

SOLVED PROBLEMS: SECTION 1.4

9. Let D be an integral domain. Prove that if the polynomial ring $D[x]$ is a principal ideal domain, then D is a field.
10. Let $f(x) = a_mx^m + \dots + a_1x + a_0$, $g(x) = b_nx^n + \dots + b_1x + b_0$, and $h(x) = c_kx^k + \dots + c_1x + c_0$ be polynomials in $\mathbf{Z}[x]$, with $f(x) = g(x)h(x)$. Let p be a prime number. Show that if b_s and c_t are the coefficients of $g(x)$ and $h(x)$ of least index not divisible by p , then a_{s+t} is the coefficient of $f(x)$ of least index not divisible by p .
11. Prove Eisenstein's irreducibility criterion, which states that if $f(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_0 \in \mathbf{Z}[x]$ and there exists a prime number p such that $a_{n-1} \equiv a_{n-2} \equiv \dots \equiv a_0 \equiv 0 \pmod{p}$ but $a_n \not\equiv 0 \pmod{p}$ and $a_0 \not\equiv 0 \pmod{p^2}$, then $f(x)$ is irreducible over \mathbf{Q} .
or, equivalently, iff $p \equiv 1 \pmod{4}$.