MAXIMAL TORSION RADICALS IN $\sigma[M]$ AND MINIMAL PRIME M-IDEALS

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ABSTRACT

If the ring R is left Noetherian, it is well-known that there is a bijective correspondence between minimal prime ideals of R and maximal torsion radicals of R–Mod. In this paper conditions are given under which this result can be extended to the category $\sigma[M]$ of modules subgenerated by a module ${}_RM$, using the notion of a prime M-ideal introduced by the author.[1]

It will be assumed throughout that R is an associative ring with identity, and that M is a fixed nonzero left R-module. A module X in R–Mod, the category of left R-modules, is said to be *M-generated* if there exists an R-epimorphism from a direct sum of copies of M onto X. The category $\sigma[M]$ of modules *subgenerated* by M is defined to be the full subcategory of R–Mod that contains all modules ${}_RX$ such that X is isomorphic to a submodule of an M-generated module. The reader is referred to the books by Dung et. al.[2] and Wisbauer[3],[4] for results on the category $\sigma[M]$. It is closed under formation of homomorphic images, submodules, and direct sums. Furthermore, it has enough injectives, since a module ${}_RX$ is injective in $\sigma[M]$

if and only if it is M-injective, and the M-injective envelope $\widehat{X}$ forms its injective envelope in $\sigma[M]$.

The results in this paper concern the analog in $\sigma[M]$ of the notion of a prime ideal of the ring R. If $\mathcal{C}$ is any collection of modules in $\sigma[M]$, we define

$$\mathrm{Ann}_M(\mathcal{C}) = \{m \in M \mid f(m) = 0 \text{ for all } f \in \mathrm{Hom}_R(M, X), \text{ for all } X \text{ in } \mathcal{C}\} ,$$

and say that a submodule N of M is an *M-ideal* if $N = \mathrm{Ann}_M(\mathcal{C})$ for some $\mathcal{C}$ in $\sigma[M]$. It has been shown[1] that if M is quasi-projective, then a submodule N of M is an M-ideal if and only if it is a fully invariant submodule of M.

The following definitions have been introduced by the author.[1] The module ${}_RX$ in $\sigma[M]$ is said to be *M-prime* if $\mathrm{Hom}_R(M, X) \neq 0$ and for all submodules $Y \subseteq X$, either $\mathrm{Ann}_M(Y) = \mathrm{Ann}_M(X)$ or $\mathrm{Ann}_M(Y) = M$. Then an M-ideal $P \subsetneq M$ is said to be a *prime M-ideal* if $P = \mathrm{Ann}_M(X)$ for some M-prime module ${}_RX$.

A subfunctor ρ of the identity on *R–Mod* is called a *radical* of *R–Mod* if $\rho(X/\rho(X)) = (0)$, for all modules ${}_RX$; it is called a *torsion radical* if, in addition, $\rho(X_0) = X_0 \cap \rho(X)$, for all submodules $X_0 \subseteq X$. If ρ and γ are radicals of *R–Mod* such that $\rho(X) \subseteq \gamma(X)$, for all modules ${}_RX$, then the notation $\rho \leq \gamma$ is used. With this notation, for any radical ρ we have $\rho \leq 1$, where 1 is the identity functor on *R–Mod*. A radical ρ of *R–Mod* is said to be a *maximal radical* if $\rho \neq 1$ and $\rho \leq \gamma$ implies $\rho = \gamma$ or $\gamma = 1$, for all radicals γ of *R–Mod*.

In Theorem 1.3 of Ref. [5] it is shown that there exists a bijective correspondence between maximal radicals of *R–Mod* and prime ideals of R. This result has been extended to $\sigma[M]$ by the author[6] under the hypothesis that M is a projective generator in $\sigma[M]$. In preparation for later results in this paper, it will be shown in Theorem 1.5 that under the weaker hypothesis that M is quasi-projective and $\mathrm{Hom}_R(M, X) \neq 0$ for all modules ${}_RX$ in $\sigma[M]$ there is still a bijective correspondence between maximal radicals of $\sigma[M]$ and prime M-ideals. If M is also Noetherian, then Theorem 1.8 characterizes the prime M-ideals as the fully invariant submodules $P \subsetneq M$ such that M/P is semi-compressible. (Recall that the module ${}_RX$ is *semi-compressible* if for each nonzero submodule $Y \subseteq X$ there is a monomorphism $f : X \to Y^k$, for some positive integer k.)

A torsion radical τ is said to be a *maximal torsion radical* of *R–Mod* if $\tau \neq 1$ and $\tau \leq \gamma$ implies $\tau = \gamma$ or $\gamma = 1$, for all torsion radicals γ of *R–Mod*. The

Walkers[7] showed that if R is a commutative Noetherian ring, then there is a bijective correspondence between minimal prime ideals of R and maximal torsion radicals of R–Mod. In a series of papers[5],[8],[9],[10] the author investigated maximal torsion radicals and showed that this bijective correspondence remains valid for any left Noetherian ring. It will be shown in Theorem 2.6 that this correspondence holds in $\sigma[M]$, provided that M is a Noetherian quasi-projective module with $\mathrm{Hom}_R(M, X) \neq 0$ for all modules ${}_RX$ in $\sigma[M]$.

1 Prime M-ideals when M is quasi-projective

The next lemma appears to be the key to strengthening earlier results.[6]

Lemma 1.1 *Assume that ${}_RM$ is quasi-projective. If $f \in \mathrm{Hom}_R(M, X)$ and $N \subseteq M$ is any M-ideal such that $f(N) = (0)$, then $N \subseteq \mathrm{Ann}_M(f(M))$.*

Proof. Let $g \in \mathrm{Hom}_R(M, f(M))$. Then since M is quasi-projective we can lift g to $\widehat{g} \in \mathrm{End}_R(M)$ with $f\widehat{g} = g$, as in the following diagram.

$$\begin{tikzcd}
 & M \arrow[d, "g"] \arrow[dl, dotted, "\widehat{g}"'] \\
M \arrow[r, "f"'] & f(M)
\end{tikzcd}$$

Because N is an M-ideal, it is a fully invariant submodule of M, and so we have $g(N) = f(\widehat{g}(N)) \subseteq f(N) = (0)$. This shows that $N \subseteq \mathrm{Ann}_M(f(M))$. □

If N and K are submodules of ${}_RM$, let $\sum_{h \in E,\, h(M) \subseteq K} h(N) = N \cdot K$, where $E = \mathrm{End}_R(M)$. If M is quasi-projective, it can be shown (as in the proof of Proposition 5.5 of Ref. [1]) that this is the same as the notation of Ref. [1].

