ON UNIVERSAL LOCALIZATION AT SEMIPRIME GOLDIE IDEALS

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Abstract

In this paper we consider an alternative to Ore localization at a semiprime ideal S of a left Noetherian ring R. In [5], P.M.Cohn introduced the universal $\Sigma(S)$-inverting ring, for the set $\Sigma(S)$ of all square matrices over R that remain regular on reduction modulo S. We give an account of this universal localization from an approach that uses general ring theoretic techniques rather than those of the theory of free rings. Together with a review of a number of known results, we present a simplification of Malcolmson's construction ([8]) of the universal Σ-inverting ring that makes use of properties particular to this situation.

We first recall some known results when P is a prime ideal of a commutative Noetherian ring R. We will use $J(R)$ to denote the Jacobson radical of a ring R. Let R_P denote the localization of R with respect to the multiplicative set $R \setminus P$, with canonical homomorphism $\lambda : R \to R_P$. It is well-known that $R_P/J(R_P)$ is isomorphic to the field of quotients of R/P. Furthermore, if $\phi : R \to T$ is any ring homomorphism such that $P = \phi^{-1}(J(T))$ and the induced mapping $\overline{\phi} : R/P \to T/J(T)$ is the embedding of R/P in its field of fractions, then $\phi(c) + J(T)$ is invertible in $T/J(T)$, for all elements $c \in R \setminus P$. Since an element invertible modulo the Jacobson radical is invertible, it follows that $\phi(c)$ is invertible in T, for all $c \in R \setminus P$. By the definition of R_P, there is a unique homomorphism $\theta : R_P \to T$ with $\theta\lambda = \phi$. It is this property that we will use to define the universal localization of a noncommutative ring R at a semiprime ideal S.

1 Properties of the universal localization

Throughout this section, R will denote a left Noetherian ring (with identity), and S will denote a semiprime ideal of R. Consider the following conditions for a ring T and ring homomorphism $\phi : R \to T$.

J_1: The ring $T/J(T)$ is a semisimple Artinian ring.

J_2: $S = \phi^{-1}(J(T))$

J_3: The ring $T/J(T)$ is a classical ring of left quotients of R/S, under the induced embedding $\overline{\phi} : R/S \to T/J(T)$.

J_4: If $\theta : R \to T'$ is a ring homomorphism such that conditions J_1, J_2, and J_3 are satisfied, then there exists a unique ring homomorphism $\theta' : T \to T'$ such that $\theta = \theta'\phi$.

Since condition J_4 states that T is universal with respect to conditions J_1 through J_3, a standard argument shows that if there exists a ring satisfying conditions J_1 through J_4, then it must be unique. Before considering the existence of such a ring, we give the relevant definition.

Definition 1.1 *Let R be a left Noetherian ring, with semiprime ideal S. A ring satisfying the above conditions J_1 through J_4 is called the universal localization of R at S, and will be denoted by R_S, with canonical homomorphism $\lambda : R \to R_S$.*

For any ideal I of R, the set of elements $c \in R$ that are regular modulo I will be denoted by $C(I)$. We need to extend this definition relative to S, as follows. For any positive integer n, let $\Sigma_n(S)$ denote the set of all matrices C such that C belongs to the $n \times n$ matrix ring $M_n(R)$ and the image of C in $M_n(R/S)$ is a regular element. This will be abbreviated by saying that C is regular modulo S. Note that $C \in \Sigma_n(S)$ if and only if the image of C is invertible under the canonical mapping from $M_n(R)$ into the left classical quotient ring $Q_{cl}(M_n(R/S)) \cong M_n(Q_{cl}(R/S))$. The union over all $n > 0$ of $\Sigma_n(S)$ will be denoted by $\Sigma(S)$.

The universal localization $R_{\Sigma(S)}$ of R at $\Sigma(S)$ is defined as the universal $\Sigma(S)$-inverting ring. It can be constructed as follows (see [4] and [5] for details). For each n and each $n \times n$ matrix $[c_{ij}]$ in $\Sigma(S)$, take a set of n^2 symbols $[d_{ij}]$, and take a ring presentation of $R_{\Sigma(S)}$ consisting of all of the elements of R, as well as all of the elements d_{ij} as generators; as defining relations take all of the relations holding in R, together with all of the relations $[c_{ij}][d_{ij}] = I$ and $[d_{ij}][c_{ij}] = I$ which define all of the inverses of the matrices in $\Sigma(S)$.

Theorem 1.2 *Let R be a left Noetherian ring. For any semiprime ideal S of R, the universal localization R_S exists, and is unique up to isomorphism.*

Proof. The uniqueness follows immediately from the definition. If $\Sigma(S)$ is the set of all square matrices that are regular modulo S, then Theorem 4.1 of [4] shows that the universal $\Sigma(S)$-inverting ring $R_{\Sigma(S)}$ satisfies properties J_1 through J_3. If $\phi : R \to T$ is any ring that satisfies conditions J_1 through J_3, then for any matrix $C \in \Sigma_n(S)$ it follows that $\phi(C)$ is invertible modulo $M_n(J(T)) = J(M_n(T))$, and hence $\phi(C)$ is invertible in $M_n(T)$. Since $R_{\Sigma(S)}$ is the universal $\Sigma(S)$-inverting ring, it satisfies condition J_4. □

Proposition 1.3 *Let R be a left Noetherian ring. If S is a localizable semiprime ideal of R, then the universal localization R_S coincides with the Ore localization of R at S.*

Proof. If $C(S)$ satisfies the left Ore condition, it is well-known that the ring of left quotients of R with respect to the multiplicative set $C(S)$ satisfies conditions J_1 through J_3. Since this ring of left quotients is universal with respect to inverting elements in $C(S)$, the argument used in the proof of the previous theorem can be repeated. □

Proposition 1.4 *Let R be a left Noetherian ring, with semiprime ideal S.*

(a) *The canonical mapping $\lambda : R \to R_S$ is an epimorphism in the category of rings.*

(b) *The ring R_S is flat as a right module over R if and only if S is a left localizable ideal.*

Proof. Part (a) follows from the characterization of R_S as the universal $\Sigma(S)$-inverting ring. Part (b) is Corollary 3.2 of [1]. □

Theorem 1.5 *Let R be left Noetherian, let N be the prime radical of R, and let $K = \ker(\lambda)$, for the canonical homomorphism $\lambda : R \to R_S$.*

(a) *The kernel K is the intersection of all ideals $I \subseteq N$ such that $C(N) \subseteq C(I)$.*

(b) *The ring R/K is a left order in a left Artinian ring, and R_N is naturally isomorphic to $Q_{cl}(R/K)$.*

Proof. Parts (a) and (b) are Proposition 1.3 and Theorem 1.4 of [2], respectively. □

It is shown in Example 4 of [1] that the universal localization at a semiprime ideal of a left Noetherian ring need not be left Noetherian. In fact, the ring given as an example is a Noetherian ring finitely generated (as a module) over its center. On the other hand, it is possible to determine conditions under which the universal localization is left Artinian.

