

# ON MAXIMAL RADICALS AND PRIME $M$ -IDEALS

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ABSTRACT: It is shown that if  $M$  is a projective generator in the category  $\sigma[M]$  of all modules subgenerated by  $M$ , then there is a bijective correspondence between the maximal radicals of  $\sigma[M]$  and the prime  $M$ -ideals of  $M$  (defined in [3]). This correspondence is used to show that if  $A$  is a (possibly nonassociative) Azumaya ring over the commutative ring  $R$ , then there is a bijective correspondence between the prime ideals of  $R$  and the prime  $A$ -ideals of  ${}_{M(A)}A$ .

It is assumed throughout the paper that  $R$  is an associative ring with identity, and that  ${}_R M$  is a fixed left  $R$ -module. In [1] and [2] the author has shown that there is a bijective correspondence between prime ideals of  $R$  and maximal radicals of  $R\text{-Mod}$ . The goal of this paper is to extend these results to the category  $\sigma[M]$  of modules subgenerated by  $M$ .

If  $\rho$  is a subfunctor of the identity on  $R\text{-Mod}$ , then  $\rho$  is called a *radical* if  $\rho(X/\rho(X)) = (0)$ , for all modules  ${}_R X$ . For a class  $\mathcal{C}$  of  $R$ -modules, the radical  $\text{rad}_{\mathcal{C}}$  of  $R\text{-Mod}$  *cogenerated* by  $\mathcal{C}$  is defined by  $\text{rad}_{\mathcal{C}}(X) = \bigcap_{f \in \text{Hom}(X, W), W \text{ in } \mathcal{C}} \ker(f)$ , for all modules  ${}_R X$ . The notation  $\text{Ann}_X(\mathcal{C}) = \text{rad}_{\mathcal{C}}(X)$  will also be used in this circumstance.

For a given radical  $\rho$  of  $R\text{-Mod}$ , a module  ${}_R X$  is said to be  $\rho$ -torsionfree if  $\rho(X) = (0)$ , and  $\rho$ -torsion if  $\rho(X) = X$ . Since  $X/\rho(X)$  is  $\rho$ -torsionfree, for all modules  ${}_R X$ , it follows that  $\rho = \text{rad}_{\mathcal{C}}(X)$  where  $\mathcal{C}$  is the class of  $\rho$ -torsionfree modules. Note that  $\rho(X)$  is the intersection of all submodules  $X'$  of  $X$  such that  $X/X'$  is  $\rho$ -torsionfree.

If  $\rho$  and  $\gamma$  are radicals of  $R\text{-Mod}$  such that  $\rho(X) \subseteq \gamma(X)$ , for all modules  ${}_R X$ , then the notation  $\rho \leq \gamma$  will be used. If the class  $\mathcal{C}$  consists of a single module  ${}_R W$ , then the notation  $\text{rad}_W = \text{rad}_{\mathcal{C}}$  will be used. Note that  $\text{rad}_W$  is the largest radical for which  $W$  is torsionfree. If  ${}_R V$  cogenerates  $W$ , then  $\text{rad}_V(W) = (0)$ , so  $\text{rad}_V \leq \text{rad}_W$ . Conversely, if  $\text{rad}_V \leq \text{rad}_W$ , then  $\text{rad}_V(W) \subseteq \text{rad}_W(W) = (0)$ , and therefore  $V$  cogenerates  $W$ .

The radical  $\rho \neq 1$  is called a *maximal* radical of  $R\text{-Mod}$  if  $\rho \leq \gamma$  implies that either  $\rho = \gamma$  or  $\gamma$  is the identity, for any radical  $\gamma$ . Theorem 1.3 of [2] shows that  $\rho$  is a maximal radical of  $R\text{-Mod}$  if and only if there exists a prime ideal  $P$  of  $R$  such that  $\rho = \text{rad}_{R/P}$ . It is this result that will be extended to  $\sigma[M]$ , provided  $M$  satisfies certain assumptions.

