

# Homework solutions, Stewart, 5th edition

## Section 7.5

#3. (a)  $\pi/3$  (b)  $-\pi/4$

#16. Graphs of  $y = \tan x$  ( $-\pi/2 < x < \pi/2$ ) and  $y = \arctan(x)$  are symmetric about the line  $y = x$ .

#26.  $f'(x) = \ln(\arctan x) + \frac{x}{(1+x^2)\arctan x}$ .

#29.  $y' = \frac{-2e^{2x}}{\sqrt{1-e^{4x}}}$ .

#30.  $y' = \frac{-x}{\sqrt{1-x^2}} + \arccos x + \frac{x}{\sqrt{1-x^2}} = \arccos x$ .

#45.  $\lim_{x \rightarrow \infty} e^x = \infty$ , so  $\lim_{x \rightarrow \infty} \arctan(e^x) = \pi/2$ .

#50. Done in class.

#66.  $u = \arctan(x)$  gives  $(\arctan x)^2/2 + C$ .

#72.  $V = \pi \int_0^2 \frac{1}{x^2+4} dx = \frac{\pi}{2} \arctan(x/2) \Big|_0^2 = \pi^2/8$ .

## Section 7.7

#6. Cancel  $x+2$  in the fraction, or use l'Hôpital to get  $-1$ .

#9. Done in class,  $-\infty$ .

#15.  $\lim_{x \rightarrow \infty} 1/x = 0$ .

#26.  $\lim_{x \rightarrow 0} \frac{\cos x - 1}{3x^2} = \lim_{x \rightarrow 0} \frac{-\sin x}{6x} = -1/6$  (I made a mistake here at first).

#28.  $\lim_{x \rightarrow \infty} 2/x = 0$ , so  $\lim_{x \rightarrow \infty} \frac{2 \ln x}{x} = 0$ .

#41. Write as  $\lim_{x \rightarrow \infty} \frac{x^3}{e^{x^2}}$  and apply l'Hôpital's rule twice to get  $\lim_{x \rightarrow \infty} \frac{3}{4xe^{x^2}} = 0$ .

#48. Write as  $\lim_{x \rightarrow 1} \frac{x-1-\ln x}{(x-1)\ln x}$ , then use l'Hôpital's rule twice (simplify as you go!) to get

$$\lim_{x \rightarrow 1} \frac{1}{2 + \ln x} = 1/2.$$

#60. Done in class;  $\lim_{x \rightarrow 0} \frac{5 \ln(\cos 3x)}{x} = 0$ , so the answer is  $e^0 = 1$ .

#88. For any  $p > 0$  we have an indeterminate form  $\infty/\infty$ , so l'Hôpital's rule applies. Since  $\lim_{x \rightarrow \infty} \frac{1}{xp x^{p-1}} = \lim_{x \rightarrow \infty} \frac{1}{px^p}$  is 0 for any positive  $p$ , the original limit was 0.

## Section 8.1

#3. Letting  $v = \frac{1}{5} \sin(5x)$  I get  $\frac{1}{5}x \sin(5x) + \frac{1}{25} \cos(5x) + C$ .

#8. Parts twice, each time expressing the trig function as a derivative  $v'$ : first write  $\cos mx = \left(\frac{1}{m} \sin mx\right)'$ , then  $\sin mx = \left(-\frac{1}{m} \cos mx\right)'$ . You should get something like

$$\frac{1}{m}x^2 \sin mx + \frac{2}{m^2}x \cos mx - \frac{2}{m^3} \sin mx + C.$$

#14. Parts three times, each time writing  $e^t = (e^t)'$ ; should be  $t^3e^t - 3t^2e^t + 6te^t - 6e^t + C$ .

#25. Use  $v' = (x)'$  and then substitution  $u = 1 - x^2$  to get  $x \arccos x - \sqrt{1 - x^2}$ . Evaluates to  $\pi/6 - \sqrt{3}/2 + 1$ .

#26. Write  $5^x = (5^x/\ln 5)'$ . I'm getting  $5/\ln 5 - 4/(\ln 5)^2$ .

#36.  $u = x^2$  to get  $\frac{1}{2} \int u^2 e^u du$ , then do just like #14.

