A NOTE ON PRIME IDEALS WHICH TEST INJECTIVITY

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It is well-known that over a commutative Noetherian ring $R$ the set of all prime ideals of $R$ is a test set for injectivity. That is, a module $R X$ is injective if and only if for any prime ideal $P \subseteq R$, any $R$-homomorphism $f : P \to X$ can be extended to $R$. Vámos [4] has shown that a set of prime ideals of a commutative Noetherian ring $R$ is a test set if and only if it contains each prime ideal $P$ of $R$ such that $R_P$ is not a field. We extend and clarify his results, showing that a set of prime ideals of a piecewise Noetherian ring is a test set if and only if it contains the set of all essential prime ideals of $R$. An example is given to show that in a piecewise Noetherian ring $R$ there may exist an essential prime ideal $P$ for which $R_P$ is a field.

A commutative ring $R$ with identity is called a piecewise Noetherian ring if it satisfies the following conditions: (i) the set of prime ideals of $R$ satisfies the ascending chain condi–
tion; (ii) for each prime ideal \( P \) of \( R \), the set of \( P \)-primary ideals satisfies the ascending chain condition; (iii) each ideal of \( R \) has only finitely many prime ideals minimal over it. This property is inherited by any localization; furthermore, it implies that any maximal ideal is finitely generated [1, Proposition 2.2]. Conditions (i) and (iii) are well-known to be equivalent to the ascending chain condition on radical ideals. Recall that a ring which satisfies the latter condition is said to have Noetherian spectrum.

**LEMMA 1.** If \( R \) is a piecewise Noetherian ring, then every nonzero \( R \)-module has an associated prime ideal.

**Proof.** We say that a module \( _RX \) has an associated prime ideal \( P \) if \( X \) contains a submodule isomorphic to \( R/P \). To prove the lemma it suffices to show that every nonzero cyclic \( R \)-module has an associated prime ideal. Suppose that this condition does not hold. Let \( S \) be the set of all radical ideals \( I \) such that \( I = \text{rad}(A) \) for some ideal \( A \) such that \( R/A \) has no associated prime ideal. Thus \( S \) is nonempty, and so it contains a maximal element \( J \), since by assumption \( R \) has Noetherian spectrum. We may assume without loss of generality that \( J \) is the nilradical \( N \) of \( R \) and that the module \( _RR \) has no associated prime ideal. If \( S(N) \) denotes the complement of the union of the minimal prime ideals of \( R \), then Proposition 1.3 of [1] shows that since \( R \) is piecewise Noetherian, the localization \( R_{S(N)} \) is an Artinian
ring. If the localization mapping from $R$ to $R_{S(N)}$ is an embedding, then for any minimal nonzero ideal $B$ of $R_{S(N)}$, it is clear that $R \cap B$ has an associated prime ideal. On the other hand, if the localization map is not an embedding, let $x$ be a nonzero element of $R$ which is torsion with respect to $S(N)$. Then $\text{rad(Ann}(x)) \supset N$, and so by assumption $Rx \cong R/\text{Ann}(x)$ has an associated prime ideal.

**THEOREM 2.** If $R$ is a piecewise Noetherian ring, then the set of all essential prime ideals of $R$ is a test set for injectivity.

Proof. The proof follows that of Theorem 3 of [3]. Let $RX$ be a module which is $P$-injective for all essential prime ideals $P$, and let $f : I \to X$ be an $R$-homomorphism from an ideal $I$ of $R$. There exists a maximal extension $f' : J \to X$, and it can easily be checked that $J$ is an essential ideal of $R$. If $J$ is a proper ideal of $R$, then it follows from Lemma 1 that $R/J$ has an associated prime ideal $P$. Then $P$ is essential since $P \supset J$, and the proof of Theorem 3 of [3] shows that $f'$ can be extended further since $X$ is $P$-injective, a contradiction.

We note that the proof of Theorem 2 requires only that the ring $R$ has the property that for any proper essential ideal $I$ of $R$, the module $R/I$ has an associated prime ideal.

**LEMMA 3.** Let $I$ be an ideal of $R$, and let $RX$ be an injective module.
(a) A submodule $Y$ of $X$ is $I$-injective if and only if for each $x \in X$, $Ix \subseteq Y$ implies there exists $x' \in x$ such that $Ix' = (0)$ and $x - x' \in Y$.

(b) If $P$ is a prime ideal and $Y$ is any submodule of $X$, then $\overline{Y} = \{x \in X \mid cx \in Y \text{ for some } c \in R \setminus P\}$ is $I$-injective for all ideals $I \not\subseteq P$.

(c) For any subset $\{a_1, \ldots, a_n\} \subseteq R$ such that $a_1$ is not a zero divisor modulo $I$, the module

$$K = \{(x_1, \ldots, x_n) \in X^n \mid \sum_{i=1}^n a_i x_i = 0\}$$

is $I$-injective.

Proof. (a) In the long exact homology sequence

$$\text{Hom}(R/I, X) \to \text{Hom}(R/I, X/Y) \to \text{Ext}(R/I, Y) \to 0$$

we have $\text{Ext}(R/I, Y) = 0$ if and only if $\text{Hom}(R/I, X) \to \text{Hom}(R/I, X/Y)$ is an epimorphism, and this translates immediately to the desired condition.

(b) In the given situation, condition (a) is satisfied by letting $x' = 0$.

