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**Universal Localization at Prime Ideals**

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Before discussing Peter Malcolmson's construction of the universal localization via ordered triples, I would like to make a few general remarks. My first taste of universal localization came in the spring of 1976, during a brief sabbatical visit to Bedford College.

Among Professor Cohn's papers on inverting matrices, I found his fundamental paper "Inversive localisation in Noetherian rings," published in 1973, to be closest in nature to the noncommutative localization that I had already been studying. The paper constructs the universal localization $R_\Gamma$ at a prime ideal $P$ of a Noetherian ring by inverting the matrices $\Gamma$ that are not zero divisors modulo $P$. For each prime ideal there is a simple Artinian classical quotient ring $Q_{cl}(R/P)$, and since its construction commutes with formation of matrix rings, that is, since $Q_{cl}(\mathrm{M}_n(R/P)) \cong \mathrm{M}_n(Q_{cl}(R/P))$, the set $\Gamma$ consists of the matrices that become invertible over $Q_{cl}(R/P)$.

Among other interesting consequences of the construction of $R_\Gamma$, Professor Cohn proved that $R_\Gamma/\mathrm{J}(R_\Gamma)$ is naturally isomorphic to $Q_{cl}(R/P)$. This models the construction in the commutative case. Motivated by my desire to put the universal localization into more familiar terms, I was able to show that $R_\Gamma$ is characterized as the ring universal with respect to this property.

In the commutative case, the inverse images of the powers of the Jacobson radical of the localization form the "symbolic powers" of the prime ideal $P$. This definition can also be given in the noncommutative case, and it turns out that $R_\Gamma/(\mathrm{J}(R_\Gamma))^n$ is an Artinian classical ring of quotients of $R/P^{(n)}$. In 1967 Alfred Goldie constructed a localization as a certain subring of the inverse limit of the rings $Q_{cl}(R/P^{(n)})$. The universal property that I mentioned above can also be used to show that Goldie's localization is a homomorphic image of the universal localization. Basically, he first factors out the intersection of the symbolic powers, and although the kernel of $\lambda : R \to R_\Gamma$ always contains the intersection of the symbolic powers, it can in general be smaller.

The first thing that I was able to prove about $R_\Gamma$ is that if $J$ is an idempotent ideal contained in $P$, then $J$ is torsion relative to $\Gamma$ and hence is contained in the kernel of the canonical map into the universal localization.

Proof: If $J$ is generated as a left ideal by $x_1, x_2, \ldots, x_n$, then we have an expression $x_i = \sum_{j=1}^{n} a_{ij} x_j$, where $a_{ij} \in J$. This leads to a matrix equation $(I - A)x = 0$, in which $I$ is the $n \times n$ identity matrix. Then $I - A$ is congruent to the identity matrix modulo $J$, so it is certainly regular modulo $J$, and thus each entry of $x$ is torsion relative to $\Gamma$. Since the generators of $J$ map to zero, we see that $J$ is contained in the kernel of the canonical mapping.

Among other consequences, this allows the computation of the universal localization at a maximal ideal of an Artinian ring. In this case $R_\Gamma = R/P^n$, if $P^{n+1} = P^n$.

If the Ore condition holds for the set of elements regular modulo $P$, then the universal localization coincides with the Ore localization at $P$. Note that $R_\Gamma$ is always an epimorphism in the category of rings, but it is a flat right $R$-module only if the Ore condition holds.

Now on to a discussion of Malcolmson's construction. The principal references I will use are:

P. Malcolmson, Construction of universal matrix localizations, Springer Lecture Notes in Math. No. 951, 1982, pp. 117-131.

J. A. Beachy, On universal localization at semiprime Goldie ideals, *Ring Theory, Proceedings of the Biennial Ohio State–Denison Conference, May, 1992*, World Scientific, 1993, pp. 41–57.

I found Malcolmson's construction via triples to be a very interesting alternative to the standard construction via generators and relations, but again many of his arguments were pretty far away from things I was used to. I looked for a way to make the construction look more like things I knew about.

