rt$ is finitely generated and there is a finitely generated ideal $N_1 \subseteq N$ such that $N + Rt = N_1 + Rt$. Since $N_1$ is finitely generated, it suffices to prove that $N$ is nilpotent modulo $N_1$, so we may assume $N_1 = 0$ and thus $N \subseteq Rt$. Since $N = \bigcap \{P : P \in \text{comp}(0)\}$ and $t$ is not contained in any minimal prime ideal of $R$, $N \subseteq Rt$ implies $N = tN$. Thus $N = N^iN$ for each $i > 0$, so $N$ is contained in every power of $M$. For any $y \in N^k$ there is $i$ with $N^i y = 0$. Since $N \subseteq M^i$, $Ny = 0$ and thus $N^{k+1} = 0$.

To prove the second assertion, assume that $R$ has finite uniform dimension. Then by Lemma 2.5, the $S(N)$-torsion ideal $T$ of $R$ is a
finite direct sum $\oplus_{i=1}^{n} T_i$, where $T_i = \{ r \in R : (M_i)^h r = 0 \text{ for some } h \}$, $M_i$ a maximal ideal of $R$ for $1 \leq i \leq n$. For any maximal ideal $M$ distinct from the $M_i$, note that since $N^k \leq T$, the localization $N^k R_M$ is zero. By the argument given in the first part of the proof, we can find an integer $j$, $j \geq k$ such that the localization of $N^j$ at $M_i$ is zero for each $i$, $1 \leq i \leq n$. This implies that $N^j = 0$. □

**Proposition 2.7** Let $R$ be a one-dimensional piecewise Noetherian ring with nilradical $N$. Then $R$ has Krull dimension as a module if and only if every factor module of $N$ has finite uniform dimension.

Proof. If $R$ has Krull dimension then every factor module of $N$ has Krull dimension, and thus by Proposition 1.4 of [7] every factor module of $N$ has finite uniform dimension.

Assume that $N$ has finite uniform dimension on each factor. By Proposition 1.5, $R/N$ is Noetherian, thus certainly has finite uniform dimension on each factor. It follows that $R$ has finite uniform dimension on each factor. Then by Theorem 1.6 of [14] we need only show that every ideal of $R$ contains a power of its radical. Let $I$ be an ideal of $R$. If $R/I$ is zero-dimensional then $R/I$ is Artinian and we are done. So we may assume that $\dim(R/I) = 1$. Then condition (ii) of Proposition 2.6 is satisfied by the ring $R/I$ and the proof is complete. □
Remark Let \( Z \) denote the integers, let \( M \) be a \( Z \)-module, and let \( Z \otimes_N M \) denote Nagata's idealization. That is, \( Z \otimes_N M \) is the (commutative) ring of all matrices of the form

\[
\begin{bmatrix}
k & 0 \\
x & k
\end{bmatrix}
\]

where \( k \in Z \) and \( x \in M \).

1. If \( \mathbb{Z}_2^\infty \) denotes the \( Z \)-injective envelope of \( \mathbb{Z}/2\mathbb{Z} \), then the ring \( R = Z \otimes_N \mathbb{Z}_2^\infty \) is piecewise Noetherian. The nilradical of \( R \) is isomorphic to \( \mathbb{Z}_2^\infty \), so it is not finitely generated. Every factor module of the nilradical has uniform dimension one, so \( R \) is a one-dimensional non-Noetherian ring which has Krull dimension.

2. If \( M \) is an infinite direct sum of copies of \( \mathbb{Z}/2\mathbb{Z} \) then \( Z \otimes_N M \) is piecewise Noetherian but fails to have Krull dimension.

