CHAPTER 2: MODULES
Review Problems

1. Let M be a left R-module. Show that M is finitely generated if there exists a submodule $N \subseteq M$ such that N and M/N are both finitely generated.

   Solution: Suppose that the cosets $\overline{x}_1, \ldots, \overline{x}_k$ generate M/N, and that the elements $y_1, \ldots, y_n$ generate N. Then the elements $x_1, \ldots, x_k, y_1, \ldots, y_n$ generate M. To show this, let $x \in M$. Then in M/N we can write $\overline{x} = \sum_{i=1}^k a_i \overline{x}_i$ for some $a_1, \ldots, a_k \in R$, and thus $x - \sum_{i=1}^k a_i x_i \in N$. But then $x - \sum_{i=1}^k a_i x_i = \sum_{j=1}^n b_j y_j$ for some $b_1, \ldots, b_n \in R$, so $x = \sum_{i=1}^k a_i x_i + \sum_{j=1}^n b_j y_j$. Thus M is finitely generated.

2. Let I, J be ideals of the ring R. Show that R/I and R/J are isomorphic as left R-modules if and only if $I = J$.

   Solution: If $I = J$, then R/I is in fact equal to R/J, not just isomorphic to it. On the other hand, if R/I and R/J are isomorphic as left R-modules, then their annihilators in R must be equal. Since I and J are two-sided ideals, we have $I = \mathrm{Ann}_R(R/I) = \mathrm{Ann}_R(R/J) = J$.

3. Show that if $x^2 = 0$ implies $x = 0$, for all x in the ring R, then all idempotent elements of R are central.

   Solution: Let $e = e^2$ be an idempotent element of R, and let $a \in R$. Then $(ea - eae)^2 = (ea - eae)(ea - eae) = eaea - eaeae - eaeea + eaeeae = eaea - eaeae - eaea + eaeae = 0$, so $ea - eae = 0$ by assumption. Similarly, $ae - eae = 0$, and so $ea = eae = ae$ for all $a \in R$, showing that e is central.

4. Let S be a simple left R-module, and let A be a minimal left ideal of R. Show that if $A \cdot S \neq (0)$, then A and S are isomorphic as left R-modules.

   Solution: If $A \cdot S \neq (0)$, then $A \cdot x \neq (0)$ for some $x \in S$. Define $f : A \to S$ by $f(a) = ax$, for all $a \in A$. Then $f(r_1 a_1 + r_2 a_2) = (r_1 a_1 + r_2 a_2)x = r_1(a_1 x) + r_2(a_2 x) = r_1 f(a_1) + r_2 f(a_2)$ for all $r_1, r_2 \in R$ and all $a_1, a_2 \in A$, so f is an R-homomorphism. Because $f \neq 0$, we have $\ker(f) \neq A$ and $\mathrm{Im}(f) \neq (0)$. Since both A and S are simple left R-modules, this implies that $\ker(f) = (0)$ and $\mathrm{Im}(f) = S$, so f is an isomorphism.

5. Let R be a commutative ring with a unique maximal ideal I, and let M be a nonzero finitely generated R-module. Show that $\mathrm{Hom}_R(M, R/I) \neq 0$.

   Solution: In general, any simple left R-module is isomorphic to R/A, for some maximal left ideal A. (See Proposition 2.1.8 (a) and Proposition 2.1.11.) Since R is a commutative ring, and I is the only maximal ideal of R, it follows that every simple R-module is isomorphic to R/I. If M is a nonzero finitely generated R-module, then by Corollary 2.1.15 it contains a maximal submodule, say N, and so M/N is a simple module. We can conclude that there exists an isomorphism $f : M/N \to R/I$. If $\pi : M \to M/N$ is the natural projection, it follows that $f\pi$ is a nonzero element of $\mathrm{Hom}_R(M, R/I)$.

Alternate solution: After studying the Jacobson radical in Chapter 3, the following proof can be given. Since I is the unique maximal ideal of R, it is the Jacobson radical of R. Nakayama's lemma implies that $IM \neq M$, since M is finitely generated and nonzero, so M/IM is a nonzero R/I-module. In fact it is a vector space over the field R/I, so there is a nonzero linear transformation $f : M/IM \to R/I$. If $\pi : M \to M/IM$ is the natural projection, it follows that $f\pi$ is a nonzero element of $\mathrm{Hom}_R(M, R/I)$.