Professor Cohn showed that elements of $\Sigma^{-1}R$ are actually entries in inverses of matrices, so $e_i C^{-1} e_j$ picks out the $(i, j)$-entry. It is easier to use triples of the form $aC^{-1}b$, where $a, b$ are any vectors over $R$. Then we might as well use triples of the form $aC^{-1}x$, where $a$ has entries in $R$ and $x$ has entries in a module.

The outline is the following: We first define an equivalence relation under which the set of ordered triples define a commutative semigroup. We then define a congruence on the semigroup that gives us an abelian group, and allows definition of a scalar multiplication. One of the main computational tools is a condition that resembles the Ore condition.

The module ${}_RX$ will be fixed; $X^n$ denotes the direct sum of $n$ copies of $X$; $I_n$ denotes the $n \times n$ identity matrix; a column vector $x \in X^n$ will be denoted by $x^t$; $\Sigma$ will denote a set of square matrices over $R$ such that

(i) $\Sigma$ contains all permutation matrices;

(ii) if $C, D \in \Sigma$, then $\begin{bmatrix} C & A \\ 0 & D \end{bmatrix} \in \Sigma$ for any matrix $A$ of the appropriate size; and

(iii) if $C, D \in \Sigma$ and $CD$ is defined, then $CD \in \Sigma$.

The set of matrices inverted by any ring homomorphism satisfies these properties. In particular, $R$ is left Noetherian and $P$ is a prime ideal of $R$, then the set $\Gamma$ of all square matrices regular modulo $P$ satisfies the above conditions.

The elements of a module of quotients, denoted by $X_\Sigma$, will be constructed as equivalence classes of ordered triples $(a, C, x^t)$, where $a \in R^n$, $C \in \Sigma_n$, and $x \in X^n$ (for any positive $n$). The ordered triples are modeled on the element $aC^{-1}x^t$, where $C$ is invertible, as would be the case over $R_\Sigma$. Let $a \in R^n$, $C \in \Sigma_n$, $x \in X^n$, $b \in R^m$, $D \in \Sigma_m$, $y \in X^m$. If $C$ and $D$ are invertible, then we have the following identities.

(i) $[a \quad b] \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}^{-1} \begin{bmatrix} x^t \\ y^t \end{bmatrix} = aC^{-1}x^t + bD^{-1}y^t$

(ii) $aU(VCU)^{-1}Vx^t = aC^{-1}x^t$, provided $U, V$ are invertible

(iii) $aC^{-1}0 = 0 = 0C^{-1}x^t$

An addition of triples is based on the first of these identities. The second motivates the definition of the initial equivalence relation for triples. We say that $(a, C, x^t) \equiv (b, D, y^t)$ if there exist invertible matrices $U, V$ in $\Sigma$ such that $b = aU$, $D = VCU$ and $y^t = Vx^t$. It is easily checked that this defines an equivalence relation. (The proof of transitivity uses the fact that $\Sigma$ is closed under products.) We note that $(a, C, x^t) \equiv (b, D, y^t)$ only if $C$ and $D$ have the same size. Equation (iii) provides the motivation for the definition of the subsemigroup that induces the final equivalence relation.

**Definition 1** *Let $(a, C, x^t)$, $(b, D, y^t)$ be ordered triples with $a, b \in R^n$, $C, D \in \Sigma_n$ and $x, y \in X^n$, for some positive integer $n$.*

*If there exist invertible $n \times n$ matrices $U, V$ in $\Sigma$ such that*

$$b = aU, \quad D = VCU, \quad \text{and} \quad y^t = Vx^t \ ,$$

*then we say that $(a, C, x^t)$ is congruent to $(b, D, y^t)$ via $U, V$, written*

$$(a, C, x^t) \equiv (b, D, y^t) \quad \text{via} \quad U, V \ .$$

*For $a \in R^n$, $C \in \Sigma_n$ and $x \in X^n$, the notation $(a : C : x^t)$ will be used for the equivalence class of the ordered triple $(a, C, x^t)$ under the equivalence relation $\equiv$. The set of all such equivalence classes, for all positive integers $n$, will be denoted by $\Sigma^{-1}X$.*

We can think of the equivalence relation as a sort of cancellation property:

$$(b, D, y^t) = (aU, VCU, Vx^t) \equiv (a, C, x^t) \ .$$

This cancellation of invertible matrices will be extended in Proposition 6 to cancellation of any matrices in $\Sigma$.

