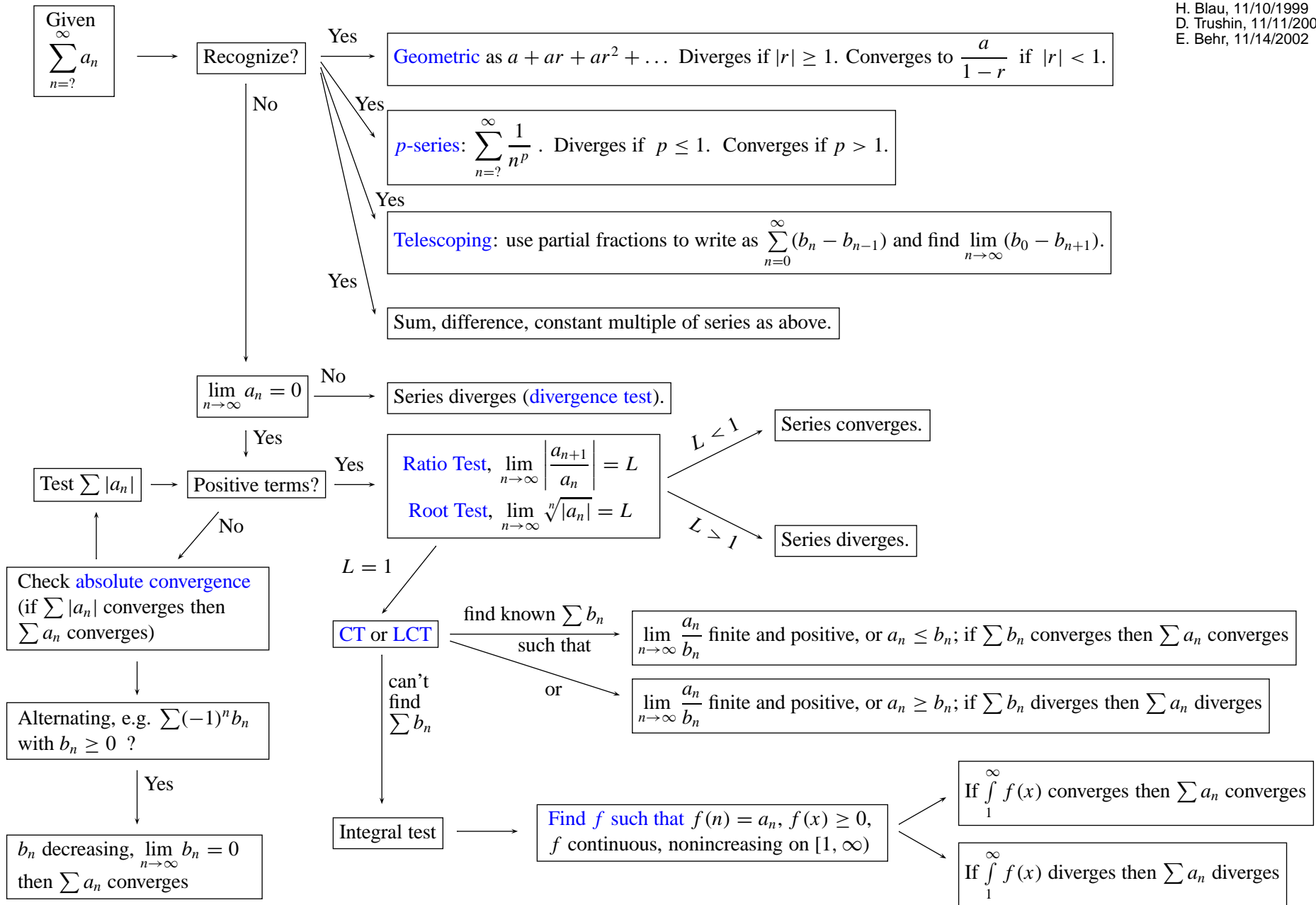


# Series Strategy

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Note that the “algorithm” described here is by far not the ‘whole story’. It covers most of what was covered in class, but the whole subject in general is extremely broad and difficult.

