Skew Polynomial Rings

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Example [Goodearl/Warfield] p. xxi

Let k be a field, and let $\mathrm{SL}_2(k)$ be the special linear group of 2×2 matrices with determinant 1. Consider the full matrix ring $M_2(k)$ as a 4-dimensional vector space with basis $X_{11}, X_{12}, X_{21}, X_{22}$, and embed $\mathrm{SL}_2(k)$ in it as the set of zeros of the polynomial

$$X_{11}X_{22} - X_{12}X_{21} - 1 .$$

Viewed in this way, $\mathrm{SL}_2(k)$ is an affine algebraic variety over k, and its coordinate ring is the algebra

$$\mathcal{O}(\mathrm{SL}_2(k)) = k[X_{11}, X_{12}, X_{21}, X_{22}] \;/\; \langle X_{11}X_{22} - X_{12}X_{21} - 1 \rangle .$$

Comments: $\mathcal{O}(M_2(k))$ is just the polynomial ring in 4 indeterminates. An affine algebraic variety is a subset of k^n that is the set of zeros of some set $\{f(x)_1, \ldots, f(x)_m\} \subseteq k[x_1, \ldots, x_n]$.

Comultiplication

The coordinate ring encodes the geometry of the set of points in $\mathrm{SL}_2(k)$. The group structure of $\mathrm{SL}_2(k)$ is encoded in certain algebra homomorphisms. The multiplication in $\mathrm{SL}_2(k)$ is a mapping $\mu : \mathrm{SL}_2(k) \times \mathrm{SL}_2(k) \to \mathrm{SL}_2(k)$, which can also be viewed as $\mu : \mathrm{SL}_2(k) \otimes \mathrm{SL}_2(k) \to \mathrm{SL}_2(k)$.

Given an element of $\mathcal{O}(\mathrm{SL}_2(k))$, it can act on $\mathrm{SL}_2(k) \times \mathrm{SL}_2(k)$ by first multiplying two elements and then substituting. This induces a mapping

$$\mathcal{O}(\mathrm{SL}_2(k)) \to \mathcal{O}(\mathrm{SL}_2(k) \times \mathrm{SL}_2(k)) \cong \mathcal{O}(\mathrm{SL}_2(k)) \otimes \mathcal{O}(\mathrm{SL}_2(k))$$

that is called a *comultiplication*. There are two more mappings, and various commutative diagrams, that give $\mathcal{O}(\mathrm{SL}_2(k))$ the structure of a *Hopf algebra*.

Quantum groups

Quote from Brown/Goodearl on *Quantum Groups*, p. vii:
"This label is attached to a large and rapidly diversifying field of mathematics and mathematical physics, originally launched by developments around 1980 in theoretical physics and statistical mechanics. ...
The algebras we study here are of two main types. On the one hand there are 'quantized coordinate rings', a term used in the literature to refer to various noncommutative algebras which are, informally expressed, deformations of the classical coordinate rings of algebraic groups or related aglebraic varieties; the adjective 'quantized' usually indicates that some solution to the quantum Yang-Baxter equation is involved in the construction and/or the representation theory of the algebra.

continued quote from Brown/Goodearl

Unfortunately, to date no axiomatic definition of this family of algebras has been given, nor a complete formulation of properties an algebra should satisfy in order to qualify as a quantum analogue of a given classical coordinate ring. This lack of defining axioms and characteristics should, in our opinion, be highlighted as a major open problem in the area. ... The second broad class consists of 'quantized enveloping algebras'– as their name suggests, these are certain deformations of the universal enveloping algebras of semisimple Lie algebras" ...

Quote from Goodearl/Warfield

Quote from Goodearl/Warfield, p. xxi:
"A phrase that captures the philosophical viewpoint of the subject is this: *Quantum groups are algebras of functions on nonexistent groups*! ... Among the algebras that arose in the theoretical physics research mentioned above was one that bears a striking resemblence to $\mathcal{O}(\mathrm{SL}_2(k))$– it is a Hopf algebra, and it has four generators that satisfy a relation very similar to the equation "determinant $= 1$" which characterizes $\mathrm{SL}_2(k)$. The only drawback is that this new algebra is not commutative, and so it cannot be an algebra of k-valued functions on anything. ... First, pick a nonzero scalar q (the 'quantum parameter') in k. (Originally, q was taken to be $e^{\hbar}$, where $\hbar$ is Planck's constant, so that q was a real number very close to 1.) Next, one forms a k-algebra with four generators ... and six relations. "

Coming soon!