Corollary 1.6 *Let R be left Noetherian, let S be a semiprime ideal of R, and let $K = \ker(\lambda)$, for the canonical homomorphism $\lambda : R \to R_S$.*

(a) *The universal localization R_S is left Artinian if and only if $S^n \subseteq K$ for some $n > 0$.*

(b) *If P is a minimal prime ideal of R, then R_P is left Artinian.*

Proof. See Theorem 1.5 and Corollary 1.6 of [1]. □

The symbolic powers of S will be defined as in the commutative situation, by extending S^n to $R_S\lambda(S^n)R_S$ and then contracting back to R.

Definition 1.7 *Let R be a left Noetherian ring, with semiprime ideal S, and let $\lambda : R \to R_S$ be the canonical homomorphism.*

The n^{th} symbolic power of S, denoted by $S^{(n)}$, is defined as

$$S^{(n)} = \lambda^{-1}(R_S\lambda(S^n)R_S) \ .$$

Proposition 1.8 *Let R be a left Noetherian ring, with semiprime ideal S, and let $\lambda : R \longrightarrow R_S$ be the canonical homomorphism.*

(a) $S^{(n)} = \lambda^{-1}(J(R_S)^n)$.

(b) *$S^{(n)}$ is the intersection of all ideals I such that $S^n \subseteq I \subseteq S$ and $C(S) \subseteq C(I)$.*

(c) *$C(S)$ is a left Ore set modulo $S^{(n)}$.*

Proof. Parts (a) and (b) follow from Proposition 2.2 of [2]. Since $S/S^{(n)}$ is the prime radical of $R/S^{(n)}$, part (c) follows from part (b) and Small's Theorem. □

A number of additional results can be proved for the symbolic powers of S. For example, for all positive integers n, m we have $S^{(n)}S^{(m)} \subseteq S^{(n+m)}$. For commutative Noetherian rings it is a standard result that $\ker(\lambda) = \cap_{n=1}^{\infty} P^{(n)}$. This fails in the noncommutative setting, as shown by the following example. Let R be the ring of lower triangular 2×2 matrices with entries from the rational numbers, in which the first entry on the diagonal has odd denominator. If S is the Jacobson radical of R, then R/S is semisimple Artinian, and so $R_S = R$ and $S^{(n)} = S^n$ for all n. Thus $\ker(\lambda) = (0) \neq \cap_{n=1}^{\infty} S^{(n)}$. The following proposition gives some positive information along these lines.

Proposition 1.9 *Let R be a left Noetherian ring, with semiprime ideal S, and let $\lambda : R \longrightarrow R_S$ be the canonical homomorphism. Then* $\ker(\lambda) \subseteq \cap_{n=1}^{\infty} S^{(n)}$.

Proof. This follows from the fact that the symbolic power $S^{(n)}$ is the kernel of the canonical homomorphism from R into $R_S/J(R_S)^n$, and this homomorphism satisfies properties J_1 through J_3 in the definition of R_S. □

Given a prime ideal P of a two-sided Noetherian ring R, and any positive integer n, the left symbolic powers H_n of P are defined by Goldie [6] as follows: $H_1 = P$, and by induction, H_n is defined as the two-sided $C(P)$ closure of PH_{n-1}. Lemma 2.3 of [2] shows that $P^{(n)} = H_n$, for any positive integer n.

Assume that R is Noetherian and let P be a prime ideal of R. For each positive integer n, let Q_n be the Artinian classical ring of quotients of $R/P^{(n)}$. Then there is a canonical epimorphism $Q_{n+1} \to Q_n$, for $n = 1, 2, \ldots$. Let $\widehat{Q}$ be the inverse limit of the rings $\{Q_n\}_{n=1}^{\infty}$ under these epimorphisms, and let $\mu : R \to \widehat{Q}$ be the induced homomorphism. Goldie's localization Q of R at P is defined as the intersection of all subrings Q' of $\widehat{Q}$ such that $Q'/J(Q')$ is simple Artinian, $\cap J(Q')^n = (0)$, and $\mu(P) \subseteq J(Q') \subseteq J(\widehat{Q})$. The proof of Theorem 1 of [7] shows that $Q/J(Q) \cong Q_{cl}(R/P)$, $\cap_{n=1}^{\infty} J(Q)^n = (0)$, and $P^{(n)} = \mu^{-1}(J(Q)^n)$.

Theorem 1.10 *Let P be a prime ideal of the Noetherian ring R. Then Goldie's localization of R at P is isomorphic to* $R_P / \cap_{n=1}^{\infty} J(R_P)^n$.

Proof. See Theorem 2.4 of [1]. □

2 Equivalence of quotients

Throughout this section ${}_RX$ will denote a fixed left R-module, and the direct sum of n copies of X will be denoted by X^n. The notation $x \in X^n$ will be used to denote a row vector with entries in X, and the corresponding column vector will be denoted by x^t. The identity of $M_n(R)$ will be denoted by I_n; the subscript will be omitted when the size is clear from the context.

Throughout the remainder of the paper, Σ will denote a set of square matrices over R such that

(i) Σ contains all permutation matrices;

(ii) if $C, D \in \Sigma$, then $\begin{bmatrix} C & A \\ 0 & D \end{bmatrix} \in \Sigma$ for any matrix A of the appropriate size; and

(iii) if $C, D \in \Sigma$ and CD is defined, then $CD \in \Sigma$.

We note that if R is left Noetherian and S is a semiprime ideal of R, then the set $\Sigma(S)$ of all square matrices regular modulo S satisfies the above conditions.

For the given set Σ, an element $x \in X$ is said to be Σ-*torsion* if x is an entry of a column vector v^t with entries in X such that $Cv^t = 0$ for some $C \in \Sigma$. The proof of Proposition 2.1 of [5] shows that the set of all Σ-torsion elements of X is a submodule, which we denote by $\mathrm{rad}_\Sigma(X)$. Then X is said to be Σ-*torsion* if $\mathrm{rad}_\Sigma(X) = X$.

It should be noted that if R is left Noetherian, S is a semiprime ideal of R, and X is finitely generated, then it is possible to give another characterization of $\Sigma(S)$-torsion modules. By Proposition 1.1 of [2], X is $\Sigma(S)$-torsion if and only if X/SX is a torsion module over R/S.

