SOLVED PROBLEMS: SECTION 2.1

13. Let M be a left R-module, and let $M_1 \subseteq M_2 \subseteq \ldots \subseteq M$ be an ascending chain of submodules of M. Prove that $\bigcup_{i=1}^{\infty} M_i$ is a submodule of M.

    Solution: It is clear that $0 \in \bigcup_{i=1}^{\infty} M_i$, and so the union is nonempty. Let $x, y \in \bigcup_{i=1}^{\infty} M_i$, and let $r \in R$. Then there exist positive integers n, k such that $x \in M_n$ and $y \in M_k$. If $n \leq k$, then $x, y \in M_k$, so $x + y \in M_k$ and $rx \in M_k$, and therefore $x + y$ and rx belong to $\bigcup_{i=1}^{\infty} M_i$. This shows that the union is a submodule of M.

14. Let ${}_RM$ be a left R-module, with submodules N and K. Show that if $N \cup K$ is a submodule of M, then either $N \subseteq K$ or $K \subseteq N$.

    Solution: If $N \subseteq K$ we are done. If not, there exists $x \in N$ with $x \notin K$. We will show that $K \subseteq N$. Let $y \in K$. Since $N \cup K$ is a submodule, either $x + y \in K$ or $x + y \in N$. The first cannot happen, since $x + y \in K$ implies $x = (x + y) - y \in K$, a contradiction. Therefore we must have $x + y \in N$, so $y = (x + y) - x \in N$.

15. Let R be a commutative ring, and let X be a subset of R that contains 1 and is closed under products. Show that if I is any ideal of R with $I \cap X = \emptyset$, then there exists a prime ideal P of R with $I \subseteq P$ and $P \cap X = \emptyset$.

    Solution: Let P be the ideal of R whose existence is guaranteed by Lemma 2.1.13. Using Proposition 1.3.2, to show that P is a prime ideal it suffices to show that if A and B are ideals that properly contain P, then AB is not contained in P. If A and B properly contain P, then by the construction of P there exist $x_1, x_2 \in X$ with $x_1 \in A$ and $x_2 \in B$. Since X is closed under multiplication, $x_1 x_2 \in X$, and so $x_1 x_2 \notin P$, and therefore AB is not contained in P.

16. An module homomorphism $f : M \to N$ is called a *monomorphism* if it satisfies the following condition: if $g, h : X \to M$ are homomorphisms with $fg = fh$, then $g = h$. Prove that f is a monomorphism if and only if it is one-to-one.

    Solution: If f is one-to-one, it is clear that it is a monomorphism. Conversely, if f is a monomorphism, choose $X = \ker(f)$, let $g : \ker(f) \to M$ be the inclusion mapping, and let h be the zero mapping. Then $fg = 0 = fh$, so the assumption that f is a monomorphism forces $g = 0$, showing that $\ker(f) = (0)$, and hence f is one-to-one.

17. An module homomorphism $f : M \to N$ is called an *epimorphism* if it satisfies the following condition: if $g, h : N \to Y$ are homomorphisms with $gf = hf$, then $g = h$. Prove that f is an epimorphism if and only if it is onto.

    Solution: If f is onto, it is clear that it is an epimorphism. Conversely, if f is an epimorphism, choose $Y = N/\operatorname{Im}(f)$, let $g : N \to Y$ be the natural projection, and let h be the zero mapping. Then $gf = 0 = hf$, so the assumption that f is an epimorphism forces $g = 0$, showing that $\operatorname{Im}(f) = N$, and hence f is onto.

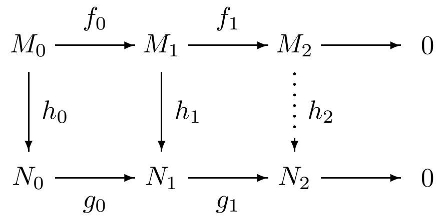
18. Show that if p, q are distinct prime numbers, then there exists a short exact sequence

    $$0 \longrightarrow \mathbf{Z}_p \xrightarrow{\ f\ } \mathbf{Z}_{pq} \xrightarrow{\ g\ } \mathbf{Z}_q \longrightarrow 0$$

    of $\mathbf{Z}$-modules.

