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RINGS WITH FINITE REDUCED RANK

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In [6], Goldie defined the reduced rank $\rho(M)$ of a finitely generated module M over a semiprime Goldie ring R to be the uniform dimension of $M/\gamma(M)$, where γ is the Goldie torsion radical defined by the set of regular elements of R. Equivalently, $\rho(M)$ is given by the length of the left Q-module $Q \otimes_R M$, where Q is the classical ring of left quotients of R, which is semisimple Artinian. The reduced rank of a finitely generated module N over a Noetherian ring R was defined by Goldie as follows: if N is the prime radical of R, then N is nilpotent, say $N^k = (0)$, and R/N is a semiprime Goldie ring, so it is possible to utilize the previous definition by considering $\sum_{i=1}^{k} \rho(N^{i-1}M/N^{i}M)$.

The purpose of this paper is to give a general definition of reduced rank and to investigate the properties of rings which have finite reduced rank under this definition. Let R be any ring, with prime radical N, let γ denote the torsion radical cogenerated by the injective envelope $\mathrm{E}(R/N)$ of the left R-module

R/N. Then a module ${}_RM$ is said to have finite reduced rank if the module of quotients $Q_\gamma(M)$ has finite length in the quotient category R–Mod/γ. This is shown to yield the usual definition for a left Noetherian ring.

This approach has the advantage of allowing one to work in a Grothendieck category. In particular, the Jordan–Hölder Theorem is valid in the quotient category R–Mod/γ, so that the length of a composition series is well-defined. The quotient functor $Q_\gamma : R\text{–}Mod \to R\text{–}Mod/\gamma$ is exact, so it follows immediately that the rank function is additive on short exact sequences of modules. The reader is referred to the paper of Goldman [8] for results related to length in quotient categories. Use will be made of a theorem of Teply and Miller [11, Theorem 1.4], which states that Hopkins' Theorem holds for quotient categories. It states that if σ is any torsion radical and $Q_\sigma(R)$ satisfies the descending chain condition in R–Mod/σ, then it must also satisfy the ascending chain condition. It is interesting to note that Theorem 1 below shows that for the particular torsion radical γ, the converse is true.

The first result of this paper gives several equivalent conditions under which the ring R has finite reduced rank. In particular, R/N must be a left Goldie ring, and so the torsion radical γ is also determined by $C(N)$, the set of elements of R regular modulo N. If R is a left order in a left Artinian ring Q, then $Q \cong Q_\gamma(R)$ and R–$Mod/\gamma \cong Q$–Mod (see [3]), so it follows immediately that R has finite reduced rank. In

Theorem 4 below it is shown that R is a left order in a left Artinian ring if and only if R has finite reduced rank on the left and $C(N) \subseteq C(0)$. This simplifies the statement of Warfield's version [17, Theorem 3] of Small's Theorem.

A commutative Noetherian ring is an order in an Artinian ring if and only if it has no embedded associated prime ideals. This result is extended in Theorem 6 to any ring which has finite reduced rank on both the left and right. Such a ring is an order in an Artinian ring if and only if the ring and certain factor rings have no embedded associated prime ideals on either the left or right.

The class of rings with finite reduced rank is extensive. In addition to all orders in Artinian rings, Lenagan [10] has shown that it contains all rings with Krull dimension. It is clear that the reduced rank of the direct sum of two rings with finite reduced rank is just the sum of the respective ranks. It is shown in Theorem 8 that if R has finite reduced rank, then the polynomial ring $R[x_1, x_2, \ldots]$ in countably many commuting indeterminates also has finite reduced rank. In fact, the polynomial ring has the same reduced rank as its coefficient ring. The condition need not be inherited by factor rings, since there are examples of prime Goldie rings which have prime factor rings which are not Goldie rings. On the other hand, if I is an ideal of R with $I \subseteq N$, then it can easily be shown that $\rho_{R/I}(R/I) \leq \rho_R(R)$. It will be shown in a subsequent paper that having finite reduced rank is a Morita invariant property.