#52.  $5 \ln x = x \ln x$  when  $x = 1$  and  $x = 5$ , and in that region  $5 \ln x \geq x \ln x$ , so we need  $\int_1^5 5 \ln x - x \ln x dx$ , which - if I'm right - is  $5(x \ln x - x) - \frac{1}{2}(x^2 \ln x - \frac{1}{2}x^2)$  evaluated between 1 and 5. Simplified that works out to be  $12.5 \ln 5 - 14 \simeq 6.118$ .

#55.  $2\pi \int_0^1 x \cos(\pi x/2) dx = 4 \left( x \sin(\pi x/2) + \frac{2}{\pi} \cos(\pi x/2) \right) \Big|_0^1$ . I hope that gives the same answer as the book.

## Section 8.2

#2. Separate one copy of the cosine and use  $u = \sin x$  to get  $\int u^6(1 - u^2) dx$

#9. Parts or double-angle formula twice:  $\sin^4(3t) = \left(\frac{1}{2}(1 - \cos(6t))\right)^2 = \frac{1}{4} - \frac{1}{2} \cos(6t) + \frac{1}{8}(1 + \cos(12t))$ , which is easy to integrate.

#24. Rewrite  $\tan^4(x) = \tan^2(x)(\sec^2(x) - 1) = \tan^2(x) \sec^2(x) - \tan^2(x) = \tan^2(x) \sec^2(x) - \sec^2(x) + 1$ . The first part is easy to integrate with  $u = \tan(x)$ , the second just gives  $\tan(x)$ , and the last gives  $x$ .

#29. Separate  $\sec(x) \tan(x)$  and do  $u = \sec(x)$ . Then you'll get  $\int \tan^2(x) du = \int u^2 - 1 du$ .

#36. To get substitution going, write  $\int \cot^2(x) \cot(x) dx = \int (\cot^2(x) - 1) \cot(x) + \cot(x) dx = -\int (\cot(x))' \cot(x) dx + \int \cot(x) dx$ . The first piece is  $-\frac{1}{2} \cot^2(x)$  by substitution  $u = \cot(x)$ , the second, by another substitution  $v = \sin x$ , is  $\ln|\sin x|$ . To evaluate use  $\cot(\pi/2) = 0$ ,  $\cot(\pi/4) = 1$ ,  $\sin(\pi/2) = 1$ ,  $\sin(\pi/4) = \sqrt{2}/2$ .

#40. Done in class; start with parts writing the integral as  $-\int (\cot x)' \csc(x) dx$ .

#60.  $V = \int_0^{\pi/4} \pi \tan^4(x) dx$  and use #24.

## Section 8.3

#4. Let  $x = 4 \sin(u)$ . This gives  $64 \int \sin^3(u) du$ , which is done using  $z = \cos(u)$ . This one works out to be  $-64(\cos(u) - \frac{1}{3} \cos^3(u))$ . Since  $u = \arcsin(x/4)$ ,  $\cos(u) = \sqrt{16 - x^2}/4$  (draw a triangle representing  $u$ ). Evaluating from 0 to  $2\sqrt{3}$  I get  $13\frac{1}{3}$ .

#7. Use  $x = 5 \sin(u)$  to get  $\frac{1}{25} \int \csc^2(u) du = -\frac{1}{25} \cot(u)$ .

#8.  $x = a \sec(u)$  gives  $dx = a \sec(u) \tan(u) du$ , so that the integral becomes  $\frac{1}{a^3} \int \sin^2(u) \cos(u) du$ . Use  $z = \sin(u)$ , then the fact that  $u = \operatorname{arcsec}(x/a)$ , so the picture shows that  $\sin(u) = \sqrt{x^2 - a^2}/x$ .

#12. No need for trig substitution, just use  $u = x^2 + 4$ .

#16. Let  $x = \frac{3}{4} \sec(u)$ , you'll get  $\frac{4}{9} \int \cos(u) du = \frac{4}{9} \sin(u) + C = \sqrt{16x^2 - 9}/9x + C$ .

#26. Complete the square and rewrite as  $\int \frac{x^2}{\sqrt{4-(x-2)^2}} dx$ , then use  $x - 2 = 2 \sin(u)$  to get  $\int (2 \sin(u) + 2)^2 du$  (because  $x = 2 \sin(u) + 2$ ). Multiply out, use a previous problem (parts, or half-angle formula) for  $\int \sin^2(u) du$ .