$\mathcal{O}_q(k^2)$

Reference: Goodearl/Warfield p. 12

Definition

Let k be a field and let $q \in k^\times$. The **coordinate ring of the quantum plane** is

$$\mathcal{O}_q(k^2) = k\langle x, y \mid xy = qyx \rangle .$$

Goodearl makes the point that this is the *free algebra* $k\langle X, Y \rangle$ on two letters X and Y (which satisfy no relations at all) modulo the ideal $\langle XY - qYX \rangle$ generated by the element $XY - qYX$. Given any k-algebra T with $u, v \in T$ such that $uv = qvu$, there is a unique k-algebra homomorphism $\phi : k\langle X, Y \rangle / \langle XY - qYX \rangle \to T$ such that $\phi(X) = u$ and $\phi(Y) = v$. This can be used to prove the following result.

$\mathcal{O}_q(k^2)$ is a skew polynomial ring

Theorem

Let k be a field and let $q \in k^\times$. Then $\mathcal{O}_q(k^2) = k[y][x; \tau]$, where τ is the automorphism of the polynomial ring $k[y]$ defined by $\tau(f(y)) = f(qy)$, for all $f(y) \in k[y]$.

Note that τ is an automorphism since adding then substituting is the same as substituting then adding, and the same holds for multiplying. The mapping is one-to-one and onto.
Goodearl/Warfield gives the following formula

$$\left(\sum_{i,j} \alpha_{ij} y^i x^j\right)\left(\sum_{s,t} \beta_{st} y^s x^t\right) = \sum_{i,j,s,t} \alpha_{ij}\beta_{st} y^{i+s} x^{j+t}$$

$$= \sum_{l,m}\left(\sum_{i+s=l;\, j+t=m} \alpha_{ij}\beta_{st} q^{js}\right) y^l x^m$$

$\mathcal{O}_q(M_2(k))$

Definition

Given $q \in k^\times$, the **quantized coordinate ring of** $M_2(k)$ is the k-algebra $\mathcal{O}_q(M_2(k))$ presented by generators $x_{11}, x_{12}, x_{21}, x_{22}$ and the relations

$$x_{11}x_{12} = qx_{12}x_{11}, \quad x_{11}x_{21} = qx_{21}x_{11}, \quad x_{12}x_{21} = x_{21}x_{12},$$
$$x_{12}x_{22} = qx_{22}x_{12}, \quad x_{21}x_{22} = qx_{22}x_{21},$$
$$x_{11}x_{22} - x_{22}x_{11} = (q - q^{-1})x_{12}x_{21}.$$

Theorem

$\mathcal{O}_q(M_2(k))$ can be expressed as an iterated skew polynomial ring of the form $k[x_{11}][x_{12}; \tau_{12}][x_{21}; \tau_{21}][x_{22}; \tau_{22}, \delta_{22}]$.

For example, since $x_{12}x_{11} = q^{-1}x_{11}x_{12}$, we define $\tau_{12} : k[x_{11}] \to k[x_{11}]$ by setting $\tau_{12}(f(x_{11})) = f(q^{-1}x_{11})$. We need δ_{22} because $x_{22}x_{11} = x_{11}x_{22} + (q^{-1} - q)x_{12}x_{21}$.

$\mathcal{O}_q(\mathrm{SL}_2(k))$

We saw that $\mathcal{O}(\mathrm{SL}_2(k)) = \mathcal{O}(M_2(k))/\langle x_{11}x_{22} - x_{121}x_{21} - 1\rangle$. In $\mathcal{O}_q(M_2(k))$ the element $x_{11}x_{22} - x_{12}x_{21}$ is replaced by the *quantum determinant* $D_q = x_{11}x_{22} - qx_{12}x_{21}$, which lies in the center of $\mathcal{O}_q(M_2(k))$.

Definition

Given $q \in k^\times$, the **quantized coordinate ring** $\mathcal{O}_q(\mathrm{SL}_2(k))$ of $\mathrm{SL}_2(k)$ is the factor algebra $\mathcal{O}_q(M_2(k))/\langle D_q - 1\rangle$.

