COMMUNICATIONS IN ALGEBRA, 15(3), 471–478 (1987)

A NOTE ON PRIME IDEALS WHICH TEST INJECTIVITY

John A. Beachy and William D. Weakley

Department of Mathematical Sciences
Northern Illinois University
DeKalb, Illinois 60115

It is well-known that over a commutative Noetherian ring R the set of all prime ideals of R is a test set for injectivity. That is, a module ${}_RX$ is injective if and only if for any prime ideal $P \subseteq R$, any R-homomorphism $f : P \to X$ can be extended to R. Vámos [**4**] has shown that a set of prime ideals of a commutative Noetherian ring R is a test set if and only if it contains each prime ideal P of R such that R_P is not a field. We extend and clarify his results, showing that a set of prime ideals of a piecewise Noetherian ring is a test set if and only if it contains the set of all essential prime ideals of R. An example is given to show that in a piecewise Noetherian ring R there may exist an essential prime ideal P for which R_P is a field.

A commutative ring R with identity is called a piecewise Noetherian ring if it satisfies the following conditions: (i) the set of prime ideals of R satisfies the ascending chain condi–

tion; (ii) for each prime ideal P of R, the set of P-primary ideals satisfies the ascending chain condition; (iii) each ideal of R has only finitely many prime ideals minimal over it. This property is inherited by any localization; furthermore, it implies that any maximal ideal is finitely generated [**1**, Proposition 2.2]. Conditions (i) and (iii) are well-known to be equivalent to the ascending chain condition on radical ideals. Recall that a ring which satisfies the latter condition is said to have Noetherian spectrum.

LEMMA 1. If R is a piecewise Noetherian ring, then every nonzero R-module has an associated prime ideal.

Proof. We say that a module ${}_RX$ has an associated prime ideal P if X contains a submodule isomorphic to R/P. To prove the lemma it suffices to show that every nonzero cyclic R-module has an associated prime ideal. Suppose that this condition does not hold. Let $\mathbf{S}$ be the set of all radical ideals I such that $I = \mathrm{rad}(A)$ for some ideal A such that R/A has no associated prime ideal. Thus $\mathbf{S}$ is nonempty, and so it contains a maximal element J, since by assumption R has Noetherian spectrum. We may assume without loss of generality that J is the nilradical N of R and that the module ${}_RR$ has no associated prime ideal. If $S(N)$ denotes the complement of the union of the minimal prime ideals of R, then Proposition 1.3 of [**1**] shows that since R is piecewise Noetherian, the localization $R_{S(N)}$ is an Artinian ring. If the localization mapping from R to $R_{S(N)}$ is an embedding, then for any minimal nonzero ideal B of $R_{S(N)}$, it is clear that $R \cap B$ has an associated prime ideal. On the other hand, if the localization map is not an embedding, let x be a nonzero element of R which is torsion with respect to $S(N)$. Then $\operatorname{rad}(\operatorname{Ann}(x)) \supset N$, and so by assumption $Rx \cong R/\operatorname{Ann}(x)$ has an associated prime ideal.

THEOREM 2. If R is a piecewise Noetherian ring, then the set of all essential prime ideals of R is a test set for injectivity.

Proof. The proof follows that of Theorem 3 of [**3**]. Let ${}_RX$ be a module which is P-injective for all essential prime ideals P, and let $f : I \to X$ be an R-homomorphism from an ideal I of R. There exists a maximal extension $f' : J \to X$, and it can easily be checked that J is an essential ideal of R. If J is a proper ideal of R, then it follows from Lemma 1 that R/J has an associated prime ideal P. Then P is essential since $P \supseteq J$, and the proof of Theorem 3 of [**3**] shows that f' can be extended further since X is P-injective, a contradiction.

We note that the proof of Theorem 2 requires only that the ring R has the property that for any proper essential ideal I of R, the module R/I has an associated prime ideal.

LEMMA 3. Let I be an ideal of R, and let ${}_RX$ be an injective module.

(a) A submodule Y of X is I-injective if and only if for each $x \in X$, $Ix \subseteq Y$ implies there exists $x' \in x$ such that $Ix' = (0)$ and $x - x' \in Y$.

(b) If P is a prime ideal and Y is any submodule of X, then $\overline{Y} = \{x \in X \mid cx \in Y \text{ for some } c \in R \backslash P\}$ is I-injective for all ideals $I \not\subseteq P$.

(c) For any subset $\{a_1, \ldots, a_n\} \subseteq R$ such that a_1 is not a zero divisor modulo I, the module

$$K = \{(x_1, \ldots, x_n) \in X^n \mid \textstyle\sum_{i=1}^n a_x x_i = 0\}$$

is I-injective.

Proof. (a) In the long exact homology sequence

$$\mathrm{Hom}(R/I, X) \to \mathrm{Hom}(R/I, X/Y) \to \mathrm{Ext}(R/I, Y) \to 0$$

we have $\mathrm{Ext}(R/I, Y) = 0$ if and only if $\mathrm{Hom}(R/I, X) \to \mathrm{Hom}(R/I, X/Y)$ is an epimorphism, and this translates immediately to the desired condition.

(b) In the given situation, condition (a) is satisfied by letting $x' = 0$.

