SOLVED PROBLEMS: SECTION 1.2

13. Check that any ring homomorphism preserves units, idempotent, and nilpotent elements.

    Solution: Let $\phi : R \to S$ be a ring homomorphism. If $a \in R$ is a unit, then there exists $a^{-1} \in R$ with $aa^{-1} = 1$ and $a^{-1}a = 1$. Then $\phi(a)\phi(a^{-1}) = \phi(aa^{-1}) = \phi(1) = 1$. Similarly, $\phi(a^{-1})\phi(a) = 1$, showing that $\phi(a^{-1}) = (\phi(a))^{-1}$. Note that I have used 1 for the identity of R and also for the identity of S. The proof requires our assumption that any ring homomorphism maps the multiplicative identity of the domain to the multiplicative identity of the codomain.

    If $e \in R$ is idempotent, the $e^2 = e$, and it follows that $(\phi(e))^2 = \phi(e^2) = \phi(e)$, so $\phi(e)$ is an idempotent element of S.

    If $a \in R$ is nilpotent, with $a^n = 0$, then it follows by an induction argument that $(\phi(a))^n = \phi(a^n)$, and therefore $(\phi(a))^n = \phi(0) = 0$, showing that $\phi(a)$ is a nilpotent element of S.

14. Show that if R is any ring, then there is a unique ring homomorphism from $\mathbf{Z}$ into R.

    Solution: Let $\phi : \mathbf{Z} \to R$ be a ring homomorphism. By definition ϕ must map $1 \in \mathbf{Z}$ to the multiplicative identity of R, so there is only the one choice for $\phi(1)$. But then the fact that ϕ preserves addition implies that $\phi(0) = 0$, and $\phi(n) = \sum_{i=1}^{n} \phi(1)$ when n is positive, while $\phi(n) = \sum_{i=1}^{|n|} -\phi(1)$ when n is negative.

15. Let R be a commutative ring, and let a be an element of R. The *annihilator* of a is $\text{Ann}(a) = \{r \in R \mid ra = 0\}$. Prove that $\text{Ann}(a)$ is an ideal of R.

    Solution: Let $A = \text{Ann}(a)$. Since $0 = 0 \cdot a$, we have $0 \in A$ and $A \neq \emptyset$. If $r, s \in A$, then $ra = 0$ and $sa = 0$, and so $(r \pm s)a = ra \pm sa = 0$. Thus $r \pm s \in A$. Let $r \in A$ and $s \in R$. Then $ra = 0$ and so $(sr)a = s(ra) = s \cdot 0 = 0$ and so $sr \in A$. Hence A is an ideal of R.

16. Show that the matrix ring $\text{M}_2(\mathbf{Q})$ is a simple ring.

    Solution: Suppose that I is a nonzero ideal in $\text{M}_2(\mathbf{Q})$, and that A is a nonzero matrix in I. Then A has at least one nonzero entry a_{ij}. Let

    $$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

    A quick computation shows that $(a_{ij})^{-1}E_{mi}AE_{jk} = E_{mk}$. Since I is a two-sided ideal, it is closed under multiplication on the left and right. Thus since I contains A, it must contain each of the matrices E_{11}, E_{12}, E_{21}, and E_{22}. Then since I is also closed under addition it is clear that any matrix $[a_{ij}]$ has the form

    $$[a_{ij}] = a_{11}E_{11} + a_{12}E_{12} + a_{21}E_{21} + a_{22}E_{22}$$

    and thus $I = \text{M}_2(\mathbf{Q})$. Therefore $\text{M}_2(\mathbf{Q})$ is a simple ring.

17. A ring homomorphism $\phi : R \to S$ is called a ring *epimorphism* if for every pair of ring homomorphisms $\theta, \psi : S \to T$, $\theta\phi = \psi\phi$ implies $\theta = \psi$. Prove that any onto ring homomorphism is a ring epimorphism. Show that the inclusion mapping $\iota : \mathbf{Z} \to \mathbf{Q}$ is a ring epimorphism.

    Solution: Let $\phi : R \to S$ be an onto ring homomorphism, and suppose that $\theta, \psi : S \to T$ are ring homomorphisms with $\theta\phi = \psi\phi$. For each $s \in S$ there exists $r \in R$ with

$s = \phi(r)$, since ϕ is onto. Thus $\theta(s) = \theta\phi(r) = \psi\phi(r) = \psi(s)$, for all $s \in S$, and so $\theta = \psi$, showing the ϕ is a ring epimorphism.