The elements of a module of quotients, denoted by X_Σ, will be constructed as equivalence classes of ordered triples (a, C, x^t), where $a \in R^n$, $C \in \Sigma_n$, and $x \in X^n$ (for any positive n). The ordered triples are modeled on the element $aC^{-1}x^t$, where C is invertible, as would be the case over R_Σ. Let $a \in R^n$, $C \in \Sigma_n$, $x \in X^n$, $b \in R^m$, $D \in \Sigma_m$, $y \in X^m$, and assume that U, V are invertible $n \times n$ matrices. If C and D are invertible, then we have the following identities.

(i) $[a \quad b] \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}^{-1} \begin{bmatrix} x^t \\ y^t \end{bmatrix} = aC^{-1}x^t + bD^{-1}y^t$

(ii) $aU(VCU)^{-1}Vx^t = aC^{-1}x^t$

(iii) $aC^{-1}0 = 0 = 0C^{-1}x^t$

An addition of triples is based on the first of these identities. The second motivates the definition of the initial equivalence relation for triples. We say that $(a, C, x^t) \equiv (b, D, y^t)$ if there exist invertible matrices U, V in Σ such that $b = aU$, $D = VCU$ and $y^t = Vx^t$. It is easily checked that this defines an equivalence relation. (The proof of transitivity uses the fact that Σ is closed under products.) We note that $(a, C, x^t) \equiv (b, D, y^t)$ only if C and D have the same size. Equation (iii) provides the motivation for the definition of the subsemigroup that induces the final equivalence relation.

Definition 2.1 *Let* (a, C, x^t), (b, D, y^t) *be ordered triples with* $a, b \in R^n$, $C, D \in \Sigma_n$ *and* $x, y \in X^n$, *for some positive integer* n. *If there exist invertible* $n \times n$ *matrices* U, V *in* Σ *such that* $b = aU$, $D = VCU$ *and* $y^t = Vx^t$, *then we say that* (a, C, x^t) *is congruent to* (b, D, y^t) *via* U, V, *written* $(a, C, x^t) \equiv (b, D, y^t)$ *via* U, V.

For $a \in R^n$, $C \in \Sigma_n$ *and* $x \in X^n$, *the notation* $(a : C : x^t)$ *will be used for the equivalence class of the ordered triple* (a, C, x^t) *under the equivalence relation* $\equiv$. *The set of all such equivalence classes, for all positive integers* n, *will be denoted by* $\Sigma^{-1}X$.

The subset of $\Sigma^{-1}X$ *consisting of all equivalence classes of elements of the form*

$$(e_1, E_1, 0), \quad \text{or} \quad (0, E_2, e_2^t), \quad \text{or} \quad \left([e_1 \quad 0], \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}, \begin{bmatrix} 0 \\ e_2^t \end{bmatrix} \right)$$

for some $e_1 \in R^m$, $E_1 \in \Sigma_m$, $E_2 \in \Sigma_n$, *and* $e_2 \in X^n$ *will be denoted by* $\Sigma_0^{-1}X$.

Proposition 2.2 *The sum of elements* $(a : C : x^t)$, $(b : D : y^t) \in \Sigma^{-1}X$ *defined by*

$$(a : C : x^t) + (b : D : y^t) = \left([a \quad b] : \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix} : \begin{bmatrix} x^t \\ y^t \end{bmatrix} \right)$$

yields an associative, commutative binary operation on $\Sigma^{-1}X$.

Proof. If $(a_1, C_1, x_1^t) \equiv (a_2, C_2, x_2^t)$ via invertible matrices $U, V \in \Sigma$ and $(b_1, D_1, y_1^t) \equiv (b_2, D_2, y_2^t)$ via invertible matrices $U', V' \in \Sigma$, then it is easy to check that the respective sums are congruent via the matrices $\begin{bmatrix} U & 0 \\ 0 & U' \end{bmatrix}$ and $\begin{bmatrix} V & 0 \\ 0 & V' \end{bmatrix}$, which are invertible and belong to Σ. Thus addition of equivalence classes is well-defined on $\Sigma^{-1}X$, and it is associative by definition. Using permutation matrices, it is straightforward to check that addition of equivalence classes is commutative. □

Proposition 2.3 *For elements* $\overline{x}, \overline{y} \in \Sigma^{-1}X$, *the relation* $\sim$ *defined by*

$$\overline{x} \sim \overline{y} \quad \text{if there exist } \overline{z}_1, \overline{z}_2 \in \Sigma_0^{-1}X \text{ such that} \quad \overline{x} + \overline{z}_1 = \overline{y} + \overline{z}_2$$

is a congruence on the semigroup $\Sigma^{-1}X$. *The set* $\Sigma^{-1}X/\sim$ *of equivalence classes of this congruence is an abelian group.*

Proof. Using permutation matrices, it is easy to check that $\Sigma_0^{-1}X$ is closed under addition. Since addition in $\Sigma^{-1}X$ is associative and commutative, it follows easily that $\sim$ is a congruence. Therefore addition of equivalence classes in $\Sigma^{-1}X/\sim$ is well-defined and satisfies the associative and commutative laws. The equivalence class of $\Sigma_0^{-1}X$ is the zero element, and the following computation shows the existence of additive inverses.

For any element $(a : C : x^t) \in \Sigma^{-1}X$, we have

$$(a, C, x^t) + (a, C, -x^t) \;=\; \left([a \;\; a], \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}, \begin{bmatrix} x^t \\ -x^t \end{bmatrix}\right) .$$

Since $[a \;\; a] \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} = [a \;\; 0]$, $\begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}$, and $\begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \begin{bmatrix} x^t \\ -x^t \end{bmatrix} = \begin{bmatrix} 0 \\ -x^t \end{bmatrix}$, we have

$$(a, C, x^t) + (a, C, -x^t) \;\equiv\; \left([a \;\; 0], \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}, \begin{bmatrix} 0 \\ -x^t \end{bmatrix}\right)$$

via $\begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}, \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}$, and the last element belongs to $\Sigma_0^{-1}X$. $\square$

If C_1, C_2 are invertible matrices such that $C_2A_1 = A_2C_1$ for matrices A_1, A_2, then $A_1C_1^{-1} = C_2^{-1}A_2$ and so $aA_1C_1^{-1}x^t = aC_2^{-1}A_2x^t$. This motivates the following lemma for triples $(aA_1 : C_1 : x^t)$ and $(a : C_2 : A_2x^t)$ such that $C_2A_1 = A_2C_1$, a situation reminiscent of the left Ore condition. This lemma will prove to be very useful computationally.