The module  ${}_R X$  is said to be *M-generated* if there exists an  $R$ -epimorphism from a direct sum of copies of  $M$  onto  $X$ . The category  $\sigma[M]$  is defined to be the full subcategory of  $R\text{-Mod}$  that contains all modules  ${}_R X$  such that  $X$  is isomorphic to a submodule of an  $M$ -generated module. The reader is referred to [6], [7], and [8] for results on the category  $\sigma[M]$ . It is shown in [7] that  $\sigma[M]$  is closed under taking homomorphic images, submodules, and direct sums.

The definition of a radical can be given in  $\sigma[M]$ , just as in  $R\text{-Mod}$ , and there is again the obvious notion of a maximal radical. In order to extend to  $\sigma[M]$  the correspondence between maximal radicals of  $R\text{-Mod}$  and prime ideals of  $R$ , it is necessary to have a notion in  $M$  that corresponds to that of a prime ideal of  $R$ . The following definitions were first given by the author in [3].

**Definition 1 ([3])** The submodule  $N$  of  $M$  is called an *M-ideal* if there is a class  $\mathcal{C}$  of modules in  $\sigma[M]$  such that  $N = \text{Ann}_M(\mathcal{C})$ .

It has been shown in Proposition 3.3 of [4] that the following conditions are equivalent for a submodule  $N \subseteq M$ : (1)  $N$  is an  $M$ -ideal; (2) there exists a radical

$\rho$  of  $R\text{-Mod}$  such that  $N = \rho(M)$ ; (3)  $g(N) = (0)$  for all  $g \in \text{Hom}_R(M, (M/N))$ ; (4)  $N = \text{Ann}_M(M/N)$ . In order to define the product of two  $M$ -ideals, a general product  $N \cdot X$  can be constructed, for any submodule  $N$  of  $M$  and any module  ${}_R X$ .

**Definition 2 ([3])** Let  $N$  be a submodule of  $M$ . For each module  ${}_R X$ , define

$$N \cdot X = \text{Ann}_X(\mathcal{C}),$$

where  $\mathcal{C}$  is the class of modules  ${}_R W$  such that  $f(N) = (0)$  for all  $f \in \text{Hom}_R(M, W)$ .

It has been shown in [3] that the following conditions hold for any submodule  $N$  of  $M$ . The submodule  $N$  is an  $M$ -ideal if and only if  $N \cdot (M/N) = (0)$ , and  $N \cdot X = (0)$  if and only if  $N \subseteq \text{Ann}_M(X)$ , for any module  ${}_R X$ . If  $\rho$  is a radical of  $R\text{-Mod}$  such that  $N \subseteq \rho(M)$ , then  $N \cdot X \subseteq \rho(X)$ , for all modules  ${}_R X$ . Thus the radical  $N \cdot (-)$  is the smallest radical  $\rho$  of  $R\text{-Mod}$  for which  $N \subseteq \rho(M)$ .

Recall that the module  ${}_R X$  is said to be a *prime* module if  $X$  is nonzero and  $\text{Ann}_R(Y) = \text{Ann}_R(X)$ , for all nonzero submodules  $Y \subseteq X$ . The definition of an  $M$ -prime module is complicated by the fact that it is possible to have  $\text{Hom}_R(M, X) = 0$  even though  $X$  is a nonzero member of  $\sigma[M]$ .

**Definition 3 ([3])** The module  ${}_R X$  is said to be  $M$ -prime if  $\text{Hom}_R(M, X) \neq 0$ , and  $\text{Ann}_M(Y) = \text{Ann}_M(X)$  for all submodules  $Y \subseteq X$  such that  $\text{Hom}_R(M, Y) \neq 0$ .

