A NOTE ON ADDITIVE RANK FUNCTIONS

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Additive rank functions have been studied for Noetherian rings by Krause. It is shown that the notion of an additive rank function can be extended to more general classes of rings, and can be used in the characterization of semiprime Goldie rings and of orders in Artinian rings.

In this paper all rings are assumed to be associative rings with an identity element, and all modules are assumed to be unital modules. The notation R–Mod is used for the category of left R-modules. The reader is referred to [1] for general terminology and notation, and to [2] for terminology and notation relating to torsion radicals and quotient categories.

In [3], A. W. Goldie defined the reduced rank $\rho(M)$ of a finitely generated module ${}_RM$ over a semiprime Goldie ring R to be the uniform dimension of $M/\gamma(M)$, where γ is the Goldie torsion radical defined by the set of regular elements of R. Equivalently, $\rho(M)$ is given by the length of the left Q-module $Q \otimes_R M$, where Q is the classical ring of left quotients of R. This length is defined for any finitely generated module over Q, since Q is a semisimple Artinian ring. For a Noetherian ring R, this definition was generalized by Goldie, as follows. If N is the prime radical of R, then N is nilpotent, say $N^k = (0)$, and R/N is a semiprime Goldie ring. Thus it is possible to extend the definition of ρ to any finitely generated left R-module M via the formula $\rho(M) = \sum_{i=1}^{k} \rho(N^{i-1}M/N^{i}M)$.

Similar functions have been studied recently by G. Krause in [4] and [5]. He defined an additive rank function to be a function λ that assigns to each finitely generated module ${}_RM$ a nonnegative integer $\lambda(M)$ such that $\lambda(M) = \lambda(K) + \lambda(N)$ for each short exact sequence $0 \to K \to M \to N \to 0$ of finitely generated modules.

In [6], the notion of reduced rank was extended by the present author to modules over any ring. If R is a ring with prime radical N, let γ be the torsion radical cogenerated by the injective envelope $\mathrm{E}(R/N)$ of the left R-module R/N. Then the module ${}_RM$ is said to have finite reduced rank if the module of quotients $Q_\gamma(M)$ has finite length in the quotient category $R\text{–}Mod/\gamma$. Assigning to M the length of $Q_\gamma(M)$ yields the usual definition of the reduced rank $\rho(M)$ over a left Noetherian ring, and provides an example of a general additive rank function, as introduced in the following definition.

Definition 1 *An* additive rank function *on R–Mod is a function that assigns to certain modules ${}_RM$ a nonnegative integer $\lambda(M)$, subject to the following conditions:*

(i) $\lambda(R)$ is defined;

(ii) if $0 \to K \to M \to N \to 0$ is any exact sequence in R–Mod, then $\lambda(M)$ is defined if and only if $\lambda(K)$ and $\lambda(N)$ are defined;

(iii) if $0 \to K \to M \to N \to 0$ is any exact sequence in R–Mod, and $\lambda(M)$ is defined, then $\lambda(M) = \lambda(K) + \lambda(N)$.

If $\lambda(M)$ is defined, we say that M has finite λ-rank.

For the general reduced rank ρ, if A is any nonzero left ideal of R/N, where N is the prime radical of R, then $Q_\gamma(A) \neq (0)$, and so $\rho(A) \neq 0$. It is helpful to have a name for such additive rank functions.

Definition 2 *Let N be the prime radical of the ring R. If λ is an additive rank function on R–Mod such that $\lambda(A) \neq 0$ for each nonzero left ideal A of R/N, then we say that λ is a* nonsingular *additive rank function.*

Lemma 3 *Let R be a ring with prime radical N, and assume that λ is a nonsingular additive rank function on R–Mod. If ${}_RM$ is any module that has finite λ-rank, and* $\mathrm{Hom}_R(M, \mathrm{E}(R/N)) \neq 0$*, then* $\lambda(M) > 0$.