We now determine which valuation domains have finite ideal-lengths and which valuation domains are piecewise Noetherian. For a nonzero prime ideal \( P \) of a valuation domain \( V \) let \( P' = \bigcup \{ Q \in \text{Spec}(V) : Q \nsubseteq P \} \). The ideal \( P' \) is the union of a chain of prime ideals and is therefore a prime ideal. Say that \( P \) is of type A, B, or C as follows:

A. \( P^2 \neq P \) or \( P = 0 \).

B. \( P^2 = P \) and \( P' = P \).

C. \( P^2 = P \) and \( P' \neq P \).

**Proposition 2.8** Let \( V \) be a valuation domain. The following conditions are equivalent:

(1) \( V \) has finite ideal-lengths;
(2) If \( P \) is a prime ideal of \( V \), then \( P \) is either of type A or type B;

(3) An ideal of \( V \) is primary if and only if it is a power of a prime ideal.

Proof. (1) implies (2). Let \( P \) be a prime ideal of \( V \); we must show that \( P \) is not of type C. Suppose it is; then \( P_P \) is of type C in \( V_P \), which has finite ideal-lengths by Proposition 1.4. But since \( (P_P)^2 = P_P \), Proposition 1.6 gives \( P_P = (P')_P \), a contradiction.

(2) implies (3). If \( P \) is a prime ideal of type B or \( P = 0 \), then \( P \) is the only \( P \)-primary ideal, so we may assume that \( P^2 \neq P \). Then the \( P \)-primary ideals of \( V \) correspond to the nonzero ideals of \( V_0 = (V/P')_P \). It is easy to check that \( V_0 \) is a Noetherian valuation domain, and thus the nonzero ideals of \( V_0 \) are powers of its maximal ideal as desired.

(3) implies (1). A valuation domain has FC. We need to show that \( V \) has a.c.c. on \( P \)-primary ideals for each prime ideal \( P \). By (3), an ascending sequence of \( P \)-primary ideals is equivalent to a descending sequence of positive integers and thus terminates. □

Rings satisfying the equivalent conditions of Proposition 2.8 have been called discrete valuation rings (see pp. 189-195 of [5]), where Gilmer proves the equivalence of (2) and (3) of Proposition 2.8. The next proposition is implicit in Lemonnier's work in [12] and [14].
Theorem 2.9  Let $V$ be a valuation domain. The following conditions are equivalent:

1. $V$ is piecewise Noetherian;

2. If $P$ is a prime ideal of $V$, then $P$ is type $A$;

3. $V$ has Krull dimension as a module.

Proof. (1) implies (2). By Proposition 2.8, $P$ is either of type $A$ or of type $B$. Since $V$ has a.c.c. on prime ideals, $P' \neq P$, so $P$ is of type $A$.

(2) implies (3). By Theorem 1.6 of [14] it suffices to show that $V$ has a.c.c. on prime ideals and that each ideal $I$ of $V$ contains a power of its radical.

If $V$ does not have a.c.c. on prime ideals then there is a strictly ascending chain $P_1 \subset P_2 \subset \ldots$ of prime ideals of $V$. But then $P = \bigcup_{i=1}^{\infty} P_i$ is a prime ideal of type $B$, contradicting (2).

Let $I$ be an ideal of $V$. Since $V$ is a valuation domain, the radical of $I$ is a prime ideal $P$, which we may assume is nonzero. It is easy to check that $\bigcap_{n=1}^{\infty} P^n$ is a prime ideal of $V$. By (2), $P$ is of type $A$, so $P' \neq P$ and thus $P' = \bigcap_{n=1}^{\infty} P^n$. Since $P$ is the radical of $I$ we have $P' \supset I$, so there is a positive integer $n$ such that $I$ is not contained in $P^n$, which in a valuation domain implies $P^n \subset I$.

(3) implies (2). This follows from Proposition 2.3. □

Thus a valuation domain having the real numbers as its value group does not have finite ideal-lengths, and a valuation domain
having $\bigoplus_{i=1}^\infty \mathbb{Z}$ (where $\mathbb{Z}$ denotes the integers and the ordering is lexicographic) as its value group has finite ideal-lengths but is not piecewise Noetherian.

Recall that $R$ is a Prüfer domain if for each maximal ideal $M$ of $R$, the localization $R_M$ is a valuation domain. The following proposition extends Exercises 7 and 11, pages 294-295 of [5].

**Proposition 2.10** Let $R$ be a Prüfer domain in which every non-zero element is contained in only finitely many maximal ideals.