Lemma 2.4 *Let $a \in R^m$, $C_1 \in \Sigma_n$, $x \in X^n$, and let A_1 be any $m \times n$ matrix over R. If there exist an $m \times n$ matrix A_2 and a matrix $C_2 \in \Sigma_m$ such that $C_2A_1 = A_2C_1$, then*

$$(aA_1 : C_1 : x^t) \sim (a : C_2 : A_2x^t) .$$

Proof. If a, C_1, C_2, A_1, A_2, and x are as stated, then

$$[a \;\; aA_1] \begin{bmatrix} I_m & -A_1 \\ 0 & I_n \end{bmatrix} = [a \;\; 0] \;\;, \;\; \begin{bmatrix} I_m & A_2 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} 0 \\ x^t \end{bmatrix} = \begin{bmatrix} A_2x^t \\ x^t \end{bmatrix}$$

and

$$\begin{bmatrix} I_m & A_2 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} C_2 & 0 \\ 0 & C_1 \end{bmatrix} \begin{bmatrix} I_m & -A_1 \\ 0 & I_n \end{bmatrix} \;=\; \begin{bmatrix} C_2 & 0 \\ 0 & C_1 \end{bmatrix} .$$

Therefore

$$\left([a \;\; aA_1], \begin{bmatrix} C_2 & 0 \\ 0 & C_1 \end{bmatrix}, \begin{bmatrix} 0 \\ x^t \end{bmatrix}\right) \;\equiv\; \left([a \;\; 0], \begin{bmatrix} C_2 & 0 \\ 0 & C_1 \end{bmatrix}, \begin{bmatrix} A_2x^t \\ x^t \end{bmatrix}\right) .$$

We then have

$$\begin{aligned} (aA_1 : C_1 : x^t) &\sim (a : C_2 : 0) + (aA_1 : C_1 : x^t) \\ &= \left([a \;\; aA_1] : \begin{bmatrix} C_2 & 0 \\ 0 & C_1 \end{bmatrix} : \begin{bmatrix} 0 \\ x^t \end{bmatrix}\right) \\ &= \left([a \;\; 0] : \begin{bmatrix} C_2 & 0 \\ 0 & C_1 \end{bmatrix} : \begin{bmatrix} A_2x^t \\ x^t \end{bmatrix}\right) \\ &= (a : C_2 : A_2x^t) + (0 : C_1 : x^t) \\ &\sim (a : C_2 : A_2x^t) . \end{aligned}$$

This completes the proof. $\square$

If S is a semiprime ideal of R for which $C(S)$ is a left denominator set, then for each $(a : C : x^t) \in \Sigma(S)^{-1}X$ there exist elements $y \in X$ and $d \in C(S)$ such that $(a : C : x^t) \sim (1 : d : y)$. To see this, let $\lambda : R \to R_S$ be the classical left localization of R at $C(S)$. We can assume without loss of generality that λ is one-to-one. Let $(a : C : x^t) \in \Sigma(S)^{-1}X$. Then $\lambda(C)$ is invertible over R_S, so it is possible to find a common denominator $d \in C(S)$ for the entries of $\lambda(a)\lambda(C)^{-1}$. Thus we have elements $d \in C(S)$ and $b \in R^n$ such that $da = bC$, and then it follows from Lemma 2.4 that $(1 \cdot a : C : x^t) \sim (1 : d : bx^t)$.

Proposition 2.5 *Let $a \in R^n$, $C \in \Sigma_n$, and $x \in X^n$.*

(a) *If $b \in R^n$ and $y \in X^n$, then*

$$(a : C : x^t) + (a : C : y^t) \sim (a : C : (x+y)^t)$$

and

$$(a : C : x^t) + (b : C : x^t) \sim (a + b : C : x^t)\ .$$

(b) *For any matrices P, Q such that $PC, CQ \in \Sigma_n$,*

$$(a : C : x^t) \sim (a : PC : Px^t) \quad \text{and} \quad (a : C : x^t) \sim (aQ : CQ : x^t)\ .$$

(c) *For any $b \in R^m$, $D \in \Sigma_m$, $y \in X^m$ and any matrices A, B of the appropriate size,*

$$(a : C : x^t) \sim \left([a \quad b] : \begin{bmatrix} C & A \\ 0 & D \end{bmatrix} : \begin{bmatrix} x^t \\ 0 \end{bmatrix} \right)$$

and

$$(a : C : x^t) \sim \left([0 \quad a] : \begin{bmatrix} D & B \\ 0 & C \end{bmatrix} : \begin{bmatrix} y^t \\ x^t \end{bmatrix} \right)\ .$$

Proof. (a) Since $C\,[I \quad I] = {}^{[I \quad I]} \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}$, it follows from Lemma 2.4 that

$$\left(a\,[I \quad I] : \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} : \begin{bmatrix} x^t \\ y^t \end{bmatrix} \right) \sim \left(a : C : {}^{[I \quad I]} \begin{bmatrix} x^t \\ y^t \end{bmatrix} \right)$$

and so $(a : C : x^t) + (a : C : y^t) \sim (a : C : (x+y)^t)$. The second half of condition (a) follows in a similar fashion.

(b) By Lemma 2.4, we have $(aI : C : x^t) \sim (a : PC : Px^t)$ since $(PC)(I) = (P)(C)$. Similarly, $(C)(Q) = (I)(CQ)$ shows that $(aQ : CQ : x^t) \sim (a : C : Ix^t)$.

(c) Since $\begin{bmatrix} C & A \\ 0 & D \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} C$, by Lemma 2.4 we have

$$\left({}^{[a \quad b]} \begin{bmatrix} I \\ 0 \end{bmatrix} : C : x^t \right) \sim \left([a \quad b] : \begin{bmatrix} C & A \\ 0 & D \end{bmatrix} : \begin{bmatrix} I \\ 0 \end{bmatrix} x^t \right)\ .$$

Finally, since $C\,[0 \quad I] = [0 \quad I] \begin{bmatrix} D & B \\ 0 & C \end{bmatrix}$, it again follows from Lemma 2.4 that

$$\left(a\,[0 \quad I] : \begin{bmatrix} D & B \\ 0 & C \end{bmatrix} : \begin{bmatrix} y^t \\ x^t \end{bmatrix}\right) \sim \left(a : C : [0 \quad I] \begin{bmatrix} y^t \\ x^t \end{bmatrix}\right) .$$

