CHAPTER 1: RINGS

Review Problems

1. Let I be an ideal of the commutative ring R. Prove that I is a prime ideal iff I is the kernel of a ring homomorphism from R into a field.

2. Let R be a commutative ring that is not a field, and let P be a maximal ideal of R. Let $I = P[x]$ be the ideal of the polynomial ring $R[x]$ consisting of all polynomials in $R[x]$ with coefficients in P. Show that I is a prime ideal that is not maximal.

3. Let D be a principal ideal domain, and let P be a prime ideal of D. Prove that D/P is a principal ideal domain.

4. Let R be a commutative ring with $1 \neq 0$. Prove that if every proper ideal of R is prime, then R is a field.

5. Let F and K be fields. Prove that if $F[x] \cong K[x]$, then $F \cong K$.

6. Let F be a field. Show that in the factor ring $F[x]/(x^n)$ an element $f(x) + (x^n)$ is invertible iff $f(0) \neq 0$.

7. Let $F = \mathbf{Z}_2$ be the field with two elements, and let R be the factor ring $F[x]/(x^2 + 1)$. Show that R has four elements, but that it is not isomorphic (as a ring) to either $\mathbf{Z}_4$ or $\mathbf{Z}_2 \oplus \mathbf{Z}_2$.

8. Let $F = \mathbf{Z}_2$ be the field with two elements. Show that the ring $R = F[x]/(x^3 + x)$ has exactly four proper nontrivial ideals.

9. Show that if R is a division ring, then for any $a \in R$ the centralizer $C(a)$ is a division ring.

10. Let D be an integral domain for which $IJ = I \cap J$ for all ideals I, J of D. Prove that D is a field.