(c) To show that condition (a) is satisfied, let $(x_1, \ldots, x_n)$ be an element of the direct sum X^n and suppose that $I(x_1, \ldots, x_n) \subseteq K$. Consider the submodule A of $(R/I)^n$ generated by $\overline{a} = (\overline{a}_1, \ldots, \overline{a}_n)$. Define a homomorphism $f : A \to X$ by $f(r\overline{a}) = r(\sum_{i=1}^n a_i x_i)$. This is well-defined since if $r(\overline{a}_1, \ldots, \overline{a}_n) = 0$, then $r \in I$ since a_1 is not a zero-divisor modulo I, and thus by assumption $f(r\overline{a}) = r(\sum_{i=1}^n a_i x_i) = 0$.

Since X is injective, there exists an extension $f' : (R/I)^n \to X$. Let $y_i = f'(e_i)$, where e_i is the element of $(R/I)^n$ with $\overline{1}$ in the i$^{\underline{\text{th}}}$ entry and 0 elsewhere, and let $x' = (y_1, \ldots, y_n)$. Now $Ix' = (0)$, since if $r \in I$, then $ry_i = f'(re_i) = 0$ for all i. Furthermore, $x - x' \in K$ since

$$\textstyle\sum_{i=1}^n a_i y_i = f(\overline{a}_1, \ldots, \overline{a}_n) = \sum_{i=1}^n a_i x_i \ .$$

THEOREM 4. Let R be a piecewise Noetherian ring and let $\mathbf{T}$ be any test set of prime ideals of R. Then $\mathbf{T}$ contains every essential prime ideal of R.

Proof. If P is an essential minimal prime ideal of R, consider $\overline{P} = \{x \in E(R) \mid cx \in P \text{ for some } c \in R \backslash P\}$, where $E(R)$ denotes the injective envelope of the module ${}_R R$. Then $\overline{P}$ is not injective since it is a proper essential submodule of $E(R)$, but by Lemma 3 (b) it is Q-injective for every prime ideal $Q \neq P$. It follows that $P \in \mathbf{T}$.

Next, assume that P is essential but not minimal. Then R_P is not a field, and is piecewise Noetherian, so PR_P is finitely generated by elements $a_1, \ldots, a_n$. Let X be the injective envelope over R_P of the unique simple R_P module S, and let

$$Y = \{(x_1, \ldots, x_n) \in X^n \mid \textstyle\sum_{i=1}^n a_i x_i = 0\} \ .$$

If $Y = X^n$, then X is annihilated by the nonzero ideal PR_P. This is a contradiction since X is a faithful R_P-module. It follows that Y is not injective as an R-module, since it must contain S^n and is therefore an essential R-submodule of the R-injective module X^n.

As an R-module, Y is Q-injective for all prime ideals $Q \not\subseteq P$, since it is a module over the localization R_P, and Lemma 3 (b) may be applied. For any prime ideal $Q \subset P$, some generator a_i of PR_P lies outside QR_P. It follows from Lemma 3 (c) that Y is QR_P-injective for all prime ideals $Q \subset P$, and it is easy to check that this implies that Y is Q-injective for all prime ideals $Q \subset P$. Again, it follows that $P \in \mathbf{T}$.

EXAMPLE 5. There exists a one-dimensional piecewise Noetherian ring R whose nilradical N is an essential minimal prime ideal such that $R_{S(N)}$ is a field.

Proof. For each positive integer i, let p_i denote the i$^{\underline{\text{th}}}$ prime number. Let $R_0 = Z[Y_1, Y_2, \ldots]$, the ring of polynomials in countably many indeterminates with integer coefficients. Let I be the ideal of R_0 generated by $\{p_i Y_i,\ Y_i^{i+1} \mid i > 0\}$. The desired example is $R = R_0/I$.

For each i, let y_i denote the image in R of Y_i. The nilradical of R is $N = (y_1, y_2, \ldots)$, and it follows that R/N is Noetherian, since $R/N \cong Z$. It is then clear that R must have Noetherian spectrum, but R is not Noetherian since N is not nilpotent. By the definition of I, each element of N is torsion with respect to $S(N)$, and thus $R_{S(N)}$ is a field.

Furthermore, the ascending chain condition on N-primary ideals holds trivially.

If P is a prime ideal of R which properly contains N, then the image of P in R/N is a nonzero prime ideal, and hence contains p_k for some integer k. Then for $i \neq k$, we have $Ry_i = (Rp_k + Rp_i)y_i = Rp_k y_i$, so $y_i \in Rp_k$. Thus the ideal (p_k, y_k) of R contains N, and in fact $P = (p_k, y_k)$, so for any P-primary ideal Q of R, R_P/QR_P is Noetherian since the prime ideal of the ring is finitely generated. It follows that R satisfies the ascending chain condition on P-primary ideals, which shows that R is piecewise Noetherian.

The set of finitely generated ideals of R cannot be a test set, and so, in particular, $\mathrm{Spec}(R)\backslash\{N\}$ is not a test set. (To show this, if the finitely generated ideals were a test set, then any direct sum of injective modules would be injective, and R would be Noetherian by Theorem 20.1 of [**2**]. This also follows from Theorem 7 (a) of [**4**].) It follows from Theorem 2 that N is an essential prime ideal of R.

REFERENCES

[**1**] J.A. Beachy and W.D. Weakley, Piecewise Noetherian rings, Commun. Algebra **12**, 1679–2706 (1984).

[**2**] C. Faith, Algebra. II. Ring Theory. Grundlehren der Mathematischen Wissenschaften, No. 101. Springer–Verlag, Berlin-New York, 1976.

[**3**] P.F. Smith, Injective modules and prime ideals, Commun. Algebra **9**, 989–999 (1981).

[**4**] P. Vámos, Ideals and modules testing injectivity, Commun. Algebra **11**, 2495–2505 (1983).

Received: April 1985