(a) The ring $R$ has finite ideal-lengths if and only if each primary ideal of $R$ is a power of a prime ideal.

(b) The ring $R$ is piecewise Noetherian if and only if $R$ has Krull dimension as a module.

Proof. Let $I$ be an ideal of $R$ and let $M$ be a maximal ideal containing $I$. Since $R_M$ is a valuation domain, there is a unique prime ideal $P$ of $R$ which is contained in $M$ and minimal over $I$. If $I$ is a nonzero ideal, then the set of maximal ideals containing $I$ is finite and thus $\text{comp}(I)$ is finite, so $R$ has FC.

The equivalence in (a) may be checked locally and thus follows from Proposition 2.8.

To prove (b), first recall that by Proposition 2.3 if $R$ has Krull dimension then $R$ is piecewise Noetherian. The local version of the converse is Proposition 2.9, so we may assume that $R_M$ has Krull dimension for every maximal ideal $M$ of $R$. We now show that $R/I$ has finite uniform dimension for each ideal $I$ of $R$; it then
follows from Theorem 1.6 of [14] that \( R \) has Krull dimension. Let \( J \) be an ideal containing \( I \) with \( J/I \) a direct sum of cyclic modules. If \( I = 0 \), then since \( R \) is a domain there is only one summand, so we may assume that \( I \) is contained in only finitely many maximal ideals, say \( M_1, \ldots, M_k \). For each \( h, 1 \leq h \leq k \), the ideals of \( R_{M_h} \) are a totally ordered set, so \( (J/I)_{M_h} \) is a cyclic module, say generated by the image of \( x_{h} \), for some \( x_{h} \) in \( J \). Let \( J_0 \) be the ideal generated by \( I \) and the \( x_{h} \)'s; since \( J/I \) and \( J_0/I \) have the same localization at each maximal ideal of the ring \( R/I \), we have \( J = J_0 \). Thus \( J/I \) is finitely generated and the direct sum \( J/I \) has finitely many nonzero summands. \( \Box \)

3. Extensions of piecewise Noetherian rings

Proposition 3.1 If every primary ideal of \( R \) has finite ideal-length, then so does every primary ideal of \( R[X] \).

Proof. Let \( P \) be a prime ideal of \( R[X] \) and let \( Q \) be a \( P \)-primary ideal. We need to show that \( (R[X]/Q)_P \) is a Noetherian ring. Let \( P' = P \cap R \) and \( Q' = Q \cap R \); without loss of generality we may assume that \( Q' = 0 \) and \( R = R_P \). Then \( P' \) is the radical of every proper ideal of \( R \); since \( P' \) is a maximal ideal, this implies that every proper ideal of \( R \) is \( P' \)-primary. Then by hypothesis \( R \) is Artinian and thus \( R[X] \) is Noetherian, which implies that \( (R[X]/Q)_P \) is Noetherian.

The next proposition is a generalization of an idea due to
Heinzer (see Exercise 26, page 65 of [9]).

Let A be a set of prime ideals of R. An ideal I of R is A-radical if for some subset B of A, I = \bigcap \{P : P \in B\}.

Proposition 3.2 Let A be a set of prime ideals of R which does not contain any infinite ascending chain. The following conditions are equivalent:

1. R has a.c.c. on A-radical ideals;
2. For any ideal I of R, \text{comp}(I) \cap A is a finite set.

Proof. (1) implies (2). If condition (2) is assumed to be false, then it fails for some A-radical ideal. By condition (1) we may find an A-radical ideal I which is maximal among A-radical ideals for which condition (2) fails. The ideal I is not prime, so we may choose c_1, c_2 \in R-I with c_1 c_2 \in I. For h = 1, 2, let J_h be the smallest A-radical ideal containing Rc_h + I. Since

\[(I + Rc_1)(I + Rc_2) \subseteq I,\]

we have

\[\text{comp}(I) \cap A \subseteq \text{comp}(J_1) \cap A \cup \text{comp}(J_2) \cap A.\]

But by the maximality of I, the right-hand side of this last is finite, which contradicts the assumption that \text{comp}(I) \cap A is infinite.