Proposition 2.2 of [3] shows that the following conditions are equivalent for any module  ${}_R X$  such that  $\text{Hom}_R(M, X) \neq 0$ : (1)  $X$  is an  $M$ -prime module; (2)  $N \cdot Y = (0)$  implies  $N \cdot X = (0)$ , for any submodule  $N \subseteq M$  and any submodule  $Y \subseteq X$  with  $M \cdot Y \neq (0)$ ; (3) for each  $m \in M \setminus \text{Ann}_M(X)$  and each  $0 \neq f \in \text{Hom}_R(M, X)$ , there exists  $g \in \text{Hom}_R(M, f(M))$  such that  $g(m) \neq 0$ ; (4)  $N \cdot Y = (0)$  implies  $N \cdot X = (0)$ , for any  $M$ -ideal  $N \subseteq M$  and any nonzero  $M$ -generated submodule  $Y \subseteq X$ .

The above definition of an  $M$ -prime module is closely related to another definition in the literature. In [5], a nonzero module  ${}_R X$  is called prime if  $\text{rad}_Y = \text{rad}_X$  for all nonzero submodules  $Y$  of  $X$ . It is shown in Proposition 2.3 of [5] that a nonzero module  $X$  satisfies this definition if and only if it is cogenerated by each of

its nonzero submodules. Proposition 2.6 of [3] shows that such modules are “universally”  $M$ -prime, since the following conditions are equivalent for a nonzero module  ${}_R X$ : (1) each nonzero submodule of  $X$  cogenerates  $X$ ; (2)  $X$  is  $M$ -prime for each module  ${}_R M$  such that  $\sigma[M] = \sigma[X]$  and  $X$  is  $M$ -generated; (3)  $X$  is  $M$ -prime for each module  ${}_R M$  such that  $\text{Hom}_R(M, X) \neq 0$ .

When considering the module  $M$  itself, the following result holds. Proposition 2.8 of [3] shows that  $M$  is an  $M$ -prime module if and only if  $f(M)$  cogenerates  $M$ , for each nonzero endomorphism  $f \in \text{End}_R(M)$ . It follows immediately that if  $M$  is monoform, then it is  $M$ -prime. When considering factor modules, Proposition 2.11 of [3] shows that if  $P \subseteq M$  is an  $M$ -ideal, then  $M/P$  is an  $M$ -prime module if and only if  $M/P$  is an  $M/P$ -prime module.

**Definition 4 ([3])** The  $M$ -ideal  $P \subseteq M$  is said to be a *prime  $M$ -ideal* if there exists an  $M$ -prime module  ${}_R X$  such that  $P = \text{Ann}_M(X)$ .

Although the above definition is given in  $R\text{-Mod}$ , it is actually a definition in  $\sigma[M]$ . It can be shown that if  $P \subseteq M$  is a prime  $M$ -ideal, then  $P = \text{Ann}_M(X)$  for an  $M$ -generated  $M$ -prime module  ${}_R X$ . Note that if  $P$  is an  $M$ -ideal, and  $M/P$  is an  $M$ -prime module, then  $P$  is a prime  $M$ -ideal since  $P = \text{Ann}(M/P)$ .

**Proposition 5** *Assume that  $\text{Hom}_R(M, X) \neq 0$  for all nonzero modules  $X$  in  $\sigma[M]$ . If  $\rho$  is a maximal radical of  $\sigma[M]$ , and  $P = \rho(M)$ , then  $P$  is a prime  $M$ -ideal such that  $\rho = \text{rad}_{M/P}$ . In this case  $M/P$  is cogenerated by each of its nonzero submodules.*

*Proof.* Let  $\rho$  be a maximal radical of  $\sigma[M]$ . Then  $\rho$  is not the identity on  $\sigma[M]$ , so there exists a nonzero  $\rho$ -torsionfree module  $X$  in  $\sigma[M]$ . By assumption,  $\text{Hom}_R(M, X) \neq 0$ , so it follows that  $M$  has a proper  $\rho$ -closed submodule, and thus  $\rho(M) \neq M$ . Letting  $\rho(M) = P$ , it follows that  $P$  is an  $M$ -ideal with  $\rho \leq \text{rad}_{M/P}$ , since  $M/P$  is  $\rho$ -torsionfree. The maximality of  $\rho$  implies that  $\rho = \text{rad}_{M/P}$ . If  $K$  is any nonzero submodule of  $M/P$ , then  $\text{rad}_K$  is a proper radical of  $\sigma[M]$  with  $\text{rad}_{M/P} \leq \text{rad}_K$ . The maximality of  $\text{rad}_{M/P}$  implies that  $\text{rad}_{M/P} = \text{rad}_K$ , and therefore  $K$  cogenerates  $M/P$ . It follows that  $M/P$  is a prime module, and thus  $P = \text{Ann}_M(M/P)$  is a prime  $M$ -ideal.  $\square$