Proposition 1.2 *If ${}_RM$ is quasi-projective, then the following conditions are equivalent for an M-ideal $P \subseteq M$.*

(1) *P is a prime M-ideal;*

(2) *for all M-ideals $N \subseteq M$ and all $f \in \mathrm{End}_R(M)$, if $N \cdot f(M) \subseteq P$, then either $N \subseteq P$ or $f(M) \subseteq P$;*

(3) *M/P is an M-prime module.*

Proof. (1) $\Longrightarrow$ (2): Since P is a prime M-ideal, there exists an M-prime module ${}_RX$ with $P = \mathrm{Ann}_M(X)$. Let $N \subseteq M$ be an M-ideal, and let $f \in \mathrm{End}_R(M)$. If $f(M) \not\subseteq P$, then there exists $g \in \mathrm{Hom}_R(M, X)$ with $gf \neq 0$, since $\mathrm{Ann}_M(X) = P$. Therefore $\mathrm{Ann}_M(gf(M)) = P$, since X is M-prime. Since $N \cdot f(M) \subseteq P$, we have $f(N) \subseteq P$, and then $gf(N) = (0)$ since $P = \mathrm{Ann}_M(X)$. It follows from Lemma 1.1 that $N \subseteq \mathrm{Ann}_M(gf(M)) = P$.

(2) $\Longrightarrow$ (3): To show that M/P is M-prime, let K/P be a nonzero submodule of M/P for which $\mathrm{Hom}_R(M, K/P) \neq 0$, and suppose that $f : M \to K/P$ is a nonzero R-homomorphism. Then $P \subseteq \mathrm{Ann}_M(K/P) \subseteq \mathrm{Ann}_M(f(M))$, and so to show that M/P is M-prime it suffices to check that $\mathrm{Ann}_M(f(M)) = P$. Let $\mathrm{Ann}_M(f(M)) = N$, and suppose that $N \neq P$.

If $\pi : M \to M/P$ is the canonical projection, then since M is quasi-projective there exists $\widehat{f} \in \mathrm{End}_R(M)$ with $\pi\widehat{f} = f$, as in the following diagram.

$$\begin{tikzcd}
 & M \arrow[dl, dotted, "\widehat{f}"'] \arrow[d, "f"] \\
M \arrow[r, "\pi"'] & M/P
\end{tikzcd}$$

Then $\widehat{f}(M) \not\subseteq P$ since $\pi\widehat{f} = f \neq 0$, so applying condition (2) to N and $\widehat{f}$ shows that $N \cdot \widehat{f}(M) \not\subseteq P$, since $N \not\subseteq P$. By the definition of $N \cdot \widehat{f}(M)$ there must exist $g \in \mathrm{End}_R(M)$ such that $g(M) \subseteq \widehat{f}(M)$ and $g(N) \not\subseteq P$. It follows that $\pi g(N) \subseteq \pi\widehat{f}(M) = f(M)$ but $\pi g(N) \neq 0$, which contradicts the fact that $N = \mathrm{Ann}_M(f(M))$. Thus $\mathrm{Ann}_M(f(M)) = P$, showing that M/P is an M-prime module.

(3) $\Longrightarrow$ (1): This follows immediately, since P is an M-ideal and therefore $P = \mathrm{Ann}_M(M/P)$. □

In the second section of the paper we require the existence of minimal prime M-ideals.

Corollary 1.3 *If ${}_RM$ is quasi-projective, then every prime M-ideal contains a minimal prime M-ideal.*

Proof. Condition (2) of Proposition 1.2 makes it possible to use the standard proof for the existence of minimal prime ideals in the ring R. Given a prime M-ideal

$P \subseteq M$, we can apply Zorn's lemma to the set $\mathcal{P}$ of prime M-ideals contained in P, directed by reverse inclusion.

Suppose that $\{Q_\alpha\}_{\alpha \in I}$ is a chain of prime M-ideals in $\mathcal{P}$ with $\cap_{\alpha \in I} Q_\alpha = Q$. We first note that Q is an M-ideal, since an intersection of fully invariant submodules is fully invariant. To show that Q is a prime M-ideal, suppose that N is a M-ideal and $f \in \mathrm{End}_R(M)$ with $N \cdot f(M) \subseteq Q$. If $f(M) \not\subseteq Q$, then there exists $\alpha \in I$ such that $f(M) \not\subseteq Q_\alpha$. Then $f(M) \not\subseteq Q_\beta$, for all $\beta \geq \alpha$, and this implies that $N \subseteq Q_\beta$, for all $\beta \geq \alpha$, since each Q_β is a prime M-ideal. Thus $N \subseteq \cap_{\alpha \in I} Q_\alpha = Q$, showing that Q is a prime M-ideal.

Since each chain of ideals in $\mathcal{P}$ has a lower bound in $\mathcal{P}$, it follows that $\mathcal{P}$ contains a minimal element, which is then minimal in the set of all prime M-ideals. □

Proposition 1.2 also allows us to sharpen Proposition 6 of Ref. [6]. (In the next proposition it is only necessary to require that M is quasi-projective, rather than requiring it to be projective in $\sigma[M]$.)