    Solution: Let $f : \mathbf{Z}_p \to \mathbf{Z}_{pq}$ be defined by $f([x]_p) = [qx]_{pq}$, for all $x \in \mathbf{Z}$. This is well-defined since if $x \equiv y \pmod{p}$, then $p \mid (x - y)$, and hence $pq \mid (qx - qy)$, showing that $f([x]_p) = f([y]_p)$. It is clear that f is one-to-one. Let $g : \mathbf{Z}_{pq} \to \mathbf{Z}_q$ be the natural projection, which is certainly onto. Then $\ker(g)$ is the set of multiples of q in $\mathbf{Z}_{pq}$, which is precisely the image of f.

19. In the following diagram, assume that the first square is a commutative diagram, and that both rows form exact sequences. Prove that there is a unique R-homomorphism $h_2 : M_2 \to N_2$ such that $h_2 f_1 = g_1 h_1$ (making the second square commutative).



Solution: We have $g_1 h_1 f_0 = g_1 g_0 h_0 = 0$, using commutativity of the diagram and the fact that $g_1 g_0 = 0$ because the bottom row is exact. This shows that $\ker(f_1) = \mathrm{Im}(f_0) \subseteq \ker(g_1 h_1)$. We define $h_2 : M_2 \to N_2$ as follows: given $x \in M_2$ there exists $x_1 \in M_1$ with $x = f_1(x_1)$, since f_1 is onto, and so we can define $h_2(x) = g_1 h_1(x_2)$. The mapping h_2 is well-defined since if $x = f_1(x_1')$, then $x_1 - x_1' \in \ker(f_1)$, so $x_1 - x_1' \in \ker(g_1 h_1)$, and therefore $h_2(x_2) = h_2(x_2')$. It is easy to check that h_2 is an R-homomorphism. Finally, uniqueness follows immediately from the fact that f_1 is an epimorphism (see Problem 17).

20. Let I be an ideal of the ring R such that $I^n = (0)$, and let M, N be left R-modules with an R-homomorphism $f : M \to N$.

(a) Prove that f induces an R-homomorphism $f' : M/IM \to N/IN$.

Solution: If $x \in IM$, then $x = \sum_{i=1}^n a_i m_i$, for elements $a_i \in I$ and $m_i \in M$. It follows that $f(x) = \sum_{i=1}^n a_i f(m_i)$, and so $f(x) \in IN$. We define $f' : M/IM \to N/IN$ by $f'(x + IM) = f(x) + IN$. This is a well-defined function since if $x_1 + IM = x_2 + IM$, then $x_1 - x_2 \in IM$, and so $f(x_1 - x_2) \in IN$, which shows that $f(x_1) + IN = f(x_2) + IN$. It is then easy to check that f' is an R-homomorphism.

(b) Prove that if f' is onto, then f is onto.

Solution: It follows, as in part (a), that $f(I^2 M) \subseteq I^2 N$, so there is also an induced R-homomorphism $f'' : M/I^2 M \to N/I^2 N$. We claim that if f' is onto, then so is f''. If $y \in N$, then since f' is onto, there exists $x \in M$ with $y + IM = f(x) + IN$. Then there exist $a_1, \ldots, a_n \in I$ and $y_1, \ldots, y_n \in N$ such that $y = f(x) + \sum_{i=1}^n a_i y_i$. We can then do the same thing for each of the elements y_i, so there exist $x_1, \ldots, x_n \in M$ with $y_i = f(x_i) + z_i$, where $z_i \in IN$. It follows that $a_i y_i = f(a_i x_i) + a_i z_i$, and then a substitution gives us $y = f(x + \sum_{i=1}^n a_i x_i) + z$, where $z \in I^2 N$. Continuing this argument inductively, we see that $f = f^{(n)}$ maps $M = M/I^n M$ onto $N = N/I^n N$.