(2) implies (1). Suppose that (1) is false and I_1 \subseteq I_2 \subseteq \ldots is a strictly ascending chain of A-radical ideals of R. For each k > 0, choose P_k \in A with I_k \subseteq P_k but I_{k+1} \nsubseteq P_k. Let T = \{j : P_j \text{ is maximal among the } P_k's\}. Since A does not contain an infinite ascending chain, every P_k is contained in some P_j, j \in T. If T were finite, there would be a largest integer h \in T, but it is
easily checked that \( P_{h+1} \neq P_j \) for \( j \leq h \). Thus \( T \) is an infinite set, and by discarding \( I_k, P_k \) for \( k \not\in T \) and renumbering we may assume \( P_1, P_2, \ldots \) are incomparable prime ideals. Let \( L \) be the ideal generated by \( I_1, P_1 I_2, P_1 P_2 I_3, \ldots \). Since each 'monomial' contains either \( P_k \) or \( I_k \), we have \( L \subseteq P_k \) for each \( k \). To show that \( P_k \in \text{comp}(L) \) for each \( k \), let \( P \) be a prime ideal of \( R \) such that \( L \subseteq P \subseteq P_k \) for some \( k \). Then \( P_1 P_2 \ldots P_k I_{k+1} \subseteq P \) and by construction \( P_1 P_2 \ldots P_{k-1} I_{k+1} \neq P \), which implies \( P_k \subseteq P \), so \( P_k = P \) is in \( \text{comp}(L) \). Thus \( \text{comp}(L) \cap A \) is infinite, contradicting the assumption that condition (2) holds. \( \square \)

**Proposition 3.3** The following conditions on \( R \) are equivalent:

(1) \( R[X] \) has FC;

(2) \( R \) has FC and a.c.c. on prime ideals;

(3) \( R[X] \) has FC and a.c.c. on prime ideals.

**Proof.** (1) implies (2): Since \( R \) is a factor ring of \( R[X] \), \( R \) inherits FC.

Let \( P_1 \subseteq P_2 \subseteq \ldots \) be an ascending chain of prime ideals of \( R \). For each \( i \) let \( D_i \) be the intersection of all the maximal ideals of \( R[X] \) which contain \( P_i \). By Theorems 26 and 27 of [2], \( P_i = R \cap D_i \) for each \( i \), so it suffices to show that the chain \( D_1 \subseteq D_2 \subseteq \ldots \) of ideals of \( R[X] \) stops. Let \( A \) be the set of maximal ideals of \( R[X] \); certainly \( A \) contains no infinite ascending chain. Since \( R[X] \) has FC, we may use Proposition 3.2 to conclude that \( R[X] \) has a.c.c. on \( A \)-radical ideals.
(2) implies (3): By Exercise 25, page 65, and Theorem 88 of [9], condition (2) is equivalent to the a.c.c. for radical ideals of R. As shown on pages 45-48 of [10], if R has a.c.c. on radical ideals then so does R[X], and by the equivalence just mentioned we have condition (3).

(3) implies (1). The proof is clear. □

**Theorem 3.4** The following conditions are equivalent:

(1) R is piecewise Noetherian;

(2) R[X] is piecewise Noetherian;

(3) R[X] has finite ideal-lengths.

Proof. (1) implies (2): Since R is piecewise Noetherian, R satisfies condition (2) of Proposition 3.3 and thus R[X] has FC and has a.c.c. on prime ideals. By Proposition 3.1, R[X] has a.c.c. on P-primary ideals for each prime ideal P, so R[X] is piecewise Noetherian.

(2) implies (3). The proof is clear.

(3) implies (1). This follows from Proposition 1.4 and Proposition 3.3. □

**Proposition 3.5** If R has Krull dimension as a module, then R[X_1,\ldots,X_n] is piecewise Noetherian for n > 0.

Proof. The result follows from Propositions 2.3 and 3.4. □

It follows easily from Theorem 3.4 that a finite integral ex-
tension of a piecewise Noetherian ring is piecewise Noetherian.
The next proposition shows that the weaker property of having finite ideal-lengths is also inherited by finite integral extensions.