Rings with finite reduced rank enjoy many of the properties of Noetherian rings. In addition to the results stated above, it will be shown that a ring with finite reduced rank on the left is left Artinian if and only if every prime factor ring is Artinian. On the other hand, an example is given to show that the prime radical of a ring with finite reduced rank need not be nilpotent. This also shows that such a ring need not be a Goldie ring.

Throughout the paper, R will be assumed to be an associative ring with identity element, and all modules will be assumed to be unital R-modules. The injective envelope of a module ${}_RM$ will be denoted by $E(M)$, and the direct sum of n isomorphic copies of M will be denoted by M^n. The reader is referred to the book by Stenström [16] for definitions and results on quotient categories and torsion radicals, and to the book by Chatters and Hajarnavis [5] for results on Noetherian rings and the rank of a module.

Any injective module ${}_RX$ is the cogenerator of a quotient category $R\text{–}Mod/\sigma$, which consists of all modules ${}_RM$ such that M and $E(M)/M$ can be embedded in a direct product of copies of X, together with all R-homomorphisms between such modules. The quotient category determines (and is determined by) the torsion radical σ, and the quotient functor $Q_\sigma : R\text{–}Mod \to R\text{–}Mod/\sigma$. For any module ${}_RM$ the submodule $\sigma(M)$ is the intersection of all kernels of homomorphisms $f \in \mathrm{Hom}_R(M, X)$, and so, in particular, $\sigma(R) = \mathrm{Ann}_R(X)$. Note that $\sigma(R)M \subseteq \sigma(M)$. The module ${}_RM$ is called σ-torsionfree if $\sigma(M) = (0)$, and σ-torsion if $\sigma(M) = M$; a submodule $M' \subseteq M$ is called σ-closed if M/M' is σ-torsionfree, and σ-dense if M/M' is σ-torsion. Thus the σ-closed left ideals of R are the left annihilators of subsets of X. In defining the quotient functor $Q_\sigma : R\text{–}Mod \to R\text{–}Mod/\sigma$ on the module ${}_RM$, the first step is to factor out the torsion submodule $\sigma(M)$. Then for $M' = M/\sigma(M)$, $Q_\sigma(M) = Q_\sigma(M')$ is defined by the identity $Q_\sigma(M')/M' = \sigma(E(M')/M')$. The subobjects of $Q_\sigma(M)$ correspond to the σ-closed submodules of M, and so for the ring R, the subobjects of $Q_\sigma(R)$ in $R\text{–}Mod/\sigma$ correspond to the left annihilators of subsets of ${}_RX$.

In the quotient category $R\text{–}Mod/\sigma$, the object $Q_\sigma(M)$ has finite length if and only if M satisfies both ascending and descending chain conditions on σ-closed submodules. In this case, $M/\sigma(M)$ must have finite uniform dimension (by [1, Lemma 1] this is true whenever $Q_\sigma(M)$ has Krull dimension in the quotient category), and if I is any σ-closed semiprime ideal of R such that $Q_\sigma(R/I)$ has finite length, then R/I must be a left Goldie ring.

The torsion radical σ is said to be prime if the cogenerating injective ${}_RX$ contains an essential submodule which is simple in the quotient category. In this case, any cogenerator of $R\text{–}Mod/\sigma$ contains an isomorphic copy of the simple object. Note that if P is a prime Goldie ideal of R (that is, R/P is a prime left Goldie ring), then $E(R/P)$ defines a prime torsion radical.

If σ is a proper torsion radical of a prime Goldie ring R and $\sigma(R) \neq (0)$, then there exists an embedding $0 \to R \to \sigma(R)^n$, since $\sigma(R)$ is a faithful left ideal and R satisfies the descending chain condition on left annihilators. This implies that $\sigma(R) = R$, contradicting the assumption that σ is not the identity functor. Thus any nonzero injective module over a prime left Goldie ring is faithful. This can be used to prove the following observation.