#27. Complete the square again, then  $x + 1 = \tan(u)$ , getting  $\int \cos^2(u) du$ , which is again doable with a half-angle formula or parts. Here  $u = \arctan(x + 1)$  (draw a picture!) so that for example  $\sin(u) = \frac{x+1}{x^2+2x+2}$ .

### Section 8.4

#5b.  $\frac{At + B}{t^2 + 1} + \frac{Ct + D}{t^2 + 4} + \frac{Et + F}{(t^2 + 4)^2}$ .

#6b.  $x^6 - x^3 = x^3(x^3 - 1) = x^3(x - 1)(x^2 + x + 1)$ , so we get  $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x - 1} + \frac{Ex + F}{x^2 + x + 1}$ .

#8.  $\int \frac{r^2}{r + 4} dr = \int r - 4 + \frac{16}{r + 4} dr = \frac{r^2}{2} - 4r + 16 \ln|r + 4| + C$ .

#10.  $\frac{1}{(t + 4)(t - 1)} = \frac{1}{5(t - 1)} - \frac{1}{5(t + 4)}$ , so the answer is  $\frac{1}{5} \ln|t - 1| - \frac{1}{5} \ln|t + 4| + C$ .

#22. I'm getting  $\frac{1}{s^2(s - 1)^2} = \frac{2}{s} + \frac{1}{s^2} - \frac{2}{s - 1} + \frac{1}{(s - 1)^2}$ .

#34.  $\frac{x^3}{x^3 + 1} = 1 - \frac{1}{x^3 + 1} = 1 - \frac{1}{x(x^2 - x + 1)} = 1 - \frac{1}{x} + \frac{x - 1}{x^2 - x + 1}$ . Of these only the

last fraction is non-trivial to integrate. It breaks up into  $\frac{x - 1/2}{x^2 - x + 1}$  (simple substitution) and

$-\frac{1/2}{(x - 1/2)^2 + 3/4}$  (which requires the trig substitution  $x - 1/2 = \frac{\sqrt{3}}{2} \tan u$ ).

#45. Using the indicated substitution we have  $x = u^6$ , i.e.  $dx = 6u^5 du$ , and the integral becomes  $6 \int u^2 + u + 1 + \frac{1}{u - 1} du$ .

### Section 8.5

#24. Parts with  $1 = (x)'$ , then  $\int \frac{2x^2}{x^2 - 1} dx$  by partial fractions. Or write  $\ln(x^2 - 1) = \ln(x + 1) + \ln(x - 1)$ , each done easily just like  $\ln(x)$ —e.g.  $\int \ln(x + 1) dx = (x + 1) \ln(x + 1) - (x + 1) + C$ .

#34. Done in class; rewriting in terms of  $\sin(x)$ ,  $\cos(x)$  and simplifying gives a fraction in which the numerator is the derivative of the denominator.

#42. Parts:  $\frac{1}{3} \int (x^3)' \arctan(x) dx = \frac{1}{3} (x^3 \arctan(x) - \int \frac{x^3}{1 + x^2} dx)$ , then write  $\frac{x^3}{1 + x^2} = x - \frac{x}{1 + x^2}$  and use  $u = 1 + x^2$  for the second term.

#60. (corrected!)  $\frac{1}{x + \sqrt[3]{x}} = \frac{1}{x^{1/3}(x^{2/3} + 1)}$ , and  $(x^{2/3} + 1)'$  is 'close' to  $\frac{1}{x^{1/3}}$ . Use  $u = x^{2/3} + 1$  to get  $\frac{3}{2} \int 1/u du$ .

$$\#66. \frac{u^3 + 1}{u^3 - u^2} = \frac{u^3 - u^2 + u^2 + 1}{u^3 - u^2} = 1 + \frac{u^2 + 1}{u^3 - u^2}, \text{ and } \frac{u^2 + 1}{u^3 - u^2} = \frac{u^2 + 1}{u^2(u - 1)} = -\frac{3}{u} - \frac{1}{u^2} + \frac{2}{u - 1}.$$

$$\#74. u = e^x \text{ will give } \int \frac{1}{u^2 - 1} du = \frac{1}{2} \int \frac{1}{u - 1} - \frac{1}{u + 1} du.$$

### Section 9.1

#2. Done in class.

$$\#4. 1 + (f'(x))^2 = \frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4} = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2, \text{ so find } \int_{1/2}^1 \frac{x^2}{2} + \frac{1}{2x^2} dx.$$