This completes the proof. $\square$

Proposition 2.6 *Let* $(a : C : x^t), (b : D : y^t) \in \Sigma^{-1}X$. *Then* $(a : C : x^t) \sim (b : D : y^t)$ *if and only if there exist vectors* e_1, u *over* R *and* e_2, v *over* X *(of the appropriate size) and matrices* $\begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}$ *and* $P, Q \in \Sigma$ *such that* $uv^t = 0$ *and*

$$(a, C, x^t) + (b, D, -y^t) + \left([e_1 \quad 0], \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}, \begin{bmatrix} 0 \\ e_2^t \end{bmatrix}\right) = (uQ, PQ, Pv^t) .$$

Proof. First assume that $(a : C : x^t) \sim (b : D : y^t)$. Then $(a : C : x^t) + (b : D : -y^t)$ is equivalent to zero, since $\Sigma^{-1}X/\sim$ is an abelian group and $(b : D : -y^t)$ represents the additive inverse of $(b : D : y^t)$. By definition of $\sim$, there exist $\overline{z}_1, \overline{z}_2 \in \Sigma_0^{-1}X$ such that $(a : C : x^t) + (b : D : -y^t) + \overline{z}_1 = \overline{z}_2$. If $\overline{z}_2 = \left([f_1 \quad 0] : \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix} : \begin{bmatrix} 0 \\ f_2^t \end{bmatrix}\right)$, then using the definition of the equivalence relation $\equiv$, there exist invertible matrices $U, V \in \Sigma$ of the appropriate size such that

$$(a, C, x^t) + (b, D, -y^t) + z_1 = \left([f_1 \quad 0]U,\ \ V \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix} U,\ \ V \begin{bmatrix} 0 \\ f_2^t \end{bmatrix}\right) .$$

Since z_1 already has the desired form, we only need to factor the right hand side. Let $Q = \begin{bmatrix} I & 0 \\ 0 & F_2 \end{bmatrix} U$ and $P = V \begin{bmatrix} F_1 & 0 \\ 0 & I \end{bmatrix}$. This factorization yields (uQ, PQ, Pv^t), for $u = [f_1 \ \ 0]$ and $v = [0 \ \ f_2]$, and then $uv^t = 0$.

Conversely, suppose that the given condition holds. Then by the definition of $\sim$ we have $(a : C : x^t) + (b : D : -y^t) \sim (uQ : PQ : Pv^t)$. Using the previous proposition and Lemma 2.4, we have

$$(uQ : PQ : Pv^t) \sim (u : I : v^t) \sim (1 : 1 : uv^t) = (1 : 1 : 0) ,$$

and thus $(a : C : x^t) \sim (b : D : y^t)$. $\square$

Recall that an element $x \in X$ is Σ-torsion if it is an entry in a vector $v \in X^n$ such that $Cv^t = 0$ for some $C \in \Sigma$. Since Σ is closed under products (when defined) and contains all permutation matrices, it can be assumed that x is the first entry of v^t.

Theorem 2.7 *Let $x, y \in X$.*

(a) *In $\Sigma^{-1}X$, $(1:1:x) \sim (1:1:y)$ if and only if $x = av^t$ and $y = bw^t$ for some $a, b \in R^n$, $v, w \in X^n$ and some $n > 0$ such that there exist $C, D, P, Q \in \Sigma_n$ satisfying $aD = bQ$, $Cv^t = Pw^t$ and $CD = PQ$.*

(b) *Furthermore, $x - y \in \mathrm{rad}_\Sigma(X)$ if and only if in condition* (a) *it is possible to take $a = b$, $C = P$ and $D = Q = I$.*

Proof. (a) If $(1:1:x) \sim (1:1:y)$, then there exist $\overline{z}_1, \overline{z}_2 \in \Sigma_0^{-1}X$ such that $(1,1,x) + z_1 \equiv (1,1,y) + z_2$. If $z_1 = \left([e_1 \;\; 0], \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}, \begin{bmatrix} 0 \\ e_2^t \end{bmatrix}\right)$, then $(1,1,x) + z_1$ has the form

$$\left([1 \;\; e_1 \;\; 0], \begin{bmatrix} 1 & 0 & 0 \\ 0 & E_1 & 0 \\ 0 & 0 & E_2 \end{bmatrix}, \begin{bmatrix} x \\ 0 \\ e_2^t \end{bmatrix}\right) .$$

This can be factored in the form (aD, CD, Cv^t) for

$$a = [1 \;\; e_1 \;\; 0],\; C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & E_1 & 0 \\ 0 & 0 & I \end{bmatrix},\; D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & E_2 \end{bmatrix},\; v = [x \;\; 0 \;\; e_2],$$

with $av^t = x$. A similar factorization of $(1,1,y) + z_2$, multiplied by the invertible matrices obtained from the definition of the relation $\equiv$, gives $(aD, CD, Cv^t) = (bQ, PQ, Pw^t)$, where $bw^t = y$.

Conversely, if the stated condition holds, then we have

$$(1:1:x) \sim (a:I:v^t) \sim (aD:CD:Cv^t) =$$

$$(bQ:PQ:Pw^t) \sim (b:I:w^t) \sim (1:1:y) .$$

(b) If $x, y \in X$ with $x - y \in \mathrm{rad}_\Sigma(X)$, then $x - y$ is the first entry of a vector u such that $Cu^t = 0$ for some $C \in \Sigma$. It is possible to write $u = v - w$, where x and y are the first entries of v and w, respectively, so that $Cv^t = Cw^t$. If a denotes the vector over R with 1 as its first entry and zeroes elsewhere, then $av^t = x$ and $aw^t = y$, giving the desired result.

Conversely, if the condition is satisfied, then $C(v^t - w^t) = 0$, so all entries of $v^t - w^t$ belong to $\mathrm{rad}_\Sigma(X)$. It follows that $x - y = a(v^t - w^t) \in \mathrm{rad}_\Sigma(X)$, completing the proof. □

Malcolmson [8] has shown that $\lambda : R \to R_\Sigma$ given by $\lambda(r) = (1:1:r)$ is a ring homomorphism, which inverts the matrices in Σ. It follows immediately that any Σ-torsion element (torsion on either left or right) must be mapped to zero by λ. The following example shows, since the left Σ-torsion ideal differs from the right Σ-torsion ideal, that it is possible to have equivalent elements $(1:1:r)$ and $(1:1:s)$ for which $r - s$ does not belong to the left Σ-torsion ideal.