Let R be a ring with prime radical N such that R/N is a left Goldie ring, and let X be a nonzero injective module such that $N^k X = (0)$ for some integer $k > 0$. Then there exists a minimal prime ideal P of R such that P is the left annihilator of a submodule of X. To show this, note that for some minimal prime P the submodule $Y = \{x \in X \mid Px = (0)\}$ is nonzero, since there exist finitely many minimal primes P_1, P_2, …, P_n (not necessarily distinct) such that $P_1 P_1 \cdots P_n X = (0)$. Then Y is an injective R/P-module since X is injective, and it is faithful since R/P is a left Goldie ring.

Theorem (1). Let N be the prime radical of the ring R. Then the following conditions are equivalent:

(1) The ring of quotients $Q_\gamma(R)$ has finite length in the quotient category R–Mod/γ cogenerated by $E(R/N)$;

(2) The set of left annihilators of subsets of $E(R/N)$ satisfies the ascending chain condition;

(3)
  (i) R/N is a left Goldie ring;
  (ii) $N^k E(R/N) = (0)$ for some integer $k > 0$;
  (iii) For any left annihilator A of a subset of $E(R/N)$, the left R-module R/A has finite uniform dimension.

Proof. The torsion radical determined by $E(R/N)$ will be denoted γ.

(1) $\Longrightarrow$ (2). The subobjects of $Q_\gamma(R)$ in R–Mod/γ are in one-to-one correspondence with the left annihilators of $E(R/N)$, and so by assumption the lattice of left annihilators of $E(R/N)$ has finite length.

(2) $\Longrightarrow$ (3). Let A be a left annihilator of $E(R/N)$. Then R/A is γ-torsionfree and the ascending chain condition on γ-closed submodules of R/A is satisfied. This implies that R/A has finite uniform dimension and satisfies the ascending chain condition on left annihilators. In particular, R/N and $R/\gamma(R)$ are left Goldie rings. It follows that the prime radical $N/\gamma(R)$ of $R/\gamma(R)$ is nilpotent, and so for some integer $k > 0$, $N^k \subseteq \gamma(R) = \operatorname{Ann}(E(R/N))$, and thus the required conditions are satisfied.

(3) $\Longrightarrow$ (1). Since R/N is a left Goldie ring, $E(R/N) \cong \oplus_{i=1}^{n} E(R/P_i)$, where P_1, P_2, …, P_n are the minimal prime ideals of R. For each index i, $1 \leq i \leq n$, the ring R/P is a left Goldie ring, and so $E(R/P_i)$ cogenerates a prime torsion radical π_i. The torsion radicals π_1, π_2, …, π_n are incomparable, and so the conditions of Theorem 3.6 of [8] are satisfied, showing that R has finite length with respect to γ if and only if it has finite length with respect to π_i, for all $1 \leq i \leq n$. Thus it follows that $Q_\gamma(R)$ has finite length in R–Mod/γ if and only for $1 \leq i \leq n$, $E(R/P_i)$ satisfies both the ascending and descending chain conditions on left annihilators. Since the descending chain condition implies the ascending chain condition [11, Theorem 1.4], it suffices to show that for any minimal prime

ideal P, the left annihilators of $E(R/P)$ satisfy the descending chain condition.

Let P be a minimal prime ideal of R, and let π be the torsion radical cogenerated by $E(R/P)$. Let $\{A_i\}_{i=1}^{n}$ be a descending chain of left annihilators of $E(R/P)$, with $A = \cap_{i=1}^{\infty} A_i$. Then A is a left annihilator of $E(R/P)$, and so by assumption it has finite uniform dimension. If U is any uniform submodule of R/A, then U is π-torsionfree, and by assumption $N^k U \subseteq \gamma(R)\, U = (0)$. It follows from the remarks preceding the theorem that there exists a minimal prime ideal Q such that Q is the annihilator of a nonzero submodule of $E(U)$. Therefore Q is an annihilator of a submodule of $E(R/P)$, which implies that $Q \subseteq P$, and then $Q = P$ since P is a minimal prime ideal. This shows that $E(U)$ cogenerates $E(R/P)$, and thus $E(U)$ defines the same torsion radical π. It follows that $E(U)$ contains a submodule which is a simple object in $R\text{–}Mod/\pi$. Thus since $E(R/A)$ has finite uniform dimension, $E(R/A) \cong \oplus_{i=1}^{m} E(U_i)$ for uniform submodules $U_i \subseteq R/A$, $1 \leq i \leq m$, such that U_i contains a submodule which maps to a simple object in $R\text{–}Mod/\pi$. This shows that in the quotient category $R\text{–}Mod/\pi$, the object $E(R/A)$ has an essential socle of finite length, and so it satisfies the finite intersection property for subobjects. Since each A_i is π-closed in R, it follows that $A = \cap_{i=1}^{t} A_i$ for some integer t. Thus the set of left annihilators of $E(R/P)$ satisfies the descending chain condition, completing the proof. $\square$