#12.  $\int_1^{\sqrt{3}} \frac{\sqrt{x^2 + 1}}{x} dx$ . Use  $x = \tan u$  first to get the integral of  $\sec^3(u)/\tan(u)$ . If you multiply the numerator and denominator by  $\tan(u)$ , you'll be able to separate  $z' = \sec(u)\tan(u)$  and use substitution  $z = \sec(u)$ . This gives  $z^2/(z^2 - 1) = 1 + 1/(z + 1)(z - 1)$ , which is done by partial fractions. There might be a simpler way, but I don't see it.

$$\#20. y = \frac{b}{a}\sqrt{a^2 - x^2} \text{ for the upper half of the ellipse. So } L = \int_{-a}^a \sqrt{1 + \frac{b^2 x^2}{a^2(a^2 - x^2)}} dx.$$

### Section 8.8

#2. (b) discontinuity/asymptote at  $x = 1/2$ ; (c) infinite interval; (d) asymptote at  $x = 1$ .

$$\#8. \lim_{t \rightarrow \infty} -\frac{1}{2t^2 + 4} + \frac{1}{4} = 1/4.$$

$$\#28. \lim_{t \rightarrow 0^+} -2/\sqrt{3} + 2/\sqrt{t} = \infty.$$

$$\#30. \lim_{t \rightarrow 9^-} \frac{3}{2}(t - 9)^{2/3} - 6 = 6.$$

#50. Since  $e^{-x} > 0$ , we have  $\frac{2 + e^{-x}}{x} > \frac{2}{x}$ , and we know that  $\int_1^{\infty} \frac{2}{x} dx$  is divergent, so the integral in question is divergent as well.

#54. For all  $x \geq 0$ ,  $e^{-x} \leq 1$ , so on the interval  $[0, 1]$  we have  $\frac{e^{-x}}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$ . Since  $\int_0^1 x^{-1/2} dx$  is convergent, the integral in the problem is also convergent.

### Section 12.1

$$\#20. a_n = \sqrt{n} \frac{1}{1 + 1/\sqrt{n}} \rightarrow \infty.$$

$$\#30. a_n = (-1)^n n, \text{ divergent.}$$

#31.  $a_n \rightarrow 0$  by the squeeze law.

#41. divergent (the even-numbered terms approach 1, the others approach  $-1$ ).

#56. Several approaches: a) you can compute  $a_{n+1} - a_n = 17/(3n + 7)(3n + 4)$  and observe that it's always positive; b) find the derivative of  $f(x) = (2x - 3)/(3x + 4)$  and verify that  $f'(x) > 0$ , so  $f$  is increasing, so  $a_n$  is increasing as well; c) write  $a_n = \frac{2}{3} \left(1 - \frac{17}{6n+8}\right)$ , which makes it obvious that as  $n$  grows,  $a_n$  also grows. The last way is the best, because it gives you the answer to the second question:  $a_n$  is always less than  $2/3$ .

## Section 12.2

#2.  $\lim_{n \rightarrow \infty} S_n = 5$ , where  $S_n = a_1 + a_2 + \dots + a_n$ .

#5.  $\lim_{n \rightarrow \infty} a_n$  doesn't exist (it isn't entirely trivial to prove!)

#8. Telescoping series, just like in example 6; convergent.

#11. Looks like  $\sum_{n=0}^{\infty} 2^n/3^{n-1}$ , geometric series with ratio  $2/3$ , although it's hard to be sure just by looking at the first four terms...

#18. Again, geometric series with  $r = 1/\sqrt{2}$ , convergent.

#20. Geometric series with  $r = e/3$ , convergent.

#30.  $a_n \rightarrow \ln(1/2) \neq 0$ , divergent.

#36.  $73/99$

#43.  $|4x| < 1$ , i.e.  $-1/4 < x < 1/4$ .

#49.  $a_n = S_n - S_{n-1} = 2/(n^2 + n)$ , and  $\lim_{n \rightarrow \infty} S_n = 1$ .

## Section 12.3 (incl. a few that weren't assigned)

#2.  $\sum_{n=2}^6 a_n \leq \int_1^6 f(x) \leq \sum_{n=1}^5 a_n$ .

#4.  $\int_1^{\infty} x^{-1/4} dx = \lim_{t \rightarrow \infty} \frac{4}{3}(t^{3/4} - 1) = \infty$ , divergent ( $\sum a_n$  is a  $p$ -series with  $p = 1/4 < 1$ ).