Now consider the inclusion mapping $\iota : \mathbf{Z} \to \mathbf{Q}$. Suppose that $\theta, \psi : \mathbf{Q} \to T$ are ring homomorphisms with $\theta\iota = \psi\iota$. For any nonzero element $q = ab^{-1}$, with $a, b \in \mathbf{Z}$, we have $\theta(q) = \theta(ab^{-1}) = \theta(a)\theta(b^{-1}) = \theta(a)(\theta(b))^{-1} = \theta\iota(a)(\theta\iota(b))^{-1} = \psi\iota(a)(\psi\iota(b))^{-1} = \psi(a)(\psi(b))^{-1} = \psi(a)\psi(b^{-1}) = \psi(ab^{-1}) = \psi(q)$, and this shows that ι is a ring epimorphism.

18. A ring homomorphism $\phi : R \to S$ is called a *ring monomorphism* if for every pair of ring homomorphisms $\theta, \psi : T \to R$, $\phi\theta = \phi\psi$ implies $\theta = \psi$. Prove that a ring homomorphism is a ring monomorphism if and only if it is one-to-one.

Hint: If $\phi : R \to S$ is a ring monomorphism, use the subring of $R \oplus R$ defined by

$T = \{(a, b) \mid a, b \in R \oplus R \text{ and } \phi(a) = \phi(b)\}$.

Solution: First, if $\phi : R \to S$ is one-to-one, and $\theta, \psi : T \to R$ are ring homomorphisms with $\phi\theta = \phi\psi$, then it follows immediately that $\theta(a) = \psi(a)$ for all $a \in T$, and so $\theta = \psi$.

Conversely, suppose that $\phi : R \to S$ is a ring monomorphism. Consider the subset $T = \{(a, b) \mid a, b \in R \oplus R \text{ and } \phi(a) = \phi(b)\}$ of the direct sum $R \oplus R$. This subset is closed under addition since if $\phi(a_1) = \phi(b_1)$ and $\phi(a_2) = \phi(b_2)$, then $\phi(a_1 + a_2) = \phi(a_1) + \phi(a_2) = \phi(b_1) + \phi(b_2) = \phi(b_1 + b_2)$. It is easily checked that it contains $(0, 0)$ and is closed under taking additive inverses. Furthermore, if $\phi(a_1) = \phi(b_1)$ and $\phi(a_2) = \phi(b_2)$, then $\phi(a_1a_2) = \phi(a_1)\phi(a_2) = \phi(b_1)\phi(b_2) = \phi(b_1b_2)$, and so the subset is closed under multiplication. The identity $(1, 1)$ belongs to the subset, so we conclude that T is a subring of the direct sum $R \oplus R$.

Define $\theta : T \to R$ by $\theta((a, b)) = a$, and define $\psi : T \to R$ by $\psi((a, b)) = b$. It can be checked easily that θ and ψ are ring homomorphism, and it follows from the definition of T that $\phi\theta((a, b)) = \phi(a) = \phi(b) = \phi\psi((a, b))$. Since ϕ is a ring monomorphism, we must have $\theta = \psi$. It follows that if $\phi(a) = \phi(b)$, then $a = b$, and therefore ϕ is one-to-one.

19. Let R be a commutative ring, let a be a unit of R, and let b be any element of R. Define a function $\phi : R[x] \to R[x]$ by $\phi(f(x)) = f(ax + b)$, for all $f(x) \in R[x]$. Show that ϕ is an automorphism of $R[x]$.

Solution: For simplicity, let $ax + b = p(x)$, so that $\phi(f(x)) = f(p(x))$, for all $f(x) \in R[x]$. If $\theta : R \to R$ is the identity mapping, and $\eta(x) = ax + b \in R[x]$, then ϕ is the unique ring homomorphism $\phi = \widehat{\theta}$ (defined in Example 1.2.1) for which $\phi(r) = r$ for all $r \in R$ and $\phi(x) = ax + b$. This is the easiest way to verify that ϕ is actually a ring homomorphism.

Now let $q(x) = a^{-1}x + a^{-1}b$ be the inverse function for $p(x)$ (that is, we have chosen $q(x)$ so that $p(q(x)) = x$ and $q(p(x)) = x$). Then define $\psi : R[x] \to R[x]$ by $\psi(f(x)) = f(q(x))$, for all $f(x) \in R[x]$. It is clear that ψ is the inverse of ϕ, so it must also be a ring homomorphism, and we have verified that ϕ is an automorphism.