Example 1. Let Λ be the ring of matrices of the form $\begin{pmatrix} a & 0 \\ x & a \end{pmatrix}$, where $a \in \mathbb{Z}$ and $x \in \mathbb{Q}/\mathbb{Z}$. The prime radical of Λ is the prime ideal $N = \begin{pmatrix} 0 & 0 \\ \mathbb{Q}/\mathbb{Z} & 0 \end{pmatrix}$, and $\gamma(\Lambda) = N$, so $Q_\gamma(\Lambda) \cong \mathbb{Q}$. The ring Λ does not have finite uniform dimension, and satisfies neither chain condition on annihilators.

Example 2. The following example shows that the prime radical of a ring which satisfies the conditions of Theorem 1 need not be nilpotent. Let F be a field, and let $F[x_1, x_2, \ldots]$ be the ring of polynomials over F in a countably infinite number of indeterminates. Let I be the ideal $< x_1x_2, x_2^2, x_1x_3, \ldots, x_1x_i, x_i^i, \ldots >$ and let $R = F[x_1, x_2, \ldots]/I$. The prime radical N of R is the ideal $(< x_2, x_3, \ldots > + I)/I$, since this is a prime ideal with nilpotent generators. The element $x_1 + I$ is regular modulo N, and so the relations $x_1x_i \in I$ for all $i > 1$ show that $\gamma(R) = N$. The localization $Q_\gamma(R)$ is just the field of fractions of $F[x_1]$, and so $Q_\gamma(R)$ has length one in the quotient category. Finally, N is not nilpotent since $(x_{k+1} + I)^k \neq (0)$ for each $k > 1$.

Definition. Let R be a ring with prime radical N, and let γ denote the torsion radical cogenerated by $E(R/N)$. The module ${}_RM$ is said to have finite reduced rank if the module of quotients $Q_\gamma(M)$ has finite length in the quotient category $R\text{–}Mod/\gamma$. In this case the length will be denoted by $\rho_R(M)$. The left reduced rank of the ring R is defined to be the reduced rank of the module ${}_RR$.

If I is an ideal of R, then the set of elements of R which are regular modulo I will be denoted by $C(I)$. The module M is said to be $C(I)$-torsion if for each $m \in M$ there exists $c \in C(I)$ such that $cm = 0$, and then a torsion radical σ can be defined for any module M by letting $\sigma(M)$ be the sum in M of all $C(I)$-torsion submodules. It is important to note that M is $C(I)$-torsionfree if and only if for each element $0 \neq m \in M$ and each $c \in C(I)$ there exists $r \in R$ such that $crm \neq 0$. (It may also happen that $cm = 0$ for some $0 \neq m \in M$ and $c \in C(I)$.)

If $C(I)$ is a left Ore set, that is, if for each $c \in C(I)$ and each $r \in R$ there exist $c' \in C(I)$ and $r' \in R$ such that $r'c = c'r$, then $\sigma(M) = \{m \in M \mid cm = 0 \text{ for some } c \in C(I)\}$. Note that the Ore condition holds if and only if R/Rc is $C(I)$-torsion for all $c \in C(I)$. The set $C(I)$ is said to be a left denominator set if it is a left Ore set and in addition, if $rc = 0$ for any $r \in R$ and $c \in C(I)$, then $c'r = 0$ for some $c' \in C(I)$. If $C(I)$ is a left denominator set, then the ring of quotients $Q_\sigma(R)$ is constructed by inverting the elements of $C(I)$, and $Q_\sigma(M) \cong Q_\sigma(R) \otimes_R M$.