Proposition 1.4 *Assume that* ${}_RM$ *is quasi-projective, and let* P *be an* M*-ideal. Then the following conditions are equivalent:*

(1) P *is a prime* M*-ideal;*

(2) *for any module* X *in* $\sigma[M]$*, if* $\mathrm{Hom}_R(M,X) \neq 0$ *and* X *is cogenerated by* M/P*, then* $\mathrm{Ann}_M(X) = P$*;*

(3) *if* ρ *is a radical of* $\sigma[M]$ *such that* $\mathrm{rad}_{M/P} \leq \rho$*, then either* $\rho(M) = P$ *or* $\rho(M) = M$*.*

Proof. (1) $\Longrightarrow$ (2): Let X be a module in $\sigma[M]$ such that $\mathrm{Hom}_R(M,X) \neq 0$ and M/P cogenerates X. Since P is an M-ideal and M/P cogenerates X, we have $P = \mathrm{Ann}_M(M/P) \subseteq \mathrm{Ann}_M(X)$. By assumption there exists a nonzero homomorphism $f : M \to X$, and then since M/P cogenerates X there exists a homomorphism $g \in \mathrm{Hom}_R(X, M/P)$ with $gf \neq 0$. Then $gf(\mathrm{Ann}_M(X)) = (0)$, so because M is quasi-projective, it follows from Lemma 1.1 that $\mathrm{Ann}_M(X) \subseteq \mathrm{Ann}_M(gf(M))$. Since M is quasi-projective, it follows from Proposition 1.2 that M/P is an M-prime module. Then $\mathrm{Ann}_M(gf(M)) \neq M$ since $gf \neq 0$, and so we must have $\mathrm{Ann}_M(gf(M)) = P$. This shows that $\mathrm{Ann}_M(X) = P$.

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (1): These implications do not require the assumption that M is quasi-projective and follow as in Proposition 6 of Ref. [6]. $\Box$

For a class $\mathcal{C}$ of R-modules, the radical $\mathrm{rad}_{\mathcal{C}}$ of R–Mod *cogenerated* by $\mathcal{C}$ is defined by setting $\mathrm{rad}_{\mathcal{C}}(X) = \mathrm{Ann}_X(\mathcal{C})$, for all modules ${}_RX$. For a given radical ρ of R–Mod, a module ${}_RX$ is said to be *ρ-torsionfree* if $\rho(X) = (0)$, and *ρ-torsion* if $\rho(X) = X$. Since $X/\rho(X)$ is ρ-torsionfree, for all modules ${}_RX$, it follows that $\rho = \mathrm{rad}_{\mathcal{C}}(X)$, where $\mathcal{C}$ is the class of ρ-torsionfree modules. Note that $\rho(X)$ is the intersection of all submodules X' of X such that X/X' is ρ-torsionfree.

If the class $\mathcal{C}$ consists of a single module ${}_RW$, then the notation $\mathrm{rad}_W = \mathrm{rad}_{\mathcal{C}}$ will be used. Note that rad_W is the largest radical for which W is torsionfree. If ${}_RV$ cogenerates W, then $\mathrm{rad}_V(W) = (0)$, so $\mathrm{rad}_V \leq \mathrm{rad}_W$. Conversely, if $\mathrm{rad}_V \leq \mathrm{rad}_W$, then $\mathrm{rad}_V(W) \subseteq \mathrm{rad}_W(W) = (0)$, and therefore V cogenerates W. The definition of a radical can be given in $\sigma[M]$, just as in R–Mod, and there is a again the obvious notion of a maximal radical.

Theorem 1.5 *Assume that ${}_RM$ is quasi-projective and that $\mathrm{Hom}_R(M, X) \neq 0$ for all nonzero modules ${}_RX$ in $\sigma[M]$. Then there is a bijective correspondence between maximal radicals of $\sigma[M]$ and prime M-ideals.*

Proof. If ρ is a maximal radical of $\sigma[M]$, then it follows from Proposition 5 of Ref. [6] that $P = \rho(M)$ is a prime M-ideal with $\rho = \mathrm{rad}_{M/P}$, since by assumption $\mathrm{Hom}_R(M, X) \neq 0$ for all modules X in $\sigma[M]$.

To prove the converse, suppose that $P \subseteq M$ is a prime M-ideal and ρ is a radical of $\sigma[M]$ with $\mathrm{rad}_{M/P} \leq \rho$. Since M is quasi-projective it follows from Proposition 1.4 that either $\rho(M) = M$ or $\rho(M) = P$. The first case is impossible since $\mathrm{Hom}_R(M, X) \neq 0$ for all modules $X \in \sigma[M]$, and the second case implies that $\rho \leq \mathrm{rad}_{M/P}$. This shows that $\mathrm{rad}_{M/P}$ is a maximal radical of $\sigma[M]$.

The correspondence that assigns to a prime M-ideal P the maximal radical $\mathrm{rad}_{M/P}$ is clearly bijective. $\Box$

The definition of an M-prime module is closely related to another definition in the literature. A nonzero module ${}_RX$ is called prime by Bican et. al.[11] if $\mathrm{rad}_Y = \mathrm{rad}_X$ for all nonzero submodules Y of X. It is shown in Proposition 2.3 of Ref. [11]

that a nonzero module X satisfies this definition if and only if it is cogenerated by each of its nonzero submodules. Proposition 2.6 of Ref. [1] shows that such modules are "universally" M-prime, since each nonzero submodule of X cogenerates X if and only if X is M-prime for each module ${}_RM$ such that $\mathrm{Hom}_R(M, X) \neq 0$.

When considering the module M itself to be M-prime, the following result holds. It is shown in Proposition 2.8 of Ref. [1] that M is an M-prime module if and only if $f(M)$ cogenerates M, for each nonzero endomorphism $f \in \mathrm{End}_R(M)$. The assumption that $\mathrm{Hom}_R(M, N) \neq 0$ for all nonzero submodules $N \subseteq M$ therefore implies that each nonzero submodule of M cogenerates M, and thus M is a prime module in the sense of Bican et. al.[11]

If R is a prime left Goldie ring, then every left ideal $A \subseteq R$ is faithful, and it follows immediately from the descending chain condition on left annihilators that R can be embedded in a finite direct sum of copies of A. Thus the left module ${}_RR$ is semi-compressible. Theorem 1.7 will extend this to certain quasi-projective M-prime modules that satisfy "Goldie-like" conditions. The condition that M is semi-compressible of course implies that M is cogenerated by each of its nonzero submodules. Furthermore, if M is semi-compressible, then it is a strongly prime module. That is, M belongs to $\sigma[N]$, for each nonzero submodule $N \subseteq M$.