(c) To show that condition (a) is satisfied, let $(x_1, \ldots, x_n)$ be an element of the direct sum $X^n$ and suppose that $I(x_1, \ldots, x_n) \subseteq K$. Consider the submodule $A$ of $(R/I)^n$ generated by $\overline{a} = (\overline{a}_1, \ldots, \overline{a}_n)$. Define a homomorphism $f : A \to X$ by $f(r\overline{a}) = r(\sum_{i=1}^n a_i x_i)$. This is well-defined since if $r(\overline{a}_1, \ldots, \overline{a}_n) = 0$, then $r \in I$ since $a_1$ is not a zero-divisor modulo $I$, and thus by assumption $f(r\overline{a}) = r(\sum_{i=1}^n a_i x_i) = 0$. 
Since $X$ is injective, there exists an extension $f' : (R/I)^n \to X$. Let $y_i = f'(e_i)$, where $e_i$ is the element of $(R/I)^n$ with $1$ in the $i\text{th}$ entry and $0$ elsewhere, and let $x' = (y_1, \ldots, y_n)$. Now $Ix' = (0)$, since if $r \in I$, then $ry_i = f'(re_i) = 0$ for all $i$. Furthermore, $x - x' \in K$ since
\[
\sum_{i=1}^{n} a_i y_i = f(\bar{a}_1, \ldots, \bar{a}_n) = \sum_{i=1}^{n} a_i x_i .
\]

**Theorem 4.** Let $R$ be a piecewise Noetherian ring and let $T$ be any test set of prime ideals of $R$. Then $T$ contains every essential prime ideal of $R$.

**Proof.** If $P$ is an essential minimal prime ideal of $R$, consider $\mathcal{P} = \{ x \in E(R) \mid cx \in P \text{ for some } c \in R \setminus P \}$, where $E(R)$ denotes the injective envelope of the module $R_R$. Then $\mathcal{P}$ is not injective since it is a proper essential submodule of $E(R)$, but by Lemma 3 (b) it is $Q$-injective for every prime ideal $Q \neq P$. It follows that $P \in T$.

Next, assume that $P$ is essential but not minimal. Then $R_P$ is not a field, and is piecewise Noetherian, so $PR_P$ is finitely generated by elements $a_1, \ldots, a_n$. Let $X$ be the injective envelope over $R_P$ of the unique simple $R_P$ module $S$, and let
\[
Y = \{ (x_1, \ldots, x_n) \in X^n \mid \sum_{i=1}^{n} a_i x_i = 0 \} .
\]
If $Y = X^n$, then $X$ is annihilated by the nonzero ideal $PR_P$. This is a contradiction since $X$ is a faithful $R_P$-module. It follows that $Y$ is not injective as an $R$-module, since it must
contain \(S^n\) and is therefore an essential \(R\)-submodule of the \(R\)-injective module \(X^n\).

As an \(R\)-module, \(Y\) is \(Q\)-injective for all prime ideals \(Q \not\subseteq P\), since it is a module over the localization \(R_P\), and Lemma 3 (b) may be applied. For any prime ideal \(Q \subset P\), some generator \(a_i\) of \(PR_P\) lies outside \(QR_P\). It follows from Lemma 3 (c) that \(Y\) is \(QR_P\)-injective for all prime ideals \(Q \subset P\), and it is easy to check that this implies that \(Y\) is \(Q\)-injective for all prime ideals \(Q \subset P\). Again, it follows that \(P \in T\).

**EXAMPLE 5.** There exists a one-dimensional piecewise Noetherian ring \(R\) whose nilradical \(N\) is an essential minimal prime ideal such that \(R_{S(N)}\) is a field.

Proof. For each positive integer \(i\), let \(p_i\) denote the \(i^{th}\) prime number. Let \(R_0 = \mathbb{Z}[Y_1, Y_2, \ldots]\), the ring of polynomials in countably many indeterminates with integer coefficients. Let \(I\) be the ideal of \(R_0\) generated by \(\{p_i Y_i, Y_i^{i+1} \mid i > 0\}\). The desired example is \(R = R_0/I\).

For each \(i\), let \(y_i\) denote the image in \(R\) of \(Y_i\). The nilradical of \(R\) is \(N = (y_1, y_2, \ldots)\), and it follows that \(R/N\) is Noetherian, since \(R/N \cong \mathbb{Z}\). It is then clear that \(R\) must have Noetherian spectrum, but \(R\) is not Noetherian since \(N\) is not nilpotent. By the definition of \(I\), each element of \(N\) is torsion with respect to \(S(N)\), and thus \(R_{S(N)}\) is a field.
Furthermore, the ascending chain condition on $N$-primary ideals holds trivially.

If $P$ is a prime ideal of $R$ which properly contains $N$, then the image of $P$ in $R/N$ is a nonzero prime ideal, and hence contains $p_k$ for some integer $k$. Then for $i \neq k$, we have $Ry_i = (Rp_k + Rp_i)y_i = Rp_k y_i$, so $y_i \in Rp_k$. Thus the ideal $(p_k, y_k)$ of $R$ contains $N$, and in fact $P = (p_k, y_k)$, so for any $P$-primary ideal $Q$ of $R$, $RP/QR$ is Noetherian since the prime ideal of the ring is finitely generated. It follows that $R$ satisfies the ascending chain condition on $P$-primary ideals, which shows that $R$ is piecewise Noetherian.

The set of finitely generated ideals of $R$ cannot be a test set, and so, in particular, Spec$(R) \setminus \{N\}$ is not a test set. (To show this, if the finitely generated ideals were a test set, then any direct sum of injective modules would be injective, and $R$ would be Noetherian by Theorem 20.1 of [2]. This also follows from Theorem 7 (a) of [4].) It follows from Theorem 2 that $N$ is an essential prime ideal of $R$.

REFERENCES


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