#5.  $\lim t \rightarrow \infty \frac{1}{3}(\ln(3t + 1) - \ln 4)$ , divergent.

#7. Compute the derivative to make sure  $f(x) = xe^{-x}$  is decreasing for  $x \geq 1$ . Integration by parts:  $\int_1^{\infty} f(x) dx = \lim_{t \rightarrow \infty} -te^{-t} - e^{-t} + 2e^{-1} = 2/e$ , because  $te^{-t} \rightarrow 0$  (e.g. by l'Hôpital's rule).

#8. Use  $f(x) = 1 + \frac{1}{x+1}$ ,  $\lim_{t \rightarrow \infty} (t + \ln(t+1) - 1 - \ln 2) = \infty$ , 'very' divergent (also note that the limit of  $a_n$  is not zero, so the integral test is an overkill here).

#11.  $p$ -series with  $p = 3$ , convergent.

#16. This would be easy if we had the comparison tests from the next section:  $(3n+2)/(n+1) > 1$  for all  $n$ , so  $a_n > 1/n$  and the series is divergent by direct comparison with the harmonic series (or use limit comparison test with  $1/n$ ). But here we'd better use the integral test. By partial fractions,  $(3x+2)/(x^2+x) = 2/x + 1/(x+1)$  which is obviously decreasing, continuous, positive, and its integral from 1 to  $\infty$  is divergent.

#20. First verify that  $f(x) = (\ln x)/x^2$  is decreasing. The integral: we could use parts, writing  $\int f(x) dx = \int (-1/x)' \ln x dx = -(\ln x)/x - 1/x$ . Now  $\lim_{t \rightarrow \infty} -(\ln t)/t - 1/t + 1 = 1$ , so the integral is convergent, and then so is the series.

#21. Divergent (done in class?)

#25. Partly done in class, and I complicated things because I was too lazy to compute the derivative of  $f(x) = 1/(x(\ln x)^p)$ . It turns out to be negative for all  $x > 1$  and all  $p$ , so we can use the integral test for all  $p$ . Substituting  $u = \ln x$  we get an antiderivative  $F(x) = \frac{1}{1-p}(\ln x)^{1-p}$  for  $p \neq 1$ , and  $F(x) = \ln(\ln x)$  for  $p = -1$ . The corresponding improper integrals  $\lim_{t \rightarrow \infty} F(t) - F(2)$  are all divergent unless  $1-p < 0$  (in which case  $(\ln x)^{1-p} \rightarrow 0$ ), so the series is convergent for  $p > 1$  only.

## Section 12.4

#2. (a)  $\sum a_n$  is then divergent because its partial sums are all greater than partial sums of  $\sum b_n$ , and those are divergent (unbounded, i.e. go to  $\infty$  as  $n \rightarrow \infty$ ).

(b) Can't say anything;  $a_n$  might be, say, one half of  $b_n$ , in which case  $\sum a_n$  will be divergent as well because its partial sums are all one-half of the (unbounded) partial sums of  $\sum b_n$ , and will be unbounded as well. On the other hand, if the  $a_n$ 's are 'significantly' smaller than the  $b_n$ 's, as in  $a_n = 1/n^2$  and  $b_n = 1/n$ , then  $\sum a_n$  may turn out to be convergent.

#3. Convergent, e.g. by direct or limit comparison with  $\sum 1/n^2$ , a  $p$ -series with  $p > 1$ .

#6. Divergent by direct (or limit) comparison with the harmonic series.

#20. Limit comparison with the geometric series with ratio  $r = 2/3 < 1$ ; or break  $a_n$  up into  $1/(1+3^n) + 2^n/(1+3^n)$  and apply direct comparison of each component with a suitable geometric series. Convergent.

#22. Convergent; e.g. limit comparison with  $\sum 1/n^2$ .

#24. Divergent, as in # 6.

#27.  $(1 + 1/n)^2$  is always at most 4. Direct comparison with  $\sum 4e^{-n}$ , a geometric series with ratio  $1/e < 1$ .

#31. Done in class - limit comparison with the harmonic series. Divergent.

## Section 12.5

#4.  $a_n = 1/\ln(n)$  are decreasing and  $\lim_{n \rightarrow \infty} a_n = 0$ , so  $\sum (-1)^n a_n$  is convergent by Leibniz's criterion.