If R is a ring with finite reduced rank, then by Theorem 1 the ring R/N is a left Goldie ring. It can be shown that the torsion radicals defined by $E(R/N)$ and $C(N)$ must be equal (see [3, Proposition 1]). This equality will be exploited in the remainder of the paper. The next proposition shows that the above definition of reduced rank coincides with Goldie's original definition of reduced rank for left Noetherian rings.

Proposition (2). Let R be a ring with finite reduced rank on the left, such that $N^k E(R/N) = (0)$ for the positive integer k. Then the module ${}_RM$ has finite reduced rank if and only if $N^{i-1}M/N^iM$ has finite reduced rank as an R/N-module for $1 \leq i \leq k$. In this case

$$\rho_R(M) = \textstyle\sum_{i=1}^{k} \rho_{R/N}(N^{i-1}M/N^iM)\ .$$

Proof. Let γ denote the torsion radical defined by $C(N)$. Then by assumption $N^kM \subseteq \gamma(R)M \subseteq \gamma(M)$, and so $Q_\gamma(M) \cong Q_\gamma(M/\gamma(M)) \cong Q_\gamma(M/N^kM))$. In R–Mod/γ, length is additive on short exact sequences, and it follows that $\rho_R(M) = \rho_R(M/N^kM) = \sum_{i=1}^{k} \rho_R(N^{i-1}M/N^iM)$ whenever the terms are finite. Finally, if X is an R/N-module, then ${}_RX$ is $C(N)$-torsionfree if and only if ${}_{R/N}X$ is $\overline{C(N)}$-torsionfree, where $\overline{C(N)} = \{c + N \mid c \in C(N)\}$ is the set of regular elements of R/N. It follows that for $1 \leq i \leq k$, $\rho_R(N^{i-1}M/N^iM) = \rho_{R/N}(N^{i-1}M/N^iM)$, completing the proof. □

Proposition (3). Let R be a ring with finite reduced rank (on the left) and prime radical N. The following conditions are equivalent:

(1) $C(N)$ is a left Ore set;

(2) For each element $c \in C(N)$, $\rho_R(Nc) = \rho_R(N)$;

(3) For any module ${}_RM$, $\rho_R(M) = 0 \Longleftrightarrow \rho_{R/N}(M/NM) = 0$.

Proof. (l) $\Longrightarrow$ (2). To show that $\rho(Nc) = \rho(N)$ it is sufficient to show that $\rho(N/Nc) = 0$. Given $a \in N$ there exist by assumption elements $c' \in C(N)$ and $a' \in R$ such that $c'a = a'c$. Since $a \in N$, $a'c \in N$, and then $a' \in N$ since c is regular modulo N. Thus $c'a = a'c \in Nc$, which shows that N/Nc is $C(N)$-torsion.

(2) $\Longrightarrow$ (3). By Proposition 2, it is sufficient to show that if N/NM is $C(N)$-torsion, then NM/N^2M is $C(N)$-torsion. Then given $am \in NM$, with $a \in N$, $m \in N$, there exists $c \in C(N)$ such that $cm \in NM$. By assumption there exists $c' \in C(N)$ such that $c'a \in Nc$, and so $c'am \in Ncm \subseteq N^2M$. It follows that NM/N^2M is $C(N)$-torsion.

(3) $\Longrightarrow$ (1). Given $c \in C(N)$, let $N = R/Rc$. Then $NM = (N + Rc)/Rc$, and so $N/NM \cong R/(N + Rc)$. But $\rho_R(R/(N + Rc)) = 0$ since $C(N)$ is a left Ore set modulo N, and so by assumption $\rho_R(R/Rc) = 0$. Thus for any $a \in R$, there exists $c' \in C(N)$ such that $c'a \in Rc$, and $C(N)$ is a left Ore set. □