**Proposition 2** *The sum of elements $(a : C : x^t)$, $(b : D : y^t) \in \Sigma^{-1}X$ defined by*

$$(a : C : x^t) + (b : D : y^t) = \left( [a \quad b] : \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix} : \begin{bmatrix} x^t \\ y^t \end{bmatrix} \right)$$

*yields an associative, commutative binary operation on $\Sigma^{-1}X$.*

**Definition 3** *The subset of $\Sigma^{-1}X$ consisting of all equivalence classes of elements of the form*

$$(e_1, E_1, 0), \quad or \quad (0, E_2, e_2^t), \quad or \quad \left( [e_1 \quad 0], \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}, \begin{bmatrix} 0 \\ e_2^t \end{bmatrix} \right)$$

*for some $e_1 \in R^m$, $E_1 \in \Sigma_m$, $E_2 \in \Sigma_n$, and $e_2 \in X^n$ will be denoted by $\Sigma_0^{-1}X$.*

We note that the first two forms could be thought of as "degenerate" versions of the third, so that really we can think of $\Sigma_0^{-1}X$ as consisting of elements of $\Sigma^{-1}X$ of the third type.

**Proposition 4** *For elements $\overline{x}, \overline{y} \in \Sigma^{-1}X$, the relation $\sim$ defined by*

$$\overline{x} \sim \overline{y} \quad \text{if there exist } \overline{z}_1, \overline{z}_2 \in \Sigma_0^{-1}X \text{ such that} \quad \overline{x} + \overline{z}_1 = \overline{y} + \overline{z}_2$$

*is a congruence on the semigroup $\Sigma^{-1}X$. The set $\Sigma^{-1}X/\sim$ of equivalence classes of this congruence is an abelian group.*

If $C, D$ are invertible matrices such that $DA = BC$ for matrices $A, B$, then $AC^{-1} = D^{-1}B$ and so $aAC^{-1}x^t = aD^{-1}Bx^t$. This motivates the following lemma for triples $(aA : C : x^t)$ and $(a : D : Bx^t)$ such that $DA = BC$, a situation reminiscent of the left Ore condition. This lemma will prove to be very useful computationally.

**Lemma 5** *Let $a \in R^m$, $C \in \Sigma_n$, $x \in X^n$, and let $A$ be any $m \times n$ matrix over $R$. If there exist an $m \times n$ matrix $B$ and a matrix $D \in \Sigma_m$ such that $DA = BC$, then*

$$(aA : C : x^t) \sim (a : D : Bx^t) \, .$$

*Proof.* If $a$, $C$, $D$, $A$, $B$, and $x$ are as stated, then

$$[a \quad aA] \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} = [a \quad 0] \quad , \quad \begin{bmatrix} I & B \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ x^t \end{bmatrix} = \begin{bmatrix} Bx^t \\ x^t \end{bmatrix}$$

and

$$\begin{bmatrix} I & B \\ 0 & I \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} = \begin{bmatrix} D & 0 \\ 0 & C \end{bmatrix} .$$

Therefore

$$\left([a \quad aA], \begin{bmatrix} D & 0 \\ 0 & C \end{bmatrix}, \begin{bmatrix} 0 \\ x^t \end{bmatrix}\right) \equiv \left([a \quad 0], \begin{bmatrix} D & 0 \\ 0 & C \end{bmatrix}, \begin{bmatrix} Bx^t \\ x^t \end{bmatrix}\right) .$$