Let R be the following ring of lower triangular matrices with entries from the ring of integers or the ring of integers modulo 2, as indicated. Let S be the prime radical of R, let Σ be the set $\Sigma(S)$, and consider the ideal I defined below.

$$R = \begin{bmatrix} Z_2 & 0 & 0 \\ Z_2 & Z & 0 \\ Z_2 & Z_2 & Z_2 \end{bmatrix} \qquad S = \begin{bmatrix} 0 & 0 & 0 \\ Z_2 & 0 & 0 \\ Z_2 & Z_2 & 0 \end{bmatrix} \qquad I = \begin{bmatrix} 0 & 0 & 0 \\ Z_2 & 0 & 0 \\ Z_2 & 0 & 0 \end{bmatrix}$$

A matrix belongs to $C(S)$ if and only if its entries on the main diagonal are all nonzero, so $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in C(S)$. This element annihilates I/SI, which shows by Proposition 1.1 of [2] that I is left $\Sigma(S)$-torsion.

Furthermore, I is the left $\Sigma(S)$-torsion ideal since S/S^2 is not $\Sigma(S)$-torsion. Arguing by symmetry, the right $\Sigma(S)$-torsion ideal is the bottom row of S, so the kernel of λ must be S. Then $(1 : 1 : x)$ is equivalent to $(1 : 1 : 0)$ for the element $x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, but $x - 0$ is not $\Sigma(S)$-torsion. This establishes a clear distinction between conditions (a) and (b) of Theorem 2.7.

3 Modules of quotients

Throughout this section ${}_RX$ will denote a fixed left R-module. We begin with the definition of a module of quotients.

Definition 3.1 *The set of equivalence classes of* $\Sigma^{-1}X/\sim$ *will be denoted by* X_Σ. *The notation* $[a : C : x^t]$ *will be used for the class of* $(a : C : x^t) \in \Sigma^{-1}X$.

Proposition 2.6 of Section 2 shows that our definition of equivalence for elements of R_Σ is the same as that of Malcolmson [8]. The multiplication about to be defined coincides with that in Malcolmson's construction, so we have in fact defined the universal Σ-inverting ring. Thus properties of R_Σ may be used in constructing modules of quotients. It should be noted that the scalar multiplication defined below can be used to construct the ring R_Σ, and the necessary properties can be verified using only the techniques of this paper.

Let $a, r \in R^n$, $C \in \Sigma_n$, $b \in R^m$, $D \in \Sigma_m$, and $y \in X^m$. If C, D are invertible, then we have the following identity, which motivates the definition of a scalar multiplication.

$$[a \;\; 0] \begin{bmatrix} C & -r^t b \\ 0 & D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ y^t \end{bmatrix} = aC^{-1}r^t \cdot bD^{-1}y^t$$

Proposition 3.2 *The scalar product of elements* $(a : C : r^t) \in \Sigma^{-1}R$ *and* $(b : D : y^t) \in \Sigma^{-1}X$ *defined by*

$$(a : C : r^t) \cdot (b : D : y^t) = \left([a \ \ 0] : \begin{bmatrix} C & -r^t b \\ 0 & D \end{bmatrix} : \begin{bmatrix} 0 \\ y^t \end{bmatrix} \right)$$

yields a well-defined, associative operation.

Proof. If $(a_1, C_1, r_1^t) \equiv (a_2, C_2, r_2^t)$ via invertible matrices $U, V \in \Sigma$ and $(b_1, D_1, y_1^t) \equiv (b_2, D_2, y_2^t)$ via invertible matrices $U', V' \in \Sigma$, then we have

$$\left([a_1 \ \ 0], \begin{bmatrix} C_1 & -r_1^t b_1 \\ 0 & D_1 \end{bmatrix}, \begin{bmatrix} 0 \\ y_1^t \end{bmatrix} \right) \equiv \left([a_2 \ \ 0], \begin{bmatrix} C_2 & -r_2^t b_2 \\ 0 & D_2 \end{bmatrix}, \begin{bmatrix} 0 \\ y_2^t \end{bmatrix} \right)$$

via the matrices $\begin{bmatrix} U & 0 \\ 0 & U' \end{bmatrix}$ and $\begin{bmatrix} V & 0 \\ 0 & V' \end{bmatrix}$.

If $\overline{p}, \overline{q} \in \Sigma^{-1}R$ and $\overline{x} \in \Sigma^{-1}X$, then $\overline{p}(\overline{q}\,\overline{x}) = (\overline{p}\,\overline{q})\overline{x}$ as a consequence of the way in which matrices are combined. $\square$

Lemma 3.3 *The following conditions hold for scalar multiplication.*

(a) *If* $(a : C : r^t) \in \Sigma^{-1}R$ *and* $(1 : 1 : x) \in \Sigma^{-1}X$*, then*

$$(a : C : r^t)(1 : 1 : x) \sim (a : C : r^t x) \, .$$

(b) *If* $(1 : 1 : s) \in \Sigma^{-1}R$ *and* $(a : C : x^t) \in \Sigma^{-1}X$*, then*

$$(1 : 1 : s)(a : C : x^t) \sim (sa : C : x^t) \, .$$

Proof. (a) Since $C \, [I \ \ 0] = [I \ \ r^t] \begin{bmatrix} C & -r^t \\ 0 & 1 \end{bmatrix}$, we have

$$\begin{aligned} (a : C : r^t)(1 : 1 : x) &= \left(a \, [I \ \ 0] : \begin{bmatrix} C & -r^t \\ 0 & 1 \end{bmatrix} : \begin{bmatrix} 0 \\ x \end{bmatrix} \right) \\ &\sim \left(a : C : [I \ \ r^t] \begin{bmatrix} 0 \\ x \end{bmatrix} \right) = (a : C : r^t x) \, . \end{aligned}$$

(b) Since $\begin{bmatrix} 1 & -sa \\ 0 & C \end{bmatrix} \begin{bmatrix} sa \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} C$, we have

$$\begin{aligned} (sa : C : x^t) &= \left([1 \ \ 0] \begin{bmatrix} sa \\ I \end{bmatrix} : C : x^t \right) \\ &\sim \left([1 \ \ 0] : \begin{bmatrix} 1 & -sa \\ 0 & C \end{bmatrix} : \begin{bmatrix} 0 \\ I \end{bmatrix} x^t \right) \\ &= (1 : 1 : s)(a : C : x^t) \end{aligned}$$

This completes the proof. $\square$

Lemma 3.4 *Let* $\overline{p} = (u : P : r^t)$, $\overline{q} = (v : Q : s^t) \in \Sigma^{-1}R$, *and let* $\overline{x} = (a : C : x^t)$, $\overline{y} = (b : D : y^t) \in \Sigma^{-1}X$. *Then*

$$(\overline{p} + \overline{q})\overline{x} \sim \overline{p}\,\overline{x} + \overline{q}\,\overline{x} \quad \textit{and} \quad \overline{q}(\overline{x} + \overline{y}) \sim \overline{q}\,\overline{x} + \overline{q}\,\overline{y}\ .$$

