

## Partial fractions decomposition – a bedtime story

**Objective:** To integrate any *rational function*, i.e.  $f(x) = \frac{g(x)}{h(x)}$ , where  $g$  and  $h$  are polynomials – expressions of the form  $a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$ .

### Procedure

If the numerator  $g(x)$  has *degree* (that is, the highest power of  $x$  which appears in it) greater than or equal to the degree of  $h(x)$ , we can use long division of polynomials to simplify the function  $f$ . A *division theorem* (dating back to antiquity in its version for whole numbers) tells us that  $g(x) = h(x)q(x) + r(x)$ , where the degree of  $r$  is less than degree of  $h$ , so that  $f(x) = \frac{g(x)}{h(x)} = \frac{h(x)q(x) + r(x)}{h(x)} = q(x) + \frac{r(x)}{h(x)}$ . The recipe for obtaining  $q(x)$  and  $r(x)$  is taught in high school algebra courses.

**Examples.** a)  $2x = 2(x+1) - 2$ , so that  $\frac{2x}{x+1} = \frac{2(x+1) - 2}{x+1} = 2 - \frac{2}{x+1}$ , which is easily integrated.

b)  $x^3 - 2x + 1 = (x^2 - 2x + 1)(x + 2) + (x - 1)$ , so  $\frac{x^3 - 2x + 1}{x^2 - 2x + 1} = (x + 2) + \frac{x - 1}{x^2 - 2x + 1}$ , with the first term easily integrated (the second term will be handled below).

Because of the above, we can focus on the case of  $f(x) = g(x)/h(x)$  such that the degree of  $g$  is smaller than the degree of  $h$ . Our problem has an easy part (integrating  $q(x)$  above) and a hard part (integrating  $r(x)/h(x)$ ); forget the easy part, and concentrate on the harder one.

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The next step is to factor the denominator. Algebra and theorems about real numbers say that every polynomial with real coefficients can be factored into parts which cannot be factored further, and each of those factors will be either *linear* (i.e. of degree 1, so they look like  $ax + b$ ), or *irreducible quadratic* (i.e. of degree 2, looking like  $ax^2 + bx + c$ , with  $b^2 - 4ac < 0$ , so it doesn't have roots and cannot be broken up into two linear factors).

In practice, factoring polynomials can be very hard. For quadratic, cubic and quartic polynomials, we have general formulas which tell us how to find roots (and hence factor). 160 years ago, shortly before dying in a duel, a 21-year old French mathematician Evariste Galois proved that for polynomials of degree 5 or higher, similar simple general formulas do not exist and cannot ever be found. But let's not worry about this too much. You aren't likely to see homework or exam problems in which this step will be a major challenge.

**Examples.** a)  $x^2 - 2x + 1 = (x - 1)(x - 1) = (x - 1)^2$ ; two copies of a linear factor

b)  $x^5 - 2x^4 + x^3 = x^3(x - 1)^2$ ; 3 copies of the linear factor  $x$  and two copies of  $x - 1$

c)  $x^2 + 2$  is an irreducible quadratic polynomial

d)  $x^3 + 6x^2 + 11x + 6 = (x + 1)(x + 2)(x + 3)$  (how can you guess this?)

e)  $x^2 - 2x - 8 = (x + 2)(x - 4)$

f)  $x^4 - 6x^2 + 8 = (x^2 + 2)(x^2 - 4) = (x^2 + 2)(x - 2)(x + 2)$ ; an irreducible quadratic factor and two distinct linear ones

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Once the denominator is broken up into factors which look like  $(ax + b)^n$  and  $(ax^2 + bx + c)^n$  (with  $ax^2 + bx + c$  irreducible), we are ready to try and write the function  $f(x)$  as a sum of more basic fractions. Another theorem whose proof is beyond the scope of this course says that  $f(x)$  can be expressed as a sum of fractions of the following form:

$\frac{A}{(ax+b)^n}$ , with  $A$  – a constant, and

$\frac{Ax+B}{(ax^2+bx+c)^n}$ , with  $A$  and  $B$  – constants.