#6. Same as #4.

#7. Divergent, terms don't converge to 0.

#14. Same as #4 (make sure  $\lim_{n \rightarrow \infty} \ln(n)/n = 0$ ).

#19. Divergent,  $\lim_{n \rightarrow \infty} n^n/n! = \infty$  because  $n^n/n! \geq n$  (e.g. see example 8 in section 12.1).

#30. Same as #4,  $1/(3^n n!)$  is decreasing and goes to 0.

#32.  $1/n^p = n^{-p}$  is decreasing for  $p \geq 0$  (look at the derivative of  $f(x) = x^{-p}$ ), so the alternating series is convergent then. For  $p < 0$  we have  $\lim_{n \rightarrow \infty} n^{-p} = \infty$ , so the series must be divergent by the  $n$ -th term test.

## Section 12.6

#3.  $|a_{n+1}|/|a_n| = 10/(n+1)$  whose limit is 0. Absolutely convergent.

#4.  $\lim_{n \rightarrow \infty} 2^n/n^4 = \infty$  (an exponential quantity grows faster than a polynomial one; e.g. apply l'Hôpital's rule to  $f(x) = 2^x/x^4$ ). Divergent.

#6. Absolutely convergent,  $\sum 1/n^4$  is a  $p$ -series with  $p > 1$ .

#8. After throwing out the minuses we get a series with behavior close to the harmonic series: e.g. apply the limit comparison test to see that  $\sum n/(n^2 + 1)$  is divergent. But the original series  $\sum (-1)^{n-1} n/(n^2 + 1)$  is convergent by Leibniz's criterion. So the series is convergent, but not absolutely (i.e. it's conditionally convergent).

#10. Ratio test:  $(n + 1)/e \rightarrow \infty$ , divergent.

#12. Direct comparison with the geometric series:  $|\sin(4n)|/4^n \leq 1/4^n$ , so absolutely convergent.

#14. Ratio test:  $2(n + 1)^2/(n^2(n + 1)) \rightarrow 0$ , absolutely convergent.

#18. Done in class; ratio test gives limit  $1/e < 1$ . Convergent (remember, for series with positive terms, ‘convergent’ and ‘absolutely convergent’ are one and the same, because in that case  $|a_n| = a_n$ ).

#20. Root test:  $\lim_{n \rightarrow \infty} \sqrt[n]{1/(\ln n)^n} = \lim_{n \rightarrow \infty} 1/\ln n = 0$ . Absolutely convergent.

#22. By the integral test  $\sum 1/(n \ln n)$  is divergent (see #21 in section 12.3), so our series isn’t absolutely convergent. But it is convergent, by Leibniz’s criterion.

#23. Root test; convergent.

### Section 12.7

#4. Leibniz’s criterion, convergent.

#6. Root test, limit  $3/8 < 1$ , convergent.

#8. Ratio test:  $\lim_{k \rightarrow \infty} 2(k + 1)/(k + 3) = 2 > 1$ , series is divergent. Or notice that  $a_k = 2^k/(k^2 + 3k + 2) \rightarrow \infty$ , not zero, so no chance for convergence.

#9. Ratio test, limit is  $1/e$ , convergent.

#10. Can use ratio test here with a little effort, but it’s easier to use the integral test with  $f(x) = x^2 e^{-x^3}$ ,  $u = x^3$ . Convergent.

#11. Leibniz’s criterion, convergent.

#12. Same as #11.

#13. Limit in the ratio test is  $0 < 1$ , convergent.

#14.  $\lim_{n \rightarrow \infty} a_n$  is not zero; series is divergent.

#15. Limit in the ratio test is  $1/3 < 1$ ; convergent.

#18. Leibniz’s criterion, convergent.

#19.  $\lim_{n \rightarrow \infty} 2^{1/n} = 1$ , so the series is divergent.

#20. Convergent by the ratio test.

#23. Sneaky... write  $\tan(1/n)$  as  $\sin(1/n)/\cos(1/n)$ , then use limit comparison with  $1/n$  (the limit will be 1). Divergent.

#24. Direct comparison with  $1/n^2$ , convergent.