The ring R is said to satisfy the regularity condition if $C(N) \subseteq C(0)$. Small [14, 15] showed that the ring R is a left order in a left Artinian ring if and only if it satisfies the regularity condition, the prime radical N of R is nilpotent, and for each integer $k > 0$, $R/(N \cap \mathfrak{r}(N^k))$ is a left Goldie ring. Recently, Warfield [17, Theorem 3] has shown that the last condition of Small's Theorem can be replaced by the assumptions that R/N is a left Goldie ring and $\sum_{i=1}^{k} \rho_{R/N}(N^{i-1}/N^i)$ is finite, where k is the index of nilpotence of N. Theorem 1 shows that these conditions can be replaced by the single condition that R has finite reduced rank on the left. This is of course equivalent to the condition that R satisfies the ascending condition on left annihilators of $E(R/N)$, which is reminiscent of the definition of a solid Goldie ring.

Theorem (4). The ring R is a left order in a left Artinian ring if and only if R satisfies the regularity condition and has finite reduced rank on the left.

Proof. Let N be the prime radical of R, and let γ be the torsion radical defined by $C(N)$. Assume that $C(N) \subseteq C(0)$ and that $\rho_R(R)$ is finite. Then R is $C(N)$-torsionfree, and for each $c \in C(N)$, right multiplication by c defines a monomorphism from N onto Nc, which shows that $\rho_R(N) = \rho_R(Nc)$, and so $C(N)$ is a left Ore set. Since R is $C(N)$-torsionfree, it satisfies the ascending chain condition on left annihilators, and so $C(N)$ is a left denominator set [16, Proposition 1.5]. Thus R–Mod/γ coincides with $Q_\gamma(R)$-Mod, and so $Q_\gamma(R)$ is a left Artinian ring of fractions, since by assumption it has finite length in the quotient category.

The converse is clear. □

Lemma (5). Let R be a ring with finite reduced rank on the left. The following conditions are equivalent for the module ${}_RM$:

(1) N is $C(N)$-torsionfree;

(2) $E(M)$ satisfies the descending chain condition on left annihilators of subsets;

(3) $E(M) \cong \oplus_{\alpha \in I} E_\alpha$, where for each $\alpha \in I$, E_α is isomorphic to a direct summand of $E(R/P_\alpha)$, for a minimal prime ideal P_α.

Proof. Let γ be the torsion radical defined by $C(N)$.

(1) $\Longrightarrow$ (2). If N is γ-torsionfree, then so is $E(M)$, and so any left annihilator of $E(M)$ is γ-closed.

(2) $\Longrightarrow$ (3). If $E(M)$ satisfies the descending chain condition on left annihilators, then by [11, Theorem 1.4] the ascending chain condition is also satisfied. Condition (3) then follows directly from Theorem 7 of [1], which characterizes injective modules with both ascending and descending chain conditions on left annihilators.

(3) $\Longrightarrow$ (1). If $E(M)$ has the given form, then it must be $C(N)$-torsionfree, since γ is defined by $E(R/N) \cong \oplus_{i=1}^{n} E(R/P_i)$, where P_1, …, P_n are the minimal prime ideals of R. □

Recall that the prime ideal P of R is said to be an associated prime ideal of the module ${}_RM$ if there exists a submodule $X \subseteq M$ such that $P = \mathrm{Ann}(X')$ for each nonzero submodule $X' \subseteq X$. If M satisfies the ascending chain condition on left annihilators, then maximal left annihilators exist, and any maximal left annihilator ideal is an associated prime ideal of M. If, in addition, M satisfies the descending chain condition on left annihilators, then P is an associated prime ideal of M if and only if $P = \mathrm{Ann}(m_1, m_2, \ldots, m_n)$ for elements $m_1, m_2, \ldots, m_n \in M$. In this case Lemma 5 shows that M is $C(N)$-torsionfree if and only if each associated prime ideal of M is minimal.

Warfield [18] has shown that if R is a Noetherian ring with prime radical N, then R is an order in an Artinian ring if and only if every associated prime ideal of ${}_RR$ and $(R/\ell(N^k))_R$ is minimal, for $k = 0, 1, \ldots$. (See [7, Theorem 3] for a proof of this theorem.) The following theorem removes the assumption that R is Noetherian, and utilizes the factor rings $R/(N \cap \ell(N^k))$ rather than $R/\ell(N^k)$. The factor rings $R/(N \cap \ell(N^k))$ were used by Small [15] to characterize right orders in right Artinian rings.