6. Let R be a ring, and let M be a left R-module with submodules N and K. Show that if N and K are Artinian, then so is $N + K$.

   Solution: The inclusion mappings $N \to M$ and $K \to M$ combine to give an onto mapping $f : N \oplus K \to N + K$. (See Proposition 2.2.4 (b), which describes homomorphisms out of a direct sum.) Since N and K are Artinian, so is $N \oplus K$, and then $N + K$ is Artinian since it is isomorphic to a factor module of $N \oplus K$. (See Proposition 2.4.5 and Corollary 2.4.6.)

7. Compute the socle of the $\mathbf{Z}$-module $\mathbf{Z}_n$.

   Solution: If n has the prime factorization $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$, where $p_1 < \ldots < p_m$ are primes, then $\mathbf{Z}_n \cong \mathbf{Z}_{p_1^{\alpha_1}} \oplus \mathbf{Z}_{p_2^{\alpha_2}} \oplus \cdots \oplus \mathbf{Z}_{p_m^{\alpha_m}}$. In each component $\mathbf{Z}_{p_k^{\alpha_k}}$ the submodules form a chain, with a minimal submodule $p_k^{\alpha_k}\mathbf{Z}_{p_k^{\alpha_k}}$ isomorphic to $\mathbf{Z}_{p_k}$. Since the socle of a direct sum is the direct sum of the socles of each summand,

$$\begin{aligned} \mathrm{Soc}(\mathbf{Z}_n) &\cong p_1^{\alpha_1-1}\mathbf{Z}_{p_1^{\alpha_1}} \oplus p_2^{\alpha_2-1}\mathbf{Z}_{p_2^{\alpha_2}} \oplus \cdots \oplus p_m^{\alpha_m-1}\mathbf{Z}_{p_m^{\alpha_m}} \\ &\cong \mathbf{Z}_{p_1} \oplus \mathbf{Z}_{p_2} \oplus \cdots \oplus \mathbf{Z}_{p_m} . \end{aligned}$$

   This completes the computation.

8. Let R be a ring, and let M be a left R-module that has a minimal submodule S such that $M/S \cong S$. Prove that either S is a direct summand of M, in which case $M \cong S \oplus S$, or else S is the only proper nontrivial submodule of M.

   Solution: Suppose that K is a proper nontrivial submodule of M. If $S \cap K \neq 0$, then $S = S \cap K$ since S is minimal, and so $S \subseteq K$. But since $M/S \cong S$, the submodule S is maximal as well as minimal, and so $S = K$. Thus the second case holds.

   Now suppose that $S \cap K = 0$. Then $S + K$ properly contains S, so we must have $S + K = M$, since S is a maximal submodule. This implies that $M \cong S \oplus K$, and so S is a direct summand of M. The inclusion mapping $i : K \to M$ followed by the projection $\pi : M \to M/S$ defines an isomorphism from K onto S, and so $M \cong S \oplus S$. Thus the first case holds.

9. Let A and B be finitely generated abelian groups. Prove that if $A \oplus A \cong B \oplus B$, then $A \cong B$.

   Solution: Using the fundamental structure theorem (Theorem 2.7.10) for finitely generated modules over a principal ideal domain, we can write

$$A \cong \mathbf{Z}^m \oplus \mathbf{Z}_{p_1^{\alpha_1}} \oplus \cdots \oplus \mathbf{Z}_{p_s^{\alpha_s}} \quad \text{and} \quad B \cong \mathbf{Z}^n \oplus \mathbf{Z}_{q_1^{\alpha_1}} \oplus \cdots \oplus \mathbf{Z}_{q_t^{\alpha_t}} ,$$

for nonnegative integers m, n and primes $p_1, \ldots, p_s$ and $q_1, \ldots, q_t$. If $A \oplus A \cong B \oplus B$, then the uniqueness in the fundamental structure theorem implies that $2m = 2n$, so $m = n$. (If either one is zero, then the other is too.) Similarly, there is a one-to-one correspondence between the primes, so $s = t$, and $p_i^{\alpha_i} = q_i^{\beta_i}$ for $1 \le i \le t$.