We then have

$$\begin{aligned} (aA : C : x^t) &\sim (a : D : 0) + (aA : C : x^t) \\ &= \left([a \quad aA] : \begin{bmatrix} D & 0 \\ 0 & C \end{bmatrix} : \begin{bmatrix} 0 \\ x^t \end{bmatrix}\right) \\ &= \left([a \quad 0] : \begin{bmatrix} D & 0 \\ 0 & C \end{bmatrix} : \begin{bmatrix} Bx^t \\ x^t \end{bmatrix}\right) \\ &= (a : D : Bx^t) + (0 : C : x^t) \\ &\sim (a : D : Bx^t) . \end{aligned}$$

This completes the proof. □

**Example 1**

If $S$ is a semiprime ideal of $R$ for which $\mathcal{C}(S)$ is a left denominator set, and $\Sigma = \Gamma(S)$, then for each $(a : C : x^t) \in \Sigma^{-1}X$ there exist elements $y \in X$ and $d \in \mathcal{C}(S)$ such that $(a : C : x^t) \sim (1 : d : y)$. To see this, let $\lambda : R \to R_S$ be the classical left localization of $R$ at $\mathcal{C}(S)$. We can assume without loss of generality that $\lambda$ is one-to-one. Let $(a : C : x^t) \in \Sigma^{-1}X$. Then $\lambda(C)$ is invertible over $R_S$, so it is possible to find a common denominator $d \in \mathcal{C}(S)$ for the entries of $\lambda(a)\lambda(C)^{-1}$. Thus we have elements $d \in \mathcal{C}(S)$ and $b \in R^n$ such that $da = bC$, and then it follows from Lemma 5 that $(1 \cdot a : C : x^t) \sim (1 : d : bx^t)$.

**Proposition 6** *Let $a \in R^n$, $C \in \Sigma_n$, and $x \in X^n$.*

**(a)** *If $b \in R^n$ and $y \in X^n$, then*

$$(a : C : x^t) + (a : C : y^t) \sim (a : C : (x + y)^t)$$

*and*

$$(a : C : x^t) + (b : C : x^t) \sim (a + b : C : x^t)\ .$$

**(b)** *For any matrices $P, Q$ such that $PC, CQ \in \Sigma_n$,*

$$(a : C : x^t) \sim (a : PC : Px^t) \quad \text{and} \quad (a : C : x^t) \sim (aQ : CQ : x^t)\ .$$

**(c)** *For any $b \in R^m$, $D \in \Sigma_m$, $y \in X^m$ and any matrices $A, B$ of the appropriate size,*

$$(a : C : x^t) \sim \left( [a \quad b] : \begin{bmatrix} C & A \\ 0 & D \end{bmatrix} : \begin{bmatrix} x^t \\ 0 \end{bmatrix} \right)$$

*and*

$$(a : C : x^t) \sim \left( [0 \quad a] : \begin{bmatrix} D & B \\ 0 & C \end{bmatrix} : \begin{bmatrix} y^t \\ x^t \end{bmatrix} \right)\ .$$

Recall that an element $x \in X$ is $\Sigma$-torsion if it is an entry in a vector $v \in X^n$ such that $Cv^t = 0$ for some $C \in \Sigma$. Since $\Sigma$ is closed under products (when defined) and contains all permutation matrices, it can be assumed that $x$ is the first entry of $v^t$.

**Theorem 7** *Let $x, y \in X$.*

**(a)** *In $\Sigma^{-1}X$, $(1 : 1 : x) \sim (1 : 1 : y)$ if and only if $x = av^t$ and $y = bw^t$ for some $a, b \in R^n$, $v, w \in X^n$ and some $n > 0$ such that there exist $C, D, P, Q \in \Sigma_n$ satisfying $aD = bQ$, $Cv^t = Pw^t$ and $CD = PQ$.*

**(b)** *Furthermore, $x - y \in \mathrm{rad}_{\Sigma(S)} X$ if and only if in condition* (a) *it is possible to take $a = b$, $C = P$ and $D = Q = I$.*