The next result requires the hypothesis that  $M$  is projective in  $\sigma[M]$ . Under this assumption, Proposition 5.1 of [3] shows that a submodule  $N$  of  $M$  is an  $M$ -ideal if and only if it is a fully invariant submodule of  $M$ . Proposition 5.4 of [3] shows that if  $N$  is any submodule of  $M$ , then the product  $N \cdot X$  is given by  $N \cdot X = \sum_{f \in \text{Hom}(M, X)} f(N)$ , and it then follows that  $N \cdot (X/Y) = (0)$  if and only if  $N \cdot X \subseteq Y$ , for any  $Y \subseteq X$  in  $\sigma[M]$ . Theorem 5.7 of [3] shows that if  $M$  is projective in  $\sigma[M]$ , then an  $M$ -ideal  $P$  is a prime  $M$ -ideal if and only if  $M/P$  is an  $M$ -prime module. Furthermore, if  $M$  is a projective generator in  $\sigma[M]$ , it follows from the same theorem that  $P$  is a prime  $M$ -ideal if and only if  $N \cdot K \subseteq P$  implies  $N \subseteq P$  or  $K \subseteq P$ , for all  $M$ -ideals  $N$  and  $K$ .

**Proposition 6** *Let  $P$  be an  $M$ -ideal, and assume that  $M$  is projective in  $\sigma[M]$ . Then the following conditions are equivalent:*

- (1)  $P$  is a prime  $M$ -ideal;
- (2) for all modules  $N$  in  $\sigma[M]$ , if  $\text{rad}_N(M) \neq M$  and  $\text{rad}_{M/P} \leq \text{rad}_N$ , then  $\text{rad}_N(M) = P$ ;
- (3) if  $\rho$  is a radical of  $\sigma[M]$  such that  $\text{rad}_{M/P} \leq \rho$ , then either  $\rho(M) = P$  or  $\rho(M) = M$ .

*Proof.* (1)  $\implies$  (2): Let  $N$  be a module in  $\sigma[M]$  with  $\text{rad}_N(M) \neq M$ , such that  $\text{rad}_{M/P} \leq \text{rad}_N$ , and let  $A = \text{rad}_N(M) = \text{Ann}_M(N)$ . Then there exists a nonzero homomorphism  $g \in \text{Hom}_R(M, N)$  since  $\text{rad}_N(M) \neq M$ , and  $M/P$  cogenerates  $N$  since  $\text{rad}_{M/P} \leq \text{rad}_N$ , so there exists a homomorphism  $f \in \text{Hom}_R(N, M/P)$  with  $fg \neq 0$ . Letting  $K = \text{Im}(f)$  defines a submodule  $K \subseteq M/P$  with  $\text{Hom}_R(M, K) \neq 0$ . Since  $M$  is projective in  $\sigma[M]$ , it follows from Theorem 5.7 of [3] that  $M/P$  is an  $M$ -prime module, and therefore  $\text{Ann}_M(K) = P$ . Because  $K \cong N/\ker(f)$ , and  $M$  is projective in  $\sigma[M]$ , it follows from Proposition 5.4 (b) of [3] that  $A \cdot K = (0)$ , and therefore  $A \subseteq P$ . By assumption,  $P \subseteq A$ , and thus  $\text{rad}_N(M) = P$ .

(2)  $\implies$  (3): If  $\rho$  is a radical of  $\sigma[M]$  with  $\text{rad}_{M/P} \leq \rho$ , let  $N = M/\rho(M)$ . If  $\rho(M) \neq M$ , then  $N$  is a module in  $\sigma[M]$  with  $\text{rad}_N(M) \neq M$  and  $\text{rad}_{M/P} \leq \text{rad}_N$ . The hypothesis implies that  $\text{rad}_N(M) = P$ , and therefore  $\rho(M) = P$ .