To prove Theorem 1.7 we need the following lemma. A submodule of M is called an *M-annihilator* if it is the intersection of kernels of a set of endomorphisms in $\mathrm{End}_R(M)$. If the set of such submodules satisfies the ascending chain condition, then M is said to have ACC on M-annihilators.

Lemma 1.6 *Let M be a quasi-projective M-prime module with $\mathrm{Hom}_R(M, N) \neq 0$ for all nonzero submodules $N \subseteq M$, and assume that M has ACC on M-annihilators. If $f \in \mathrm{End}_R(M)$ and $f(K) \neq (0)$ for the submodule $K \subseteq M$, then $K \cap \ker(f)$ is not an essential submodule of K.*

Proof. Let $f \in \mathrm{End}_R(M)$, with $f(K) \neq (0)$ for the submodule $K \subseteq M$. Since M has ACC on M-annihilators, the nonempty set

$$\{h \in \mathrm{End}_R(M) \mid h(K) \neq (0) \text{ and } \ker(h) \supseteq \ker(f)\}$$

must contain an element whose kernel is maximal in the set. If we show that $K \cap \ker(h)$ is not essential in K, then it follows that $K \cap \ker(f) \subseteq K \cap \ker(h)$ is not essential in K. Therefore we can assume without loss of generality that $\ker(f)$ is maximal among R-homomorphisms such that $f(K) \neq (0)$.

Let $\pi_1 : M \to M/(K \cap \ker(f))$ and $\pi_2 : M/(K \cap \ker(f)) \to M/\ker(f)$ be the natural projections, and let $i : K/(K \cap \ker(f)) \to M/(K \cap \ker(f))$ be the inclusion. Let $\overline{f}$ be the factorization of f through $\pi_2\pi_1$, so that we have $\pi_2\pi_1\overline{f} = f$. Then $\pi_2 i$ is a monomorphism, and so the module $K/(K \cap \ker(f))$ is isomorphic to a nonzero submodule of M. By the comments preceding the theorem, the hypothesis implies that M is cogenerated by each of its nonzero submodules. It follows that there exists a homomorphism $g \in \mathrm{Hom}_R(M, K/(K \cap \ker(f)))$ with $gf(K) \neq (0)$. Since M is quasi-projective, the homomorphism $ig : M \to M/(K \cap \ker(f))$ can be lifted to $\widehat{g} \in \mathrm{End}_R(M)$ with $\pi_1\widehat{g} = ig$. This leads to the following commutative diagram.

$$\begin{tikzcd}
M \arrow[r, "f"] & M \arrow[r, "\widehat{g}"] \arrow[d, "g"'] & M \arrow[r, "f"] \arrow[d, "\pi_1"'] & M \\
& K/K \cap \ker(f) \arrow[r, "i"] & M/K \cap \ker(f) \arrow[r, "\pi_2"] & M/\ker(f) \arrow[u, "\overline{f}"]
\end{tikzcd}$$

Because $\overline{f}$ and $\pi_2 i$ are monomorphisms and $gf(K) \neq (0)$, we have

$$f\widehat{g}f(K) = \overline{f}\pi_2\pi_1\widehat{g}f(K) = \overline{f}\pi_2 igf(K) \neq (0) ,$$

and then since $\ker(f) \subseteq \ker(f\widehat{g}f)$, it follows from the maximality of $\ker(f)$ that $\ker(f\widehat{g}f) = \ker(f)$. We claim that $(K \cap \ker(f)) \cap \widehat{g}f(K) = (0)$, which will show that $K \cap \ker(f)$ is not essential in K. (We note that $\widehat{g}(M) \subseteq K$, since we have $\pi_1\widehat{g}(M) = ig(M) \subseteq K/(K \cap \ker(f))$.) If $y \in (K \cap \ker(f)) \cap \widehat{g}f(K)$, then $y = \widehat{g}f(x)$ for some $x \in K$. But then $f(y) = 0$ since $y \in \ker(f)$, and so $f\widehat{g}f(x) = f(y) = 0$ implies $f(x) = 0$ since $\ker(f\widehat{g}f) = \ker(f)$. We conclude that $y = 0$, completing the proof. □

Theorem 1.7 *Let M be a quasi-projective M-prime module with* $\mathrm{Hom}_R(M, N) \neq 0$ *for all nonzero submodules $N \subseteq M$. If M has* ACC *on M-annihilators and finite uniform dimension, then M is semi-compressible.*

Proof. Let N be any nonzero submodule of M. Since X cogenerates M, there exists a nonzero homomorphism $f_1 : M \to N$. If f_1 is a monomorphism, then we are done. If not, then $K_1 \neq (0)$ for $K_1 = \ker(f_1)$, and so there exists $f_2 : M \to N$ with $f_2(K_1) \neq (0)$. If $K_2 = K_1 \cap \ker(f_2)$, then it follows from Lemma 1.6 that K_2 is not an essential submodule of K_1, since M has ACC on M-annihilators. Since M has finite uniform dimension, the preceding remarks show that the uniform dimension of K_2 is strictly less than the uniform dimension of K_1. Therefore the descending chain we are constructing must terminate in (0) after some finite number k of steps, at which point $\cap_{i=1}^{k} \ker(f_i) = (0)$. The homomorphisms $\{f_i\}_{i=1}^{k}$ can then be combined to give the necessary embedding $M \to N^k$. □

The next theorem is the main goal of this section, as it will be used in the next section to show that maximal torsion radicals of $\sigma[M]$ correspond to minimal prime M-ideals.