To each linear factor  $ax+b$  with  $n$  copies of it appearing in the denominator, there will correspond a sum of fractions  $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \dots + \frac{A_n}{(ax+b)^n}$ .

**Examples.** a) if  $x+2$  appears in the denominator once, the partial fraction decomposition of  $f(x)$  will include the term  $\frac{A}{x+2}$  with some constant  $A$

b) if  $x-1$  appears in  $h(x)$  three times, we need to include  $\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$  in the decomposition

c) even though  $x^2$  looks like a quadratic factor – it is not: it is the linear factor  $x$  repeated twice. We take care of it with two fractions  $\frac{A}{x} + \frac{B}{x^2}$

d)  $f(x) = \frac{??}{x^2(x+2)(x-1)^3}$  will be equal to a sum  $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2} + \frac{D}{x-1} + \frac{E}{(x-1)^2} + \frac{F}{(x-1)^3}$   
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Time for the irreducible quadratic factors. To each quadratic factor  $ax^2+bx+c$  with  $n$  copies of it appearing in the denominator, there will correspond a sum  $\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \dots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$  (a minor modification of this will be mentioned later).

**Examples.** a) if  $x^2+1$  appears in the denominator twice, we include  $\frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$

b)  $f(x) = \frac{??}{x^2(x+2)(2x^2+2x+2)^2}$  can be decomposed into the sum  $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2} + \frac{Dx+E}{2x^2+2x+2} + \frac{Fx+G}{(2x^2+2x+2)^2}$   
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The pattern is simple enough, I hope. We will now look at the method of finding the constants.

Given the decomposition, say,  $f(x) = \frac{x^2+3x-1}{x^2(x^2+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+2}$ , we pull the terms on the right hand side into a single fraction by finding a common denominator:  $\frac{Ax(x^2+2) + B(x^2+2) + (Cx+D)x^2}{x^2(x^2+2)}$ .

Now we use the fact that two rational functions with identical denominators are equal precisely when their numerators are the same, too; so we are trying to guarantee that  $x^2+3x-1 = Ax(x^2+2) + B(x^2+2) + (Cx+D)x^2$ .

Two polynomial functions, like the two numerators at hand, are equal when they produce identical results for all values of  $x$ . Higher algebra tells us that in our situation, when we deal with real numbers, this will happen precisely when *all corresponding coefficients* of the powers of  $x$  must be the same (e.g. coefficient of  $x^2$  on the left is the same as that on the right).

The first condition says that if we substitute any  $x$  into the two sides, the results must be equal. Do this wisely: choose values of  $x$  which will simplify things. For example, taking  $x=0$  we get  $-1=2B$ , and we now know that  $B=-\frac{1}{2}$  (so that  $A=2$ ). Using the second approach and comparing the coefficients of  $x^3$

we get  $0 = A + C$ , i.e.  $C = -2$ . Finally, coefficients of  $x^2$  on both sides tell us that  $1 = B + D$ , so that  $D = \frac{3}{2}$ . We now know that

$$f(x) = \frac{x^2 + 3x - 1}{x^2(x^2 + 2)} = \frac{2}{x} - \frac{1}{2x^2} + \frac{-2x + 3/2}{x^2 + 2}.$$

You can use either of these approaches, or a mixture of them, to determine all of the unknown constants  $A, B, C, \dots$

We are ready to integrate  $f(x)$ : the first term will produce a natural logarithm, the second is done using the power rule; the third fraction can be written as two, one of them integrated using a substitution  $u = x^2 + 2$ , and the second one will involve  $\arctan(x/\sqrt{2})$ .

### Integrating the resulting partial fractions

Fractions of the form  $\frac{A}{(ax + b)^n}$  are easily integrated using the power rule (if  $n > 1$ ), or the natural logarithm (when  $n = 1$ ).

Fractions of the form  $\frac{Ax + B}{(ax^2 + bx + c)^n}$  are a little harder. First, we try to use substitution  $u = ax^2 + bx + c$ , with  $u' = 2ax + b$ . For instance,  $\frac{4x}{(x^2 - x + 2)^3} = \frac{4x - 2 + 2}{(x^2 - x + 2)^3} = \frac{4x - 2}{(x^2 - x + 2)^3} + \frac{2}{(x^2 - x + 2)^3}$ .