(3)  $\implies$  (1): Let  $K$  be any nonzero submodule of  $M/P$  such that  $\text{Hom}_R(M, K)$  is nonzero. Then  $\rho = \text{rad}_K$  defines a radical of  $\sigma[M]$  for which  $\text{rad}_{M/P} \leq \rho$  and

$\rho(M) \neq M$ , so  $\rho(M) = P$ , showing that  $\text{Ann}_M(K) = P$ . It follows that  $M/P$  is an  $M$ -prime module, and hence  $P$  is a prime  $M$ -ideal.  $\square$

**Theorem 7** *If  $M$  is a projective generator in  $\sigma[M]$ , then there is a bijective correspondence between maximal radicals of  $\sigma[M]$  and prime  $M$ -ideals.*

*Proof.* Assume that  $M$  is a projective generator in  $\sigma[M]$ . If  $\rho$  is a maximal radical of  $\sigma[M]$ , then it follows from Proposition 5 that  $P = \rho(M)$  is a prime  $M$ -ideal with  $\rho = \text{rad}_{M/P}$ . Conversely, if  $P \subseteq M$  is a prime  $M$ -ideal, it follows from Proposition 6 that  $\text{rad}_{M/P}$  is a maximal radical of  $\sigma[M]$ . The correspondence that assigns to a prime  $M$ -ideal  $P$  the maximal radical  $\text{rad}_{M/P}$  is clearly bijective.  $\square$

It is evident that the notion of a maximal radical is categorical. That is, any equivalence of categories must preserve maximal radicals. This fact can be used to exploit the correspondence given in Theorem 7. In particular, the ring  $R$  will now be assumed to be a commutative ring. In the terminology of [8], a unital  $R$ -module  $A$  with an  $R$ -bilinear map  $\mu : A \times A \rightarrow A$  is called an  $R$ -algebra. It is called a *central*  $R$ -algebra if the natural mapping of  $R$  into the centroid of  $A$  is an isomorphism. The  $R$ -subalgebra of  $\text{End}_R(A)$  generated by all left and right multiplications (and the identity) is called the multiplication algebra of  $A$ , denoted by  $M(A)$ . It is possible to consider  $A$  as a left  $M(A)$ -module. If  $A$  is a central  $R$ -algebra, then every algebra ideal is a fully-invariant  $M(A)$ -submodule of  $A$ . The subcategory  $\sigma[A]$  of  $M(A)\text{-Mod}$  is shown in [8] to play an important role in studying nonassociative algebras. Finally, a central  $R$ -algebra is defined in [8] to be an *Azumaya ring* (over  $R$ ) if  $A$  is finitely generated, self-projective, and a self-generator as an  $M(A)$ -module.

**Corollary 8** *If  $A$  is an Azumaya ring over the commutative ring  $R$ , then there is a bijective correspondence between the prime ideals of  $R$  and the prime  $A$ -ideals of  $M(A)A$ .*

*Proof.* If  $A$  is a central  $R$ -algebra that is finitely generated as an  $M(A)$ -module, then Theorem 26.4 of [8] shows that the following conditions are equivalent: (1)  $A$  is an Azumaya ring; (2)  $A$  is a projective generator in  $\sigma[A]$ ; and (3) the functor

$\text{Hom}_{M(A)}(A, -) : \sigma[A] \rightarrow R\text{-Mod}$  is an equivalence of categories. It follows that the maximal radicals of  $R\text{-Mod}$  are in one-to-one correspondence with the maximal radicals of  $\sigma[M]$ . Since the prime ideals of  $R$  correspond to the maximal radicals of  $R\text{-Mod}$ , Theorem 7 implies that the prime ideals of  $R$  can be put in one-to-one correspondence with the prime  $A$ -ideals of  $A$ , considered as a left  $M(A)$ -module.  $\square$

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