Theorem 1.8 *Let M be a quasi-projective module such that* $\mathrm{Hom}_R(M, X) \neq 0$ *for all nonzero modules ${}_RX$ in $\sigma[M]$. If M is Noetherian, then a submodule $P \subseteq M$ is a prime M-ideal if and only if P is a fully invariant submodule of M such that M/P is semi-compressible.*

Proof. Proposition 5.1 of Ref. [1] shows that since M is quasi-projective, a submodule is an M-ideal if and only if it is fully invariant. If M/P is a semi-compressible module, then it is certainly M-prime, and so $P = \mathrm{Ann}_M(M/P)$ is a prime M-ideal.

To show the converse, suppose that P is a prime M-ideal. Then Proposition 1.2 shows that M/P is an M/P-prime module. By assumption, for each nonzero submodule N/P of M/P there exists $0 \neq f \in \mathrm{Hom}_R(M, N/P)$. Since P is an M-ideal, it follows that $f(P) = (0)$, and so $\mathrm{Hom}_R(M/P, N/P) \neq 0$. Finally, M/P is Noetherian since M is Noetherian, and so M/P has finite uniform dimension and ACC on M/P-annihilators. Thus M/P satisfies the hypotheses of Theorem 1.7, and so it is semi-compressible. □

2 Maximal torsion radicals in $\sigma[M]$

It is well-known that if ${}_RW$ is an injective module, then rad_W defines a torsion radical of R–*Mod*, and that every torsion radical of R–*Mod* is of this form. For a module ${}_RX$, the torsion radical $\mathrm{rad}_{E(X)}$ is the largest torsion radical τ (with respect to the relation $\leq$ defined earlier) for which X is τ-torsionfree, where $\mathrm{E}(X)$ denotes the injective envelope of X (in R–Mod). Thus for any torsion radical τ, we have $\tau(X) = 0$ if and only if $\tau \leq \mathrm{rad}_{E(X)}$.

Since $\sigma[M]$ is a Grothendieck category, it is possible to extend many of the torsion-theoretic concepts to $\sigma[M]$. Proposition 9.3 of Ref. [4] shows that every torsion radical of $\sigma[M]$ has the form rad_W, where ${}_RW$ is an M-injective module. Since $\mathrm{Hom}_R(M, W) \neq 0$ for any nonzero M-injective module, it follows that rad_W is the identity on $\sigma[M]$ if and only if $W = (0)$. In fact, this remark also implies that a torsion radical τ of R–Mod restricts to the identity on $\sigma[M]$ if and only if $\tau(M) = M$.

For a torsion radical τ of $\sigma[M]$, we say that a module ${}_RX$ is τ-torsionfree if $\tau(X) = (0)$, and τ-torsion if $\tau(X) = X$. A submodule $Y \subseteq X$ is said to be τ-*closed* if X/Y is τ-torsionfree; in general, the τ-closure of X in Y is defined as the intersection of all τ-closed submodules of X which contain Y. We can define the relation $\leq$ for torsion radicals in $\sigma[M]$, just as in R–Mod, and then $\tau = \mathrm{rad}_{\widehat{X}}$ is the largest torsion radical τ for which X is τ-torsionfree. Here $\widehat{X}$ is used to denote the M-injective envelope of X, which is constructed as the largest M-generated submodule of the R-injective envelope $\mathrm{E}(X)$ of X.

Lemma 2.1 *Let τ be a torsion radical of $\sigma[M]$. If $P \subseteq M$ is a maximal proper τ-closed M-ideal, then M/P is an M-prime module.*

Proof. If K/P is any nonzero submodule of M/P with $\mathrm{Hom}_R(M, K/P) \neq 0$, then $\mathrm{Ann}_M(K/P)$ is a proper M-ideal, and it is certainly τ-closed, since M/P is assumed to be τ-torsionfree. Furthermore, $P = \mathrm{Ann}_M(M/P)$ since P is an M-ideal, so $P \subseteq \mathrm{Ann}_M(K/P)$. By assumption $P = \mathrm{Ann}_M(K/P)$, which shows that M/P is an M-prime module. $\square$

Theorem 2.2 *Let M be a module such that* $\mathrm{Hom}_R(M,X) \neq 0$ *for all nonzero modules ${}_RX$ in $\sigma[M]$. Let P be a proper M-ideal, and let $\tau = \mathrm{rad}_{\widehat{M/P}}$ be the torsion radical determined by the M-injective envelope $\widehat{M/P}$ of M/P. Then the following conditions are equivalent.*

(1) *M/P is an M-prime module;*

(2) *P contains all proper τ-closed M-ideals.*

Proof. (1) $\Longrightarrow$ (2): Let N be a proper τ-closed M-ideal of M, and let $\pi : M \to M/N$ be the canonical projection. Then M/N is cogenerated by $\widehat{M/P}$ since it is τ-torsionfree, and so there exists a nonzero homomorphism $f : M/N \to \widehat{M/P}$. Since M/P is an essential submodule of $\widehat{M/P}$, we have $f\pi(M) \cap (M/P) \neq (0)$, and then by hypothesis $\mathrm{Hom}_R(M, f\pi(M) \cap (M/P)) \neq 0$, so

$$\mathrm{Ann}_M(f\pi(M)) \subseteq \mathrm{Ann}_M(f\pi(M) \cap (M/P)) = P\ ,$$

since M/P is an M-prime module. Since $f\pi(N) = (0)$, it follows from Lemma 1.1 that $N \subseteq \mathrm{Ann}_M(f\pi(M))$, and therefore $N \subseteq P$ (as required).

(2) $\Longrightarrow$ (1): This follows from Lemma 2.1. □

The following characterization of prime M-ideals via torsion radicals of $\sigma[M]$ should be compared with Proposition 1.4.