Proof. We have the following equalities.

$$(\overline{p} + \overline{q})\overline{x} = \left([u \ \ 0 \ \ v \ \ 0] \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} : \begin{bmatrix} P & 0 & -r^t a \\ 0 & Q & -s^t a \\ 0 & 0 & C \end{bmatrix} : \begin{bmatrix} 0 \\ 0 \\ x^t \end{bmatrix} \right)$$

$$\overline{p}\,\overline{x} + \overline{q}\,\overline{x} = \left([u \ \ 0 \ \ v \ \ 0] : \begin{bmatrix} P & -r^t a & 0 & 0 \\ 0 & C & 0 & 0 \\ 0 & 0 & Q & -s^t a \\ 0 & 0 & 0 & C \end{bmatrix} : \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ x^t \end{bmatrix} \right)$$

The two expressions are equal by Lemma 2.4, since

$$\begin{bmatrix} P & -r^t a & 0 & 0 \\ 0 & C & 0 & 0 \\ 0 & 0 & Q & -s^t a \\ 0 & 0 & 0 & C \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} P & 0 & -r^t a \\ 0 & Q & -s^t a \\ 0 & 0 & C \end{bmatrix}.$$

Finally, by Lemma 2.4 the expressions given below for $\overline{q}(\overline{x}+\overline{y})$ and $\overline{q}\,\overline{x}+\overline{q}\,\overline{y}$ are equal since we have the following identity.

$$\begin{bmatrix} Q & -s^t a & -s^t b \\ 0 & C & 0 \\ 0 & 0 & D \end{bmatrix} \begin{bmatrix} I & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} Q & -s^t a & 0 & 0 \\ 0 & C & 0 & 0 \\ 0 & 0 & Q & -s^t b \\ 0 & 0 & 0 & D \end{bmatrix}$$

$$\overline{q}\,\overline{x} + \overline{q}\,\overline{y} = \left([v \ \ 0 \ \ 0] \begin{bmatrix} I & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} : \begin{bmatrix} Q & -s^t a & 0 & 0 \\ 0 & C & 0 & 0 \\ 0 & 0 & Q & -s^t b \\ 0 & 0 & 0 & D \end{bmatrix} : \begin{bmatrix} 0 \\ x^t \\ 0 \\ y^t \end{bmatrix} \right)$$

$$\overline{q}(\overline{x} + \overline{y}) = \left([v \ \ 0 \ \ 0] : \begin{bmatrix} Q & -s^t a & -s^t b \\ 0 & C & 0 \\ 0 & 0 & D \end{bmatrix} : \begin{bmatrix} I & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ x^t \\ 0 \\ y^t \end{bmatrix} \right)$$

This completes the proof. □

Theorem 3.5 *For any module* ${}_R X$, *the set* X_Σ *is a unital left module over* R_Σ.

Proof. We have shown in Proposition 2.3 that X_Σ is an abelian group under addition. To show that the multiplication defined in Proposition 3.2 respects the equivalence relation $\sim$ on $\Sigma^{-1}X$, let $\overline{p}, \overline{q} \in \Sigma^{-1}R$ and $\overline{x}, \overline{y} \in \Sigma^{-1}X$ with $\overline{p} \sim \overline{q}$ and $\overline{x} \sim \overline{y}$. Then there exist $\overline{z}_1, \overline{z}_2 \in \Sigma_0^{-1}R$ and $\overline{z}_3, \overline{z}_4 \in \Sigma_0^{-1}X$ with $\overline{p}+\overline{z}_1 = \overline{q}+\overline{z}_2$ and $\overline{x}+\overline{z}_3 = \overline{y}+\overline{z}_4$. Thus we have

$$(\overline{p}+\overline{z}_1)\overline{x} = (\overline{q}+\overline{z}_2)\overline{x} \qquad \text{and} \qquad \overline{q}(\overline{x}+\overline{z}_3) = \overline{q}(\overline{y}+\overline{z}_4)\ .$$

It follows from Lemma 3.4 that

$$\overline{p}\,\overline{x}+\overline{z}_1\overline{x} \sim \overline{q}\,\overline{x}+\overline{z}_2\overline{x} \qquad \text{and} \qquad \overline{q}\,\overline{x}+\overline{q}\,\overline{z}_3 \sim \overline{q}\,\overline{y}+\overline{q}\,\overline{z}_4\ .$$

By definition of scalar multiplication, we have $\overline{z}_1\overline{x}, \overline{z}_2\overline{x}, \overline{q}\,\overline{z}_3, \overline{q}\,\overline{z}_4 \in \Sigma_0^{-1}X$, and so we obtain $\overline{p}\,\overline{x} \sim \overline{q}\,\overline{y}$.

The distributive laws hold by Lemma 3.4. Finally, X_Σ is a unital left R_Σ-module by Lemma 3.3. $\square$

Proposition 3.6 *For the module ${}_RX$, define the mapping $\eta : X \to X_\Sigma$ by $\eta(x) = [1 : 1 : x]$, for all $x \in X$. Then η is an R-homomorphism.*

Proof. The ring R acts on X_Σ via the homomorphism $\lambda : R \to R_\Sigma$ defined by $\lambda(r) = [1 : 1 : r]$. Thus by Lemma 3.3 (a), for any $r \in R$ and any $x \in X$ we have $\eta(rx) = [1 : 1 : rx] = [1 : 1 : r][1 : 1 : x] = \lambda(r)\eta(x)$. $\square$

Proposition 3.7 *Let $[a : C : x^t] \in X_\Sigma$. Then $[a : C : x^t] = \lambda(a)\lambda(C)^{-1}\eta(x^t)$ for the canonical mappings $\lambda : R \to R_\Sigma$ and $\eta : X \to X_\Sigma$.*

Proof. Let e_i denote the vector with 1 in the ith entry and zero elsewhere. Assume that $C \in \Sigma_n$, and let $C = [c_{ij}]$ for elements $c_{ij} \in R$. It follows from Lemma 3.3 (b) that for a fixed $k > 0$,

$$\begin{aligned}\sum_{i=1}^{n}[1 : 1 : c_{ki}][e_i : C : e_j^t] &= \sum_{i=1}^{n}[c_{ki}e_i : C : e_j^t] \\ &= [e_kC : C : e_j^t] = [e_k : I : e_j^t] \\ &= [1 : 1 : e_ke_j^t] = \delta_{kj}\ .\end{aligned}$$

(Proposition 2.5 (a) and (b) and Lemma 2.4 have also been used.)