**Proposition 3.6** A finite integral extension of a ring with finite ideal-lengths has finite ideal-lengths.

Proof. Let \( R \) be a ring with finite ideal-lengths and let \( T \) be a finite integral extension of \( R \). Since \( R \) has finite ideal-lengths, every primary ideal of \( R \) has finite ideal-length. The ring \( T \) is a factor ring of \( R[X_1, \ldots, X_n] \) for some \( n \), so by Proposition 3.1 every primary ideal of \( T \) has finite ideal-length. By Theorem 1 of [8], \( T \) inherits FC from \( R \), so \( T \) has finite ideal-lengths. \( \square \)

**Remark.** Let \( \bar{Z} \) be the integral closure of \( Z \) in the complex numbers and let \( M \) be a maximal ideal of \( \bar{Z} \). It is easy to check that every element of \( M \) has a square root in \( M \), so \( M^2 = M \) and by Nakayama's Lemma, \( M \) is not finitely generated. Since \( \dim(\bar{Z}) = 1 \), \( M \) is not the union of a strictly ascending chain of prime ideals, so by Proposition 1.6, \( \bar{Z} \) does not have finite ideal-lengths. Thus an integral extension of a Noetherian ring need not have finite ideal-lengths.

The next proposition gives three equivalent statements of an open question. Statement (2) is the question as posed on page 372 of Heinzer's 1973 paper [8].
Proposition 3.7 The following statements are equivalent:

(1) The integral closure of any Noetherian domain is piecewise Noetherian;

(2) The maximal ideals of the integral closure of any Noetherian domain are finitely generated;

(3) For any local Noetherian domain $D$ (with maximal ideal $m$) whose integral closure $\overline{D}$ has a unique maximal ideal $M$, the module $M/\overline{m}$ is finitely generated over $\overline{D}$.

Proof. (1) implies (2). This follows from Proposition 2.2.

(2) implies (3). The proof is clear.

(3) implies (1). Let $R$ be a Noetherian domain with integral closure $\overline{R}$. By Theorem 2 of [8], $\overline{R}$ has FC, so we need only show that for each prime ideal $P$ and $P$-primary ideal $Q$ of $\overline{R}$, the ring $\overline{R}/Q$ has a.c.c. on $(P/Q)$-primary ideals. Let $p = R \cap P$ and $S = R - P$.

Since $p_S$ is a maximal ideal of $R_S$, any prime ideal of $(\overline{R})_S$ which contains $p_S$ must contract to $p_S$. Since $\overline{R}$ has FC, there are only finitely many prime ideals of $(\overline{R})_S$ which contain $p_S$, say $P_1 = p_S$, $P_2, \ldots, P_k$, and these are all the maximal ideals of $(\overline{R})_S$. Choose $a \in P_1 - \bigcup_{i=2}^k P_i$. Then $R_1 = R_S[a]$ is a Noetherian domain. Choose a maximal ideal, say $m_1$, which contains $a$. Since $(\overline{R})_S$ is the integral closure of $R_1$, there must be a maximal ideal of $(\overline{R})_S$ which lies over $m_1$; clearly only $(P)_S$ will do. Let $T = R_1 - m_1$ and set $D = (R_1)_T$. Then $D$ is a local Noetherian domain with maximal ideal $m = (m_1)_T$ and the integral closure $\overline{D}$ of $D$ is $(\overline{R})_S)_T$, which
has unique maximal ideal $M = (P_S)_T$. Since $D$ is Noetherian, $\overline{M}$ is a
finitely generated ideal of $\overline{D}$. By our assumption (3), the module
$M/\overline{M}$ is also finitely generated, so $M$ is finitely generated over
$\overline{D}$. The ideal $Q' = (Q_S)_T$ of $\overline{D}$ is $M$-primary, so by Cohen's Theorem
the ring $\overline{D}/Q'$ is Noetherian, which implies that $\overline{R}/Q$ has a.c.c on
P-primary ideals as desired. □

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