Corollary 2.3 *Assume that M is a quasi-projective module with* $\mathrm{Hom}_R(M,X) \neq 0$ *for all nonzero modules ${}_RX$ in $\sigma[M]$.*

(a) *A proper M-ideal P is a prime M-ideal if and only if it is a maximal τ-closed M-ideal for some torsion radical τ of $\sigma[M]$.*

(b) *A proper M-ideal P is a prime M-ideal if and only if* $\mathrm{Ann}_M(X) \subseteq P$ *for all modules ${}_RX$ in $\sigma[M]$ cogenerated by $\widehat{M/P}$.*

Proof. We first note that it follows from Proposition 1.2 that an M-ideal P is a prime M-ideal if and only if M/P is an M-prime module, since M is assumed to be quasi-projective.

(a) This follows immediately from Theorem 2.2 and Lemma 2.1.

(b) Let τ be the torsion radical determined by the M-injective envelope $\widehat{M/P}$ of M/P. If P is a prime M-ideal and ${}_RX$ is cogenerated by $\widehat{M/P}$, then $\mathrm{Ann}_M(X)$ is τ-closed, and so it follows from Theorem 2.2 that $\mathrm{Ann}_M(X) \subseteq P$.

Conversely, if $\mathrm{Ann}_M(X) \subseteq P$ for all modules ${}_RX$ in $\sigma[M]$ cogenerated by $\widehat{M/P}$, then P contains every τ-closed M-ideal, and so it follows from Theorem 2.2 that P is a prime M-ideal. □

Definition 2.4 *A proper torsion radical μ of $\sigma[M]$ is said to be* maximal *if $\mu \leq \tau$ implies $\mu = \tau$, for all proper torsion radicals τ of $\sigma[M]$.*

The Walkers[7] showed that the maximal torsion radicals of a commutative Noetherian ring are in one-to-one correspondence with its minimal prime ideals. This result has been extended to the noncommutative case.[5] It is also of interest to know when every proper torsion radical is contained in a maximal torsion radical, and in the following lemma it is convenient to combine the two conditions.

Lemma 2.5 *If every proper torsion radical of $\sigma[M]$ is contained in a maximal torsion radical, and the maximal torsion radicals of $\sigma[M]$ correspond to the minimal prime M-ideals, then every nonzero M-injective module contains a submodule whose left annihilator in M is a minimal prime M-ideal.*

The converse holds if M is a module such that $\mathrm{Hom}_R(M, X) \neq 0$ for all nonzero modules ${}_RX$ in $\sigma[M]$, and for each minimal prime M-ideal P the factor module M/P is an M-prime module.

Proof. Suppose that every proper torsion radical of $\sigma[M]$ is contained in a maximal torsion radical, and the maximal torsion radicals of $\sigma[M]$ correspond to the minimal M-prime ideals of M. If ${}_RW$ is a nonzero module that is injective in $\sigma[M]$, then rad_W is a proper torsion radical of $\sigma[M]$, so by assumption there exists a prime M-ideal P that is minimal among prime M-ideals and has the property that $\mathrm{rad}_W \leq \mu$, for the torsion radical $\mu = \mathrm{rad}_{\widehat{M/P}}$ determined by the M-injective envelope $\widehat{M/P}$. Then P must be rad_W-closed since it is μ-closed, and therefore P is the left annihilator of a submodule of W.

Conversely, suppose that M is a module such that $\mathrm{Hom}_R(M,X) \neq 0$ for all nonzero modules ${}_RX$ in $\sigma[M]$, and for each minimal prime M-ideal P the factor module M/P is an M-prime module. Assume that each nonzero M-injective module contains a submodule whose left annihilator in M is a minimal prime M-ideal. If τ is a proper torsion radical of $\sigma[M]$, then $\tau = \mathrm{rad}_W$ for a nonzero M-injective module W. If P is a minimal prime M-ideal which is the annihilator in M of a submodule of W, then $\mathrm{rad}_W(M/P) = 0$ implies that $\mathrm{rad}_W \leq \mu$, for $\mu = \mathrm{rad}_{\widehat{M/P}}$.

If P is any minimal prime M-ideal, let $\mu = \mathrm{rad}_{\widehat{M/P}}$. If $\mu \leq \tau$ for a proper torsion radical τ, then as above, there is a minimal prime M-ideal P' such that $\tau \leq \mu'$, where $\mu' = \mathrm{rad}_{\widehat{M/P'}}$. But then P' is μ-closed, and so it follows from Theorem 2.2 that $P' \subseteq P$, since by assumption M/P is an M-prime module. Since P is minimal we must have $P' = P$, and thus $\tau = \mu$. It follows that every minimal prime M-ideal defines a maximal torsion radical of $\sigma[M]$, and then every proper torsion radical of $\sigma[M]$ is contained in a maximal torsion radical of $\sigma[M]$.

As above, if $\mathrm{rad}_{\widehat{M/P}} = \mathrm{rad}_{\widehat{M/P'}}$, where P and P' are minimal prime M-ideals, then $P = P'$, and this establishes the one-to-one correspondence between minimal prime M-ideals and maximal torsion radicals of $\sigma[M]$. □

Theorem 2.6 *Let M be a quasi-projective module such that* $\mathrm{Hom}_R(M,X) \neq 0$ *for all nonzero modules ${}_RX$ in $\sigma[M]$. If M is Noetherian, then every proper torsion radical of $\sigma[M]$ is contained in a maximal torsion radical of $\sigma[M]$, and the maximal torsion radicals of $\sigma[M]$ are in one-to-one correspondence with the minimal prime M-ideals.*

Proof. Since M is quasi-projective, Proposition 1.2 shows that an M-ideal P is a prime M-ideal if and only if M/P is an M-prime module. Thus by Lemma 2.5 it suffices to show that if ${}_RX$ is any nonzero M-injective module, then X contains a submodule whose left annihilator in M is a minimal prime M-ideal.