A similar argument holds on the other side, showing that the entries of $\lambda(C)^{-1}$ are just the elements $[e_i : C : e_j^t]$ in R_Σ. Having found $\lambda(C)^{-1}$, it follows that

$$\lambda(a)\lambda(C)^{-1}\eta(x^t) = \sum_{j=1}^{n}\left(\sum_{i=1}^{n}[1 : 1 : a_i][e_i : C : e_j^t]\right)[1 : 1 : x_j]$$

$$
\begin{aligned}
&= \sum_{j=1}^{n}\left(\sum_{i=1}^{n}[a_ie_i : C : e_j^t]\right)[1:1:x_j] \\
&= \sum_{j=1}^{n}[a : C : e_j^t][1:1:x_j] = \sum_{j=1}^{n}[a : C : e_j^t x_j] \\
&= [a : C : x^t]\ .
\end{aligned}
$$

This completes the proof. □

We say that the module ${}_RX$ is Σ-*torsionfree* if $Cx^t = 0$ implies $x = 0$, for all $C \in \Sigma_n$ and all $x \in X^n$. We say that X is Σ-*divisible* if for each $x \in X^n$ and each $C \in \Sigma$ there exists $y \in X^n$ such that $Cy^t = x^t$.

Theorem 3.8 *The homomorphism $\eta : X \to X_\Sigma$ is an isomorphism if and only if X is Σ-torsionfree and Σ-divisible.*

Proof. If η is an isomorphism, then X has a natural structure as a left R_Σ-module, and so $Cx^t = 0$ implies $x^t = \lambda(C)^{-1}Cx^t = 0$ for any $x \in X^n$, showing that X is Σ-torsionfree. Similarly, X is Σ-divisible since for any $x \in X^n$ we have $x^t = C(\lambda(C)^{-1}x^t)$.

Conversely, suppose that X is Σ-torsionfree and Σ-divisible. For any element $[a : C : x^t] \in X_\Sigma$, there exists y such that $x^t = Cy^t$, and then

$$[a : C : x^t] = [a : C : Cy^t] = [a : I : y^t] = [1 : 1 : ay^t]$$

by Proposition 2.5 (b) and Lemma 2.4. Thus $[a : C : x^t] = \eta(ay^t)$ and so η is an epimorphism. Now let $x \in \ker(\eta)$. By Theorem 2.7 (a) there exist $a, b \in R^n$, $v, w \in X^n$, and $C, D, P, Q \in \Sigma_n$ such that $x = av^t$, $bw^t = 0$, $aD = bQ$, $Cv^t = Pw^t$, and $CD = PQ$. Since X is Σ-divisible there exist $v_1, w_1 \in X^n$ such that $v^t = Dv_1^t$ and $w^t = Qw_1^t$. Therefore $CDv_1^t = Cv^t = Pw^t = PQw_1^t$, and so $v_1^t = w_1^t$ since $CD = PQ$ and X is Σ-torsionfree. But then $x = av^t = a(Dv_1^t) = (bQ)w_1^t = bw^t = 0$, and so η is a monomorphism. □

As in Theorem 4.9 of [9], the following corollary implies that if R is a left hereditary ring, then so is the universal localization R_Σ. The corollary can be proved using an argument similar to the standard one for the class of torsionfree divisible modules over an integral domain.

Corollary 3.9 *In the category of left R-modules, the class of left R_Σ-modules is closed under extensions.*

Theorem 3.10 *For any module ${}_RX$, the module of quotients X_Σ is naturally isomorphic to $R_\Sigma \otimes_R X$.*

Proof. Let $\epsilon : X \longrightarrow R_\Sigma \otimes_R X$ be the natural homomorphism defined by $\epsilon(x) = 1 \otimes x$, for all $x \in X$. Then since X_Σ is an R_Σ-module, we can define $\theta : R_\Sigma \otimes_R X \longrightarrow X_\Sigma$ with $\theta(q \otimes x) = q\eta(x)$, for all $q \in R_\Sigma$ and $x \in X$, where η is the canonical mapping from X to X_Σ. Then $\theta\epsilon = \eta$, and for $x \in X$ and $\lambda(a)\lambda(C)^{-1}\lambda(r^t) \in R_\Sigma$, we have

$$\begin{aligned}\theta\left(\lambda(a)\lambda(C)^{-1}\lambda(r^t) \otimes x\right) &= \lambda(a)\lambda(C)^{-1}\lambda(r^t)\eta(x) \\ &= \lambda(a)\lambda(C)^{-1}\eta(r^t x)\ .\end{aligned}$$

It follows that given $\lambda(a)\lambda(C)^{-1}\eta(x^t) \in X_\Sigma$, we have

$$\lambda(a)\lambda(C)^{-1}\eta(x^t) = \theta\left(\sum_{i=1}^{n} \lambda(a)\lambda(C)^{-1}\lambda(e_i^t) \otimes e_i x^t\right)\ ,$$

which shows that θ is onto.

To show that θ is one-to-one, we will show that it has an inverse. For $(a : C : x^t) \in \Sigma^{-1}X$, define

$$\phi((a : C : x^t)) = \sum_{i=1}^{n} \lambda(a)\lambda(C)^{-1}\lambda(e_i^t) \otimes e_i x^t\ .$$

It is clear that ϕ is well-defined on $\Sigma^{-1}X$ and additive. To show that it is well-defined on X_Σ, by Proposition 2.6 it suffices to show that $\phi((a : C : x^t)) = 0$ for any element $(a : C : x^t)$ of the form $(uQ : PQ : Pv^t)$ with $P, Q \in \Sigma$ and $uv^t = 0$. We have

$$\begin{aligned}\phi((uQ : PQ : Pv^t)) &= \sum_{i=1}^{n} \lambda(uQ)\lambda(PQ)^{-1}\lambda(e_i^t) \otimes e_i P v^t \\ &= \sum_{i=1}^{n} \lambda(uQ)\lambda(PQ)^{-1}\lambda(e_i^t) \otimes \left(\sum_{j=1}^{n}(e_i P e_j^t) e_j v^t\right) \\ &= \sum_{j=1}^{n} \lambda(u)\lambda(Q)\lambda(Q)^{-1}\lambda(P)^{-1}\lambda(P)\lambda(e_j^t) \otimes e_j v^t \\ &= \sum_{j=1}^{n} \lambda(u) e_j^t \otimes e_j v^t = \sum_{j=1}^{n} 1 \otimes u e_j^t e_j v^t = 1 \otimes uv^t\ .\end{aligned}$$

This expression is equal to zero whenever $uv^t = 0$. It can be shown easily that $\phi\theta = 1$, and so θ is an R_Σ-isomorphism.

It is clear that θ is a natural transformation. □

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