Malcolmson has shown that the mapping $\lambda : R \to R_\Sigma$ given by $\lambda(r) = (1 : 1 : r)$ is a ring homomorphism which inverts the matrices in $\Sigma$. Theorem 7 provides a connection with the description of the kernel of $\lambda : R \to R_\Sigma$ given in Corollary 11.9 of the second edition of *Free Rings and Their Relations.*

**Corollary 8** *Let $\lambda : R \to R_\Sigma$ be the canonical homomorphism. Then for $r \in R$ we have $r \in \ker(\lambda)$ if and only if there is a relation of the form*

$$\begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

*with $A_{21}, A_{22}, B_{12}, B_{22} \in \Sigma$.*

*Proof.* If $r \in \ker \lambda$, then $(1 : 1 : r) \sim (1 : 1 : 0)$, and so it follows from part (a) of the theorem that for some $n > 0$ there exist $a, b, c, d \in R^n$ and $C, D, P, Q \in \Sigma_n$ such that $r = ab^t$, $0 = cd^t$, $aD = cQ$, $Cb^t = Pd^t$ and $CD = PQ$. Then

$$\begin{bmatrix} a & c \\ C & P \end{bmatrix} \begin{bmatrix} b^t & -D \\ -d^t & Q \end{bmatrix} = \begin{bmatrix} ab^t - cd^t & -aD + cQ \\ Cb^t - Pd^t & -CD + PQ \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix}.$$

Conversely, suppose that there is a relation of the form

$$\begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

with $A_{21}, A_{12}, B_{12}, B_{22} \in \Sigma$. Let $e_1$ be the unit vector and define $a = e_1 A_{11}$, $c = e_1 A_{12}$, $b = e_1 B_{11}^t$, $d = -e_1 B_{21}^t$, $C = A_{21}$, $P = A_{22}$, $D = -B_{12}$, $Q = B_{22}$, $s = ab^t$, and $t = cd^t$. The conditions in part (a) of Theorem 7 can easily be checked, and so $(1 : 1 : s) \sim (1 : 1 : t)$. It follows that $\lambda(r) = \lambda(s - t) = \lambda(s) - \lambda(t) = 0$. □

Let $a, r \in R^n$, $C \in \Sigma_n$, $b \in R^m$, $D \in \Sigma_m$, and $y \in X^m$. If $C, D$ are invertible, then we have the following identity, which motivates the definition of a scalar multiplication.

$$[a \quad 0] \begin{bmatrix} C & -r^t b \\ 0 & D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ y^t \end{bmatrix} = aC^{-1}r^t \cdot bD^{-1}y^t$$

**Proposition 9** *The scalar product of elements* $(a : C : r^t) \in \Sigma^{-1}R$ *and* $(b : D : y^t) \in \Sigma^{-1}X$ *defined by*

$$(a : C : r^t) \cdot (b : D : y^t) = \left( [a \;\; 0] : \begin{bmatrix} C & -r^t b \\ 0 & D \end{bmatrix} : \begin{bmatrix} 0 \\ y^t \end{bmatrix} \right)$$

*yields a well-defined, associative operation.*

**Lemma 10** *The following conditions hold for scalar multiplication.*

**(a)** *If* $(a : C : r^t) \in \Sigma^{-1}R$ *and* $(1 : 1 : x) \in \Sigma^{-1}X$, *then*

$$(a : C : r^t)(1 : 1 : x) \sim (a : C : r^t x) \; .$$

**(b)** *If* $(1 : 1 : s) \in \Sigma^{-1}R$ *and* $(a : C : x^t) \in \Sigma^{-1}X$, *then*

$$(1 : 1 : s)(a : C : x^t) \sim (sa : C : x^t) \; .$$

**Lemma 11** *Let* $\overline{p} = (u : P : r^t)$, $\overline{q} = (v : Q : s^t) \in \Sigma^{-1}R$, *and let* $\overline{x} = (a : C : x^t)$, $\overline{y} = (b : D : y^t) \in \Sigma^{-1}X$. *Then*

$$(\overline{p} + \overline{q})\overline{x} \sim \overline{p}\,\overline{x} + \overline{q}\,\overline{x} \quad \text{and} \quad \overline{q}(\overline{x} + \overline{y}) \sim \overline{q}\,\overline{x} + \overline{q}\,\overline{y}\ .$$

