SOLVED PROBLEMS: SECTION 1.3

13. Let P be a prime ideal of the commutative ring R. Prove that if P is a prime ideal of R, then $A \cap B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$, for all ideals A, B of R. Give an example to show that the converse is false.

14. Show that in the polynomial ring $\mathbf{Z}[x]$, the ideal (n, x) generated by $n \in \mathbf{Z}$ and x is a prime ideal if and only if n is a prime number.

15. Let R be a Boolean ring (see Exercise 1.1.11 in the text) and let P be a prime ideal of R. Prove that P is maximal, and that $R/P \cong \mathbf{Z}_2$.

16. Let R be a commutative ring. Then R is called a *local ring* if it has a proper ideal P such that $P \supseteq I$, for all proper ideals I of R. Prove that the following conditions are equivalent for R.

    (1) R is a local ring;

    (2) the set of nonunits of R forms an ideal;

    (3) there exists a maximal ideal P of R such that $1 + x$ is a unit, for all $x \in P$.

17. Prove that any nonzero homomorphic image of a local ring is again a local ring.

18. Show that the ring R defined in Exercise 1.2.9 of the text is a local ring.