Since M is Noetherian, it follows from Proposition 3.4 of Ref. [1] that X has an associated prime M-ideal P. (We note that it is always true that $\mathrm{Hom}_R(M,X) \neq 0$ if X is a nonzero M-injective module.) Since $P = \mathrm{Ann}_M(Y)$ for a nonzero submodule $Y \subseteq X$, it follows that $\mathrm{rad}_X(M/P) = (0)$. By Corollary 1.3 there is a minimal prime M-ideal Q that is contained in P. Let K be the preimage in M of $\mathrm{rad}_X(M/Q)$, so that $K/Q = \operatorname{rad}_X(M/Q)$. Since $Q \subseteq P$ and $\operatorname{rad}_X(M/P) = (0)$, it follows that M/Q is not rad_X-torsion, and so $K \neq M$. If $K \neq Q$, then by Theorem 1.8 there is an embedding $M/Q \to (K/Q)^k$ for some positive integer k, since M/Q is semi-compressible. This forces M/Q to be rad_X-torsion, a contradiction. We conclude that M/Q is rad_X-torsionfree, and therefore Q is the annihilator of a nonzero submodule of X. This completes the proof. □

The next example exhibits a projective module ${}_RM$ of finite length such that $\sigma[M] = R\text{–}Mod$ has two maximal torsion radicals and yet M has only one minimal prime M-ideal. The example shows that in the statement of Theorem 2.6 the hypothesis that $\operatorname{Hom}_R(M,X) \neq 0$ for all nonzero modules ${}_RX$ in $\sigma[M]$ is necessary.

In the example, the ring R is left Artinian. In this case, if $\operatorname{Hom}_R(M,X) \neq 0$ for all nonzero modules ${}_RX$ in $\sigma[M]$, then it is easy to check that an M-ideal P is a prime M-ideal if and only if M/P is a homogeneous semisimple module. Thus every prime M-ideal is both maximal and minimal. An analysis of the proof of Theorem 2.6 shows that the theorem remains valid under the assumption that R is a left Artinian ring and $\operatorname{Hom}_R(M,X) \neq 0$ for all nonzero modules ${}_RX$ in $\sigma[M]$.

Example 2.1

Let F be a field, and let $R = \begin{bmatrix} F & 0 \\ F & F \end{bmatrix}$ be the ring of lower triangular 2×2 matrices over F. Let M be the left ideal $\begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix}$, and let N be the subset $\begin{bmatrix} 0 & 0 \\ F & 0 \end{bmatrix}$. The lattice of submodules of the projective module M is just $(0) \subset N \subset M$, and (0) and N are M-ideals since they are fully invariant. Since M/N is a simple R-module, it follows that N is an M-prime ideal. On the other hand, $\operatorname{Ann}_M(N) = M$ since $\operatorname{Hom}_R(M,N) = 0$, and it follows that (0) is also a prime M-ideal. Then (0) is the only minimal prime M-ideal.

There are two isomorphism classes of simple left R-modules, represented by N and M/N, and it can be shown that the injective envelopes in $\sigma[M]$ are $\widehat{N} = M$ and $\widehat{M/N} = M/N$, since M and M/N are injective left R-modules. The torsion radicals rad_M and $\operatorname{rad}_{M/N}$ are both maximal.

If we let N(R) denote the prime radical of R, then Proposition 2.7 of Ref. [12] shows that every proper torsion radical of R–Mod is contained in a maximal torsion radical, and the maximal torsion radicals of R–Mod correspond to the minimal prime ideals of R if and only if $R/\mathrm{N}(R)$ satisfies the same condition and N(R) is right T-nilpotent. We recall that an ideal A is right T-nilpotent if for each sequence $\{a_i\}_{i=1}^{\infty}$ of elements of A there exists an integer n such that $a_n a_{n-1} \cdots a_1 = 0$. Another characterization of T-nilpotence states that A is right T-nilpotent if and only if $AX \neq (0)$ for any nonzero left R-module X, and this occurs if and only if $\mathrm{Hom}_R(R/A, X) \neq 0$ for any nonzero module X. With this characterization of T-nilpotence, Proposition 2.7 of Ref. [12] can be extended to $\sigma[M]$, letting Rad(M) denote the intersection of all prime M-ideals of M.

Proposition 2.7 *Let M be a quasi-projective module with* $\mathrm{Hom}_R(M, X) \neq 0$ *for all nonzero modules ${}_RX$ in $\sigma[M]$.*

Every proper torsion radical of $\sigma[M]$ is contained in a maximal torsion radical of $\sigma[M]$, and the maximal torsion radicals of $\sigma[M]$ correspond to the minimal prime M-ideals of $M \iff \sigma[M/\mathrm{Rad}(M)]$ satisfies the same condition, and for all modules ${}_RX$ in $\sigma[M]$, $\mathrm{Hom}_R(M/\mathrm{Rad}(M), X) \neq 0$.

Proof. $\Longrightarrow$). We first show that $M/\mathrm{Rad}(M)$ is a quasi-projective module with $\mathrm{Hom}_R(M/\mathrm{Rad}(M), X) \neq 0$ for all modules ${}_RX$ in $\sigma[M]$. Since Rad(M) is the intersection of all M-prime ideals of M, it is also the intersection of the annihilators in M of all M-prime modules in $\sigma[M]$. Thus Rad(M) is a fully invariant submodule of M, and so $M/\mathrm{Rad}(M)$ is quasi-projective since M is quasi-projective. For any nonzero module ${}_RX$ in $\sigma[M]$, by Lemma 2.5 there is a submodule Y of $\widehat{X}$ whose annihilator in M is a minimal M-prime ideal P. Then $X \cap Y \neq 0$, so by assumption $\mathrm{Hom}_R(M, X \cap Y) \neq 0$. If $f \in \mathrm{Hom}_R(M, X \cap Y)$, then $f(P) = (0)$, so $f(\mathrm{Rad}(M)) = (0)$, which implies that $\mathrm{Hom}_R(M/\mathrm{Rad}(M), X \cap Y) \neq 0$.