10. Let M be a finitely generated projective module over a principal ideal domain D. Prove that M is a free D-module.

Solution: Using the fundamental structure theorem (Theorem 2.7.10) for finitely generated modules over a principal ideal domain, we can write

$$M \cong D^m \oplus (D/p_1^{\alpha_1}D) \oplus (D/p_2^{\alpha_2}D) \oplus \cdots \oplus (D/p_k^{\alpha_k}D) ,$$

for some positive integer m and some irreducible elements $p_1, \ldots, p_k$. If M is projective, then each of its summand must also be projective, so this forces the summands of the form $D/p_i^{\alpha_i}D$ to be projective. But then the natural projection $\pi : D \to D/p_i^{\alpha_i}D$ would split, leading to a decomposition of D as a direct sum of two nonzero ideals. This is impossible since D is an integral domain.

11. Let R be a commutative ring, and let M and N be R-modules. Show that $M \otimes_R N$ is isomorphic to $N \otimes_R M$.

Solution: We will assume that the statement implies that M and N are both left and right R-modules, with $rx = xr$ for all $r \in R$ and all $x \in M$, $x \in N$. Let $\tau : M \times N \to M \otimes_R N$ and $\sigma : N \times M \to N \otimes_R M$ be the R-bilinear mappings associated with the respective tensor products. Define $\beta : M \times N \to N \otimes_R M$ by $\beta(x, y) = y \otimes x$, for all $x \in M$ and $y \in N$. Then $\beta(x_1 + x_2, y) = y \otimes (x_1 + x_2) = y \otimes x_1 + y \otimes x_2 = \beta(x_1, y) + \beta(x_2, y)$, for all $x_1, x_2 \in M$ and $y \in N$. A similar argument shows that $\beta(x, y_1 + y_2) = \beta(x, y_1) + \beta(x, y_2)$, for all $x \in M$ and $y_1, y_2 \in N$. Furthermore, $\beta(xr, y) = \beta(rx, y) = y \otimes rx = yr \otimes x = \beta(x, yr) = \beta(x, ry)$ for all $r \in R$, $x \in M$, and $y \in N$. This shows that β is an R-bilinear mapping, and so there exists a unique $\mathbf{Z}$-homomorphism $f : M \otimes_R N \to N \otimes_R M$ such that $f\tau = \beta$.

Similarly, we can define an R-bilinear mapping $\gamma : N \times M \to M \otimes_R N$ by $\gamma(y, x) = x \otimes y$, for all $x \in M$, $y \in N$. Again, there exists a unique $\mathbf{Z}$-homomorphism $g : N \otimes_R M \to M \otimes_R N$ such that $g\sigma = \gamma$. For each element $y \otimes x = \sigma(y, x)$ in $N \otimes_R M$, we have $g(y \otimes x) = g\sigma(y, x) = \gamma(y, x) = x \otimes y$. It follows that $g\beta(x, y) = x \otimes y$, for all $x \in M$ and $y \in N$, and so it is easy to check that $g\beta$ is an R-bilinear mapping. The identity mapping $i : M \otimes_R N \to M \otimes_R N$ satisfies the condition $i\,\tau = g\beta$. But we also have $gf\tau = g\beta$, and so uniqueness forces $gf = i$. Similarly, we can show that fg is the identity mapping, and this establishes the desired isomorphism.

12. Let A be a nonzero injective $\mathbf{Z}$-module. Prove that A cannot be finitely generated.

Solution: Suppose that A is finitely generated. Then Corollary 2.1.15 implies that there exists a maximal submodule $B \subset A$. Thus A/B is a simple abelian group, so it is isomorphic to $\mathbf{Z}_p$ for some prime p. Exercise 2.3.11 in the text states that over a principal ideal domain a module is injective if and only if divisible. Exercise 2.3.10 shows that any homomorphic image of a divisible module is divisible. We conclude that $\mathbf{Z}_p$ must be divisible. This is an obvious contradiction since the equation $px \equiv 1 \pmod{p}$ has no solution.

Alternate solution: It is possible to use the fundamental structure theorem for modules over a principal ideal domain to write A as a direct sum of cyclic groups. This leads to a contradiction, since each summand must be divisible, but it is easy to show that no finite abelian group can be divisible, and no abelian group isomorphic to $\mathbf{Z}$ can be divisible.