Theorem (6). Let R be a ring with prime radical N. The following conditions are equivalent:

(1) R is a left and right order in an Artinian ring;

(2) (i) R has finite reduced rank on the left and right:

(ii) for each left ideal A of R, there exits a prime ideal P such that $P = \ell(a_1, a_2, \ldots, a_n)$ for elements $a_1, a_2, \ldots, a_n \in A$;

(iii) each associated prime ideal of ${}_RR$ and $(R/(N \cap \ell(N^k)))_R$ is minimal, for $k = 0, 1, \ldots$.

Proof. (1) $\Longrightarrow$ (2). If R is an order in an Artinian ring, then R has finite reduced rank on the left and right, and satisfies the ascending and descending chain conditions on left annihilators, so conditions (i) and (ii) hold. It is known that for $k = 0, 1, \ldots$, $R/(N \cap \ell(N^k))$ is an order in an Artinian ring [15, Lemma 1], but a proof that condition (iii) holds will be given without making use of this fact.

By assumption ${}_RR$ is $C(N)$-torsionfree, and $C(N)$ is a left Ore set. The ideals $\ell(N^k)$, for $k = 0, 1, \ldots$, are $C(N)$-closed on the left, since they are left annihilators, and so the ideals $N \cap \ell(N^k)$, $k = 0, 1, \ldots$ are $C(N)$-closed on the left. Thus the ring $R/(N \cap \ell(N^k))$ satisfies the ascending chain condition for left annihilators, for $k = 0, 1, \ldots$, and so $C(N)$ is a left denominator set modulo $N \cap \ell(N^k)$. Since $R/(N \cap \ell(N^k))$ is $C(N)$-torsionfree on the left, it must be $C(N)$-torsionfree on the right, for $k = 0, 1, \ldots$. It follows from Lemma 5 that the associated prime ideals of R and $(R/(N \cap \ell(N^k)))_R$, $k = 0, 1, \ldots$, must he minimal.

(2) $\Longrightarrow$ (1). If conditions (ii) and (iii) hold, then R is $C(N)$-torsionfree by Lemma 5, and it follows that ${}_R(R/(N \cap \ell(N^k)))$ is $C(N)$-torsionfree for $k = 0, 1, \ldots$. Thus for $k = 0, 1, \ldots$, $R/(N \cap \ell(N^k))$ satisfies the ascending and descending chain conditions on left annihilators, and therefore satisfies the ascending and descending chain conditions on right annihilators. Thus each right ideal of $R/(N \cap \ell(N^k))$ has an associated prime ideal, and then condition(iii) implies that $R/(N \cap \ell(N^k))$ is $C(N)$-torsionfree for $k = 0, 1, \ldots$.

Since R has finite reduced rank on both sides, and the regularity condition is symmetric, all that remains to be shown is that $C(N)$ is a left Ore set. This will be done by using Proposition 3, and to show that $\rho({}_R N) = \rho({}_R Nc)$ for all $c \in C(N)$, it is sufficient to show that $xc \neq 0$ for all $c \in C(N)$ and $0 \neq x \in N$. Suppose that $xc = 0$ for $x \in N$ and $c \in C(N)$. If $x \neq 0$, then since N is nilpotent there exists an integer $k \geq 0$ such that $xN^k \neq (0)$ but $xN^{k+1} = (0)$. The coset $x + (N \cap \ell(N^k))$ is a nonzero element of the right R/N-module

$(N \cap \ell(N^{k+1}))/(N \cap \ell(N^k))$, which has been shown to be $C(N)$-torsionfree. Since $C(N)$ is a right Ore set modulo N, this contradicts the assumption that $xc = 0$ □

Corollary (7). Let R be a left and right Noetherian ring with prime radical N. Then R is an order in an Artinian ring if and only if every associated prime ideal of ${}_RR$ and $(R/(N \cap \ell(N^k)))_R$, is a minimal prime ideal, for $k = 0, 1, \ldots$.