I’ll be using the shorthand  $\sum a_n$  for the series  $\sum_{n=1}^{\infty} a_n$  (or whatever else the starting  $n$  is). As I said in class many times, the first 5 or 100,000 terms do not affect convergence; and the topic is interesting (and difficult) precisely because we are attempting to add up infinitely many terms. So the ‘bottom limit’ in the  $\sum$  doesn’t matter much, while the ‘top’ will always be infinity.

First, we know that if  $\sum a_n$  is convergent, i.e. its partial sums  $S_n$  have a finite limit, then  $\lim_{n \rightarrow \infty} a_n$  must be zero. You can say this in many equivalent ways:

- if  $\sum a_n$  is convergent then  $\lim_{n \rightarrow \infty} a_n = 0$
- for  $\sum a_n$  to be convergent, we *must* have  $\lim_{n \rightarrow \infty} a_n = 0$
- if  $\lim_{n \rightarrow \infty} a_n$  is not zero, then  $\sum a_n$  cannot possibly converge.

But remember that this is strictly a one-way street: if the  $a_n$ ’s do *not* approach zero,  $\sum a_n$  must be divergent; but if the  $a_n$ ’s *do* approach zero, we still don’t know anything about  $\sum a_n$ , other than there is a *chance* of it being convergent. [back to the diagram...](#)

The next quick check should be whether  $\sum a_n$  looks like a **geometric series**  $\sum a_0 q^n$ , simply because we know all there is to know about it. If  $-1 < q < 1$  then the series is convergent (and we even know that its sum is  $a_0 \frac{1}{1-q}$ ). For all other values of  $q$  it is divergent. End of story. The only complication here is that we may see a geometric series written in such a way that it doesn’t look like one at the first glance, e.g. something like  $\sum 3\sqrt[3]{e^{n-1}}$ .

We now approach the big divide between the case when all terms  $a_n$  are nonnegative (since those that are zero can be simply thrown out of the partial sums, we usually say ‘positive’ instead), and the situation when some might be negative.

If all  $a_n$ ’s are nonnegative, then the partial sums can only grow, and a basic property of real numbers is that the sequence of  $S_n$ ’s which keeps going up will be convergent if and only if that sequence is *bounded*, i.e. will never exceed some finite value. The job of analyzing this case is comparatively straightforward.

But if some terms can be negative, then all sorts of things can happen because of cancellation: the values of  $S_n$  are no longer ‘marching forward’, but can be hesitating, lunging back and forth, or zipping backwards. It’s a bit as if your bathtub faucet sometimes gushed water, sometimes dripped, and sometimes sucked it back out of the tub.

If you know that all terms of the series you are dealing with are positive, you can fast-forward to the next section.

### Arbitrary terms

We have seen very few general methods of dealing with series whose terms may be negative. One is the geometric series (with  $q < 0$ ) which was discussed above.

The second is when the series has a particular form where some of the positive and negative terms cancel out, so that we can even find an expression for  $S_n$ , as in  $\sum \frac{1}{n(n+2)}$  or  $\sum \ln\left(\frac{n}{n+1}\right)$ . This type of series is sometimes called ‘**telescoping**’. Once we have a formula for  $S_n$ , we can use everything we know about sequences to figure out the limit of the partial sums.

The third, most general situation is the **alternating series**, in which every other term is positive and the others are negative. In the language of partial sums, this means that we make one step forward, one step backwards, another step forward, etc. Now if the steps are getting smaller, we are trapped in a small space, between the place where we started, and the first step we made. And if those steps are getting arbitrarily small, then

we are zeroing in on some specific point in our walk. That's precisely what Leibniz's criterion says: given  $\sum (-1)^n a_n$  with  $a_n \geq 0$  (those are the lengths of our steps), if  $a_n$ 's are non-increasing and  $\lim_{n \rightarrow \infty} a_n = 0$ , then the series in question must be convergent.