The first term is integrated by substitution, producing  $\int u^{-3} du = -\frac{1}{2}u^{-2}$ . Not bad. The second term is more difficult; we'll take care of it last.

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In fact, we can slightly change the partial fraction decomposition process to make our life a bit easier at this stage. Instead of using fractions of the form  $\frac{Ax + B}{(ax^2 + bx + c)^n}$  to begin with, you can try setting up  $\frac{A(2ax + b) + B}{(ax^2 + bx + c)^n}$  and finding the appropriate constants just like we did before.

This takes into account the derivative of the denominator  $u(x)$ , and we right away get fractions of the type  $A \frac{u'(x)}{(u(x))^n}$  (ready to be integrated by substitution), and the nasty terms of the form  $A \frac{1}{(ax^2 + bx + c)^n}$ .

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We come to the last obstacle: a fraction which looks like  $\frac{A}{(ax^2 + bx + c)^n}$ . Obviously, we can pull  $A$  out of the integral; moreover, we can simplify the denominator by factoring out  $a$ , so that the fraction becomes  $\frac{A}{a^n} \frac{1}{(x^2 + dx + e)^n}$ , where  $d$  and  $e$  are obtained by dividing  $b$  and  $c$  by  $a$ . For example:

$$\frac{3}{(2x^2 + 4x + 5)^3} = \frac{3}{8} \frac{1}{(x^2 + 2x + 2.5)^3}$$

Of course, the numbers you get for  $d$  and  $e$  may be quite ugly, but that's the price of making the quadratic polynomial in the denominator start with just  $x^2$ . What is the purpose of all that? It allows us to concentrate now on fractions of the form  $\frac{1}{(x^2 + bx + c)^n}$ .

We integrate them by completing the square in the denominator. If we write  $x^2 + bx + c = (x + \frac{b}{2})^2 + (c - \frac{b^2}{4})$  (why?), and using the substitution  $u = x + \frac{b}{2}$ , we make the fraction look like  $\frac{1}{(u^2 + p)^n}$ , where  $p = c - \frac{b^2}{4}$ . If

we started with an irreducible quadratic polynomial  $x^2 + bx + c$ , i.e.  $b^2 - 4c < 0$ , it will always turn out that the term  $p$  above is positive! So if we use  $q = \sqrt{p}$ , the fraction finally becomes  $\frac{1}{(u^2 + q^2)^n}$ . As explained in class (and in the section on trigonometric substitutions), this type of function is integrated by making yet another substitution:  $u = q \tan w$ . This changes the integral into a trigonometric one, which – hopefully – we can solve using methods we already learned.

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The above looks quite bad, because we tried to show that a certain method can always be used. Let's look at the concrete example of a term which looks like  $\frac{3}{(2x^2 + 6x + 6)^4}$ . First, take the  $2^4 = 16$  out of the denominator, so  $3/16$  can be pulled out of the integral. We are now left with  $\frac{1}{(x^2 + 3x + 3)^4}$ . Completing the square we get a somewhat ugly  $\frac{1}{((x + \frac{3}{2})^2 + \frac{3}{4})^4}$ .

Now forget what lead us to this point, and recall similar functions from the section on inverse trigonometric substitution. We can make this expression a little easier by setting  $u = x + \frac{3}{2}$  (thus getting  $\frac{1}{(u^2 + \frac{3}{4})^4}$ ), or we can skip that intermediate substitution and let  $x + \frac{3}{2} = \frac{\sqrt{3}}{2} \tan z$  so that the denominator will turn into  $\frac{81}{256} \sec^8 z$ . Since  $dx = \frac{\sqrt{3}}{2} \sec^2 z dz$ , we are now dealing with  $\int \frac{128}{27\sqrt{3}} \cos^6 z dz$ . This is a pretty bad one, involving a fairly high even power of the cosine, but by now we know that (given enough time and persistence) we can handle it with repeated integration by parts, or by means of the double-angle formula. Yay, we're done!