**Theorem 12** *For any module* ${}_RX$, *the set* $X_\Sigma$ *is a unital left module over* $R_\Sigma$.

We say that the module ${}_RX$ is $\Sigma$-*torsionfree* if $Cx^t = 0$ implies $x = 0$, for all $C \in \Sigma_n$ and all $x \in X^n$. We say that $X$ is $\Sigma$-*divisible* if for each $x \in X^n$ and each $C \in \Sigma$ there exists $y \in X^n$ such that $Cy^t = x^t$.

**Theorem 13** *The homomorphism* $\eta : X \to X_\Sigma$ *is an isomorphism if and only if* $X$ *is* $\Sigma$*-torsionfree and* $\Sigma$*-divisible.*

**Theorem 14** *For any module* ${}_RX$, *the module of quotients* $X_\Sigma$ *is naturally isomorphic to* $R_\Sigma \otimes_R X$.

Proof of Lemma 11. We have the following equalities.

$$(\overline{p}+\overline{q})\overline{x} = \left( \begin{matrix} [u \quad 0 \quad v \quad 0] \\ \\ \\ \\ \end{matrix} \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} : \begin{bmatrix} P & 0 & -r^t a \\ 0 & Q & -s^t a \\ 0 & 0 & C \end{bmatrix} : \begin{bmatrix} 0 \\ 0 \\ x^t \end{bmatrix} \right)$$

$$\overline{p}\,\overline{x} + \overline{q}\,\overline{x} = \left( [u \ \ 0 \ \ v \ \ 0] : \begin{bmatrix} P & -r^t a & 0 & 0 \\ 0 & C & 0 & 0 \\ 0 & 0 & Q & -s^t a \\ 0 & 0 & 0 & C \end{bmatrix} : \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ x^t \end{bmatrix} \right)$$

The two expressions are equal by Lemma 5, since

$$\begin{bmatrix} P & -r^t a & 0 & 0 \\ 0 & C & 0 & 0 \\ 0 & 0 & Q & -s^t a \\ 0 & 0 & 0 & C \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} P & 0 & -r^t a \\ 0 & Q & -s^t a \\ 0 & 0 & C \end{bmatrix} .$$

Finally, by Lemma 5 the expressions given below for $\overline{q}(\overline{x}+\overline{y})$ and $\overline{q}\,\overline{x}+\overline{q}\,\overline{y}$ are equal since we have the following identity.

$$\begin{bmatrix} Q & -s^t a & -s^t b \\ 0 & C & 0 \\ 0 & 0 & D \end{bmatrix} \begin{bmatrix} I & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} =$$

$$\begin{bmatrix} I & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} Q & -s^t a & 0 & 0 \\ 0 & C & 0 & 0 \\ 0 & 0 & Q & -s^t b \\ 0 & 0 & 0 & D \end{bmatrix}$$

$$\overline{q}\,\overline{x}+\overline{q}\,\overline{y} =$$

$$\left( \begin{matrix} [v \quad 0 \quad 0] \\ \\ \\ \\ \end{matrix} \begin{bmatrix} I & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} : \begin{bmatrix} Q & -s^t a & 0 & 0 \\ 0 & C & 0 & 0 \\ 0 & 0 & Q & -s^t b \\ 0 & 0 & 0 & D \end{bmatrix} : \begin{bmatrix} 0 \\ x^t \\ 0 \\ y^t \end{bmatrix} \right)$$

$$\overline{q}(\overline{x}+\overline{y}) = \left( [v \ \ 0 \ \ 0] : \begin{bmatrix} Q & -s^t a & -s^t b \\ 0 & C & 0 \\ 0 & 0 & D \end{bmatrix} : \begin{bmatrix} I & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ x^t \\ 0 \\ y^t \end{bmatrix} \right)$$

This completes the proof.

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