Proof. If R is right Noetherian, then R satisfies the descending chain condition on left annihilators, and so condition 2 (ii) of Theorem 6 is satisfied. □

Theorem (8). Let R be a ring with finite reduced rank on the left. Then the ring of polynomials $R[x_1, x_2, \ldots]$ in commuting indeterminates $x_1, x_2, \ldots$ has finite reduced rank on the left. Moreover, $\rho(R) = \rho(R[x_1, x_2, \ldots])$.

Proof. Let N be the prime radical of R, and let γ be the torsion radical defined by the prime radical $N[x_1, x_2, \ldots]$ of $R[x_1, x_2, \ldots]$. If K is the $C(N)$-torsion ideal of R, then $K[x_1, x_2, \ldots]$ is easily seem to be γ-torsion. It is well-known that the powers $(N[x_1, x_2, \ldots])^k$ are just $N^k[x_1, x_2, \ldots]$, and since for some integer $m > 0$, $N^k \subseteq K$, it follows that $(N[x_1, x_2, \ldots])^k \subseteq K[x_1, x_2, \ldots]$. In general, if the submodule X of M is $C(N)$-torsion, then $\rho(M) = \rho(M/X)$. This means that it is sufficient to show that

$$\sum_{i=1}^{m} \rho((N^{i-1}/N^i)[x_1, x_2, \ldots]) = \sum_{i=1}^{m} \rho((N^{i-1}[x_1, x_2, \ldots])/(N^i[x_1, x_2, \ldots]))$$

is finite. The reduced rank of these factors can be computed over $R/N[x_1, x_2, \ldots]$, which is a semiprime Goldie ring. Recall that over such a ring, the reduced rank of a module M is the uniform dimension of $M/Z(M)$, where $Z(M)$ is the singular submodule of M.

Shock [13] has shown that the singular ideal of $R[x_1, x_2, \ldots]$ is $Z(R)[x_1, x_2, \ldots]$, and that if R has finite uniform dimension, then $R[x_1, x_2, \ldots]$ has equal uniform dimension. His proofs can be extended to show that the singular submodule of $A/B[x_1, x_2, \ldots]$ is $Z(A/B)[x_1, x_2, \ldots]$, if $B \subseteq A$ are ideals of R, and that if A/B has finite uniform dimension, then $A/B[x_1, x_2, \ldots]$ has equal dimension. Thus $\sum_{i=1}^{m} \rho((N^{i-1}/N^i)[x_1, x_2, \ldots]) = \sum_{i=1}^{m} \rho(N^{i-1}/N^i)$, and so the reduced rank of $R[x_1, x_2, \ldots]$ is equal to that of R. □

Theorem (9). Let R be a ring with finite reduced rank on the left. The following conditions are equivalent:

(1) R is left Artinian;

(2) For each prime ideal P of R, R/P is Artinian;

(3) Each prime ideal of R is maximal, and for each finitely generated module M there exist elements $x_1, x_2, \ldots, x_n \in M$ such that $\mathrm{Ann}(x_1, x_2, \ldots, x_n) = \mathrm{Ann}(M)$.

Proof. Theorem 2.7 of [2] states that the given conditions are equivalent for any ring R such that for each submodule $X \subseteq E(R/N)$ there exist $x_1, \ldots, x_n \in X$ such that $\mathrm{Ann}(x_1, \ldots, x_n) = \mathrm{Ann}(X)$. This condition on submodules of $E(R/N)$ follows immediately from the descending chain condition on left annihilators of $E(R/N)$. □

Corollary (10). A left or right perfect ring with finite reduced rank on the left is Artinian.

Proof. This follows from the fact that any left or right perfect ring satisfies condition (2) of the previous theorem. An alternate proof can be given by observing that if J is the Jacobson radical of a left or right perfect ring, then $J = N$ and R/J is semisimple Artinian, so that $E(R/N)$ is a cogenerator for R–*Mod*. Thus every left ideal of R must be $C(N)$-closed, and R is left Artinian.□

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