#27. Tough, if you don’t have a good feeling for the rates of growth of various functions. Reasoning:  $\ln k$  grows quite a bit more slowly than  $k$ , even more slowly than  $\sqrt{k}$ , so the numerator grows more slowly than  $k^{3/2}$ . On the other hand, the denominator grows as a cube of  $k$ . The net result is that the terms of this series should ‘favorably compare’ with the  $p$ -series  $\sum 1/k^{3/2}$ , and the series should converge.

To clear things up use limit comparison with  $\sum \ln(k)/k^2$  (the limit of the quotients is 1, so either both series diverge or both converge). I now claim that the series  $\sum \ln(k)/k^2$  is convergent: from some point on  $\ln k \leq \sqrt{k}$ , so we can use direct comparison with  $\sum \sqrt{k}/k^2 = \sum 1/k^{3/2}$ . How do I know that  $\ln k \leq \sqrt{k}$  (or, for that matter, any positive power of  $k$ ) from some  $k$  on? Remember,  $\ln k/k \rightarrow 0$  as  $k \rightarrow \infty$ . So  $\ln k/\sqrt{k} = 2 \ln(\sqrt{k})/\sqrt{k}$  also tends to 0, and that can only happen if  $\ln k$  is smaller than  $\sqrt{k}$  for large  $k$ 's. See also problem 88 in section 7.7. Maybe there is a shorter explanation? I can't think of one now.

#28. Try the limit comparison with  $\sum 1/n^2$  (or direct comparison with  $\sum e/n^2$ , since roots of  $e$  are less than or equal to  $e$  itself). Convergent.

#32. Root test, limit is  $0 < 1$ , convergent.

#35. Root test:  $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^n = 1/e < 1$ , convergent.

## Section 12.8

#7. All  $x$  (done in class).

#8. Similar one done in class: limit in the ratio test is  $\lim_{n \rightarrow \infty} (n+1)(1+1/n)^n x$ , which is infinity for any nonzero  $x$ . So this series converges only for  $x = 0$ .

#12. Apply ratio test to  $\sum |a_n|$  to get convergence when  $|x| < 5$  (so the radius of convergence is 5). Now at  $x = 5$  the series is  $\sum 1/n^5$ , which is convergent. At  $x = -5$  it's the alternating series  $\sum (-1)^n/n^5$ , also convergent. The interval of convergence is  $[-5, 5]$ .

#13.  $\lim_{n \rightarrow \infty} \sqrt[n]{\ln n} = 1$  (e.g. look at  $f(x) = e^{\ln(\ln x)/x}$  using l'Hôpital's rule). So the root test tells us that  $\sum |a_n|$  is convergent when  $|x| < 4$ . For  $x = 4$  we get the alternating series  $\sum (-1)^n 1/\ln(n)$ , which is convergent by Leibniz's criterion. But for  $x = -4$  the series is  $\sum 1/\ln n$ , which is divergent e.g. by direct comparison with  $\sum 1/n$  (as above, from some point on  $\ln n \leq n$ , so  $1/\ln n \geq 1/n$ ). The interval of convergence is  $(-4, 4]$ .

#18. This is a power series centered at  $a = -3$ . Ratio test gives convergence for  $|x+3| < 0.5$ , i.e. between  $-3.5$  and  $-2.5$ . For  $x = -3.5$  the series equals  $\sum 1/\sqrt{n}$ , which is divergent. For  $x = -2.5$  it's an alternating series satisfying Leibniz's criterion. The interval of convergence is  $(-3.5, -2.5]$ .

#27. Root test says that the series converges for all  $x$ .

#28. Ratio test, radius of convergence is 1. For  $x = 1$  or  $x = -1$  note that the monster fraction is always greater than 1 (as a product of 2 times  $4/3$  times  $6/5$  etc.), so there is no chance of it having limit 0. The series is divergent then, so the interval of convergence is  $(-1, 1)$ .

#30. The series is centered at 0, and thus has interval of convergence from  $-R$  to  $R$  (and who knows about the endpoints). Since the series is convergent at  $-4$ ,  $R$  is at least 4. Since it's divergent for  $x = 6$ ,  $R$  must be at most 6. So the best case scenario for the interval of convergence (largest possible) is  $[-6, 6)$ , while the worst case is  $[-4, 4)$ . (a) is when  $x = 1$ , which is certainly inside the interval of convergence, so we're OK. (b) is when  $x = 8$ , can't be convergent (else it would be convergent for  $x = 6$  as well). (c) is OK just like (a), and (d) is the case  $x = -9$  which must be divergent like (b).