If X is an $M/\mathrm{Rad}(M)$-injective module, then by assumption its M-injective envelope $\widehat{X}$ has a submodule Y whose annihilator $\mathrm{Ann}_M(Y)$ is a minimal prime M-ideal P. Without loss of generality we can assume that X is M-generated. Since M/P is a actually a factor module of $M/\mathrm{Rad}(M)$, it follows that X is $M/\mathrm{Rad}(M)$-generated, and thus X belongs to $\sigma[M/\mathrm{Rad}(M)]$.

$\Longleftarrow$). If ${}_RW$ is a nonzero M-injective module, then by assumption we have $\mathrm{Hom}_R(M/\mathrm{Rad}(M), W) \neq 0$. Thus W contains a nonzero submodule belonging to $\sigma[M/\mathrm{Rad}(M)]$, so by assumption there is a submodule of W whose left annihilator in $M/\mathrm{Rad}(M)$ is a minimal prime $M/\mathrm{Rad}(M)$-ideal. But then the annihilator is in fact a minimal prime M-ideal, and so the proof can be completed by applying Lemma 2.5. □

Our final results concern a characterization of maximal torsion radicals via the quotient categories that they define. The theory of quotient categories of $\sigma[M]$ is developed in Section 9 of Ref. [4]. If τ is a torsion radical in $\sigma[M]$, an module ${}_RW$ in $\sigma[M]$ is said to be (M,τ)-injective if it is injective relative to all exact sequences $0 \to X \to Y$ in $\sigma[M]$ such that the image of X is τ-dense in Y. That is, if $i : X \to Y$ is one-to-one in $\sigma[M]$, and $i(X)$ is σ-dense in Y, then any R-homomorphism $f : X \to W$ can be extended to $\widehat{f} : Y \to W$ with $\widehat{f}i = f$. We note that there is a version of Baer's criterion for (M,τ)-injective modules. It has been shown[13] that W is (M,τ)-injective if and only if for any R-homomorphism $f : A \to W$ such that A is a τ-dense left ideal of R and $R/\ker(f)$ belongs to $\sigma[M]$ there exists an element $w \in W$ with $f(a) = aw$ for all $a \in A$.

The quotient category of $\sigma[M]$ determined by τ is the full subcategory of all τ-torsionfree (M,τ)-injective modules, denoted by $\sigma[M]/\tau$. There is an associated quotient functor $Q_\tau : \sigma[M] \to \sigma[M]/\tau$ defined on the module ${}_RX$ in $\sigma[M]$ by letting $Q_\tau(X)$ be the (M,τ)-injective envelope of the factor module $X/\tau(X)$.

We say that an M-ideal $K \subseteq M$ is a *torsion M-ideal* if $K = \tau(M)$ for some torsion radical τ of $\sigma[M]$. This is equivalent to the condition that $K = \mathrm{Ann}_M(W)$ for a module ${}_RW$ injective in *R–Mod*, and also equivalent to the condition that $f(K) = 0$ for all R-homomorphisms $f : M \to \widehat{M/K}$ (see Ref. [14]).

Proposition 2.8 *Let M be any left R-module. The following conditions are equivalent for a torsion radical τ of $\sigma[M]$ with $\tau(M) = K$.*

(1) *The torsion radical τ is a maximal torsion radical of $\sigma[M]$;*

(2) *every injective object in the quotient category $\sigma[M]/\tau$ is a cogenerator;*

(3) *the torsion radical τ is defined by $\widehat{M/K}$ and K is maximal in the set of proper τ-closed torsion M-ideals.*

Proof. (1) $\Longrightarrow$ (2): In general, if W is any injective object in $\sigma[M]/\tau$, then $\tau \leq \mathrm{rad}_W$ since W is τ-torsionfree. It follows that if τ is a maximal torsion radical, then for any injective object W we must have $\mathrm{rad}_W = \tau$, and hence W is a cogenerator in $\sigma[M]/\tau$.

(2) $\Longrightarrow$ (3): Since $W = \widehat{M/K}$ is injective in $\sigma[M]/\tau$, it is a cogenerator and hence $\mathrm{rad}_W = \tau$. If L is any torsion M-ideal with $K \subseteq L$ and $Z = \widehat{M/L}$, then as before we must have $\tau = \mathrm{rad}_Z$, and follows that $L = \mathrm{rad}_Z(M) = \tau(M) = K$.

(3) $\Longrightarrow$ (1): If γ is a torsion radical of $\sigma[M]$ with $\tau \leq \gamma$, then $M/\gamma(M)$ is τ-torsionfree since it is γ-torsionfree, and thus $\gamma(M)$ is a τ-closed torsion M-ideal with $K \subseteq \gamma(M)$. By hypothesis we must have $K = \gamma(M)$, and this implies that $\tau = \gamma$, because τ is defined by $\widehat{M/K}$. □

Corollary 2.9 *If M is Noetherian and τ is a maximal torsion radical of $\sigma[M]$, then $\sigma[M]/\tau$ has an injective cogenerator that is an essential extension of a simple object.*

Proof. If M is Noetherian, then there exists a maximal τ-closed submodule N of M. It follows that the quotient module $Q_\tau(M/N)$ is simple in $\sigma[M]/\tau$. The desired conclusion is a consequence of Proposition 2.8, since $\widehat{M/N}$ is a cogenerator for $\sigma[M]/\tau$. □

Corollary 2.10 *If M is Noetherian and τ is a maximal torsion radical of $\sigma[M]$, Assume that M is a quasi-projective module with* $\mathrm{Hom}_R(M, X) \neq 0$ *for all nonzero modules ${}_RX$ in $\sigma[M]$. Let τ be a torsion radical of $\sigma[M]$, let $\tau(M) = K$, and assume further that τ is defined by $\widehat{M/K}$. If K is a prime M-ideal, the τ is a maximal torsion radical of $\sigma[M]$.*

Proof. It follows from Corollary 2.3 (b) that K is a maximal τ-closed M-ideal. The desired conclusion is therefore a consequence of condition (3) of Proposition 2.8. □

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