Proof. Let f be a nonzero element of $\mathrm{Hom}_R(M, \mathrm{E}(R/N))$. Since $f(M) \neq (0)$ and $\mathrm{E}(R/N)$ is an injective envelope of R/N, we have $f(M) \cap (R/N) \neq (0)$. It follows that $\lambda(f(M) \cap (R/N)) \neq 0$, since λ is a nonsingular additive rank function. Because λ is additive, we must have $\lambda(f(M)) \neq 0$, and therefore $\lambda(M) \neq 0$. □

Theorem 4 *The ring R has finite reduced rank if and only if there exists a nonsingular additive rank function on R–Mod.*

Proof. If R has finite reduced rank, then the reduced rank function ρ is a nonsingular additive rank function.

Conversely, suppose that λ is a nonsingular additive rank function on R–Mod. Let N be the prime radical of R. We will verify the conditions of Theorem 1 of [6], where it is proved that R has finite reduced rank if and only if the set of left annihilators of subsets of $\mathrm{E}(R/N)$ satisfies the ascending chain condition. Let A and B be left ideals of R that are left annihilators of subsets of $\mathrm{E}(R/N)$. If A properly contains B, then there exists $x \in \mathrm{E}(R/N)$ with $Bx = (0)$ but $Ax \neq (0)$. Let $f : A \to \mathrm{E}(R/N)$ be the R-homomorphism defined by $f(a) = ax$, for all $a \in A$. Then f is nonzero, but $f(B) = (0)$, and so $\mathrm{Hom}_R((A/B), \mathrm{E}(R/N)) \neq 0$. It follows from Lemma 3 that $\lambda(A/B) > 0$. This implies that any properly ascending chain of left annihilators of $\mathrm{E}(R/N)$ has at most $\lambda(R)$ terms, completing the proof. □

Theorem 5 *Let R be a semiprime ring. Then R is a left Goldie ring if and only if there exists a nonsingular additive rank function on R–Mod.*

Proof. Theorem 1 of [6] shows that if R has finite reduced rank, and N is the prime radical of R, then R/N is a semiprime Goldie ring. □

We note that Theorem 5 has a more direct proof. It is clear that if a module has finite λ-rank, then it must have finite uniform dimension. As in the proof of Theorem 4 it is possible to show that if R is semiprime and there exists a nonsingular additive rank function on R–Mod, then R satisfies the descending chain condition on left annihilators of faithful left ideals. It then follows from Theorem 2.5 of [7] that R is a semiprime Goldie ring.

If I is an ideal of R, then the set of elements of R that are regular modulo I will be denoted by $\mathcal{C}(I)$. If N is the prime radical of R, the ring R is said to satisfy the regularity condition if $\mathcal{C}(N) \subseteq \mathcal{C}(0)$.

Theorem 6 *The ring R is a left order in a left Artinian ring if and only if R satisfies the regularity condition and there exists a nonsingular additive rank function on R–Mod.*

Proof. The theorem follows from Theorem 4, since it is proved in Theorem 4 of [6] that R is a left order in a left Artinian ring if and only if R satisfies the regularity condition and has finite reduced rank on the left. □

We end this note with a comment on atomic additive rank functions. Krause has shown in [4] that every additive rank function on a left Noetherian ring is constructed from atomic additive rank functions defined by the minimal prime ideals of R. For certain additive rank functions we can extend this result to the general setting.

Without restrictions on the ring R, suppose that λ is an additive rank function defined by a torsion radical of R–Mod. That is, we assume that σ is a torsion radical such that the localization $Q_\sigma(R)$ has finite length in the quotient category R–Mod/σ, and that $\lambda(M)$ is the length of $Q_\sigma(M)$ in the quotient category (whenever this is defined for the module ${}_RM$). Theorem 7 of [8] shows that σ is defined by finitely many σ-closed prime ideals, and these are minimal in the set of prime Goldie ideals of R. These prime ideals define maximal torsion radicals, with associated atomic rank functions.

References

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