The following can be proved by using the fact that $\mathcal{O}_q(M_2(k))$ is an iterated skew polynomial ring.

Theorem

$\mathcal{O}_q(\mathrm{SL}_2(k))$ *is a Noetherian domain.*

Existence of skew polynomial rings

Theorem

Let R be a ring, and let δ be a derivation of R. The set $R[x;\delta]$ of skew polynomials over R is a ring.

Proof: Let t be an indeterminate, and consider the endomorphism ring $E = \mathrm{End}_Z(R[t])$ of $R[t]$ as an abelian group. The plan is to show that there is an additive isomorphism between the additive group of $R[x;\delta]$ and a subring of E.

The multiplication on the subring can be transferred to $R[x;\delta]$, so that we can avoid many of the details necessary to prove directly that $R[x;\delta]$ is a ring. For any element $a \in R$, define $\lambda_a : R[t] \to R[t]$ by $\lambda_a(f(t)) = af(t)$. Since λ_a defines an element of the endomorphism ring E, the mapping $\phi : R \to E$ defined by $\phi(a) = \lambda_a$ is a one-to-one ring homomorphism. We will identify R with the subring $\phi(R)$ of E.

We can extend δ to a derivation of $R[t]$, by defining

$$\delta(a_0 + a_1 t + \ldots + a_n t^n) = \delta(a_0) + \delta(a_1)t + \ldots + \delta(a_n)t^n \ .$$

It can be checked that $\delta(f(t)g(t)) = f(t)\delta(g(t)) + \delta(f(t))g(t)$, since $\delta(a+b) = \delta(a) + \delta(b)$ and $\delta(ab) = \delta(a)b + a\delta(b)$, for all $a, b \in R$.

We next define an element of E, which we denote by z, by letting $z(f(t)) = f(t)t + \delta(f(t))$. For any $a \in R$, we have

$$\begin{aligned}
[za](f(t)) &= z(af(t)) \ = \ (af(t))t + \delta(af(t)) \\
&= (af(t))t + \delta(af(t)) \ = \ a(f(t)t) + \delta(a)f(t) + a\delta(f(t)) \\
&= a(f(t)t + \delta(f(t))) + \delta(a)f(t) \ = \ a(zf(t)) + \delta(a)f(t) \ .
\end{aligned}$$

Thus, as functions in E, the actions of za and $az + \delta(a)$ are equal.

Proof continued

Let S be the subset $\sum_{i=0}^{\infty} Rz^i$ of E. We claim that S is a subring E isomorphic to $R[x;\delta]$. It is clear that S is a subgroup under addition, and from the relation $zR \subseteq Rz + R$ it follows by induction that $z^i R \subseteq Rz^i + Rz^{i-1} + \ldots Rz + R$, for all i. This implies, in turn, that S is closed under multiplication. Now define $\Phi : R[x;\delta] \to E$ by letting

$$\Phi(a_0 + \ldots + a_n x^n) = \phi(a_0) + \ldots + \phi(a_n) z^n ,$$

for all $a_0 + \ldots + a_n x^n \in R[x;\delta]$. Because of the identification we have made, we can write this as

$$\Phi(a_0 + \ldots + a_n x^n) = a_0 + \ldots + a_n z^n .$$

If $\Phi(a_0 + \ldots + a_n x^n) = 0$, then the endomorphism $a_0 + \ldots + a_n z^n$ must be identically zero in E.

Proof continued

The action of z on the constant polynomial 1 is given by $z(1) = 1 \cdot t + \delta(1) = t$, since $\delta(1) = 0$. Thus the action of $a_0 + \ldots + a_n z^n$ on 1 yields $a_0 + \ldots + a_n t^n$, and if this is zero, then $a_0 = \ldots = a_n = 0$. This shows that Φ is one-to-one, and since the image of Φ is S, we can use Φ^{-1} to transfer the multiplication of S back to $R[x;\delta]$. Finally, we have $\Phi(x) = z$, and since $za = az + \delta(a)$ in S, the corresponding identity holds in $R[x;\delta]$. $\square$

The next step will be to show that if R is a left Noetherian ring and τ is an automorphism, then the skew polynomial ring $R[x;\tau,\delta]$ is left Noetherian.