But that's about all we know right now. So what if the various tests mentioned above are inconclusive, or simply do not apply? Well, we use a 'detour'. Instead of looking at the original  $\sum a_n$  we switch to  $\sum |a_n|$  (which will of course have positive terms), and we apply one of the several methods mentioned in the next section to it.

If we then determine that  $\sum |a_n|$  is convergent, then the original series had to be convergent. This case is so nice that it has its own name – we then say that  $\sum a_n$  is *absolutely* convergent.

However, if one of the tests described below fails or even shows that  $\sum |a_n|$  diverges, then (with two exceptions which will be mentioned below) we still don't know what  $\sum a_n$  is doing, and we have to keep trying.

### Positive terms

In this section we assume that all terms of the series are positive. We have two main 'external' criteria for convergence (in which we somehow compare our  $\sum a_n$  with some other known series), and two 'internal' tests in which we do not refer to any auxiliary series. There is also a test involving a relation between the limit of a series and an improper integral.

**Direct comparison** of  $\sum a_n$  with  $\sum b_n$  works as follows: if we know that  $\sum b_n$  is convergent (which, in case of positive terms, can be very vaguely expressed as "the  $b_n$ 's get small enough fast enough"), and we also can show that (at least from some  $n$  on)  $a_n \leq b_n$  always holds, then  $\sum a_n$  has to be convergent. Conversely, if  $\sum b_n$  does *not* converge and the terms of  $\sum a_n$  are even bigger than the  $b_n$ 's, then  $\sum a_n$  must be divergent as well.

**Limit comparison** of  $\sum a_n$  with  $\sum b_n$  relies on the fact that, again very vaguely speaking, if the terms  $a_n$  are roughly  $k$  times the  $b_n$ 's (at least from some  $n$  on), and  $k$  is some non-zero constant, then adding those  $a_n$ 's will produce results which are roughly  $k$  times the result of adding the  $b_n$ 's. So if the partial sums of  $\sum b_n$  are convergent then the partial sums of  $\sum a_n$  will converge – possibly to a very different value of course; and if the partial sums of  $\sum b_n$  grow without bounds, then so will those of  $\sum a_n$  (remember that for positive series this is the only way they can diverge).

Expressing this precisely: if  $\lim_{n \rightarrow \infty} a_n/b_n = k$  for some (finite)  $k > 0$ , then  $\sum a_n$  and  $\sum b_n$  behave the same way – either both converge, or both diverge.

If the terms  $a_n$  can be expressed as values of some function  $f(x)$  computed at integer  $x$ 's, i.e.  $a_n = f(n)$ , then we can try to compare the series with the area under the graph of  $f$  (the **integral test**). This *only works* when  $f$  behaves well enough: it must be continuous, positive and non-increasing. In this situation if we manage to find the improper integral  $\int_a^\infty f(x) dx$ , it will tell us what the series is doing: both are convergent, or both divergent. Note that the *value* of the integral and the sum of the series will in general be different.

A special case is the ' $p$ -series'  $\sum n^{-p}$ . Using the integral test we showed that it's convergent when  $p > 1$  and divergent otherwise. This and the geometric series are two special cases where we know precisely what happens.

We end with the two 'internal' tests: the **ratio test** and the **root test**. In both cases we compute a certain limit  $L$ , and draw conclusions about the series on the basis of its value. In the ratio test we find  $L = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$ ; in the root test we compute  $L = \lim_{n \rightarrow \infty} \sqrt[n]{a_n}$  instead. The first one is typically easier to deal with.

Both of them work by, in essence, checking if the terms  $a_n$  are ultimately sufficiently close in their behavior to a geometric series. For example, if  $\sum a_n$  is a geometric series with quotient  $q$ , then it's obvious that in both tests  $L$  will simply be  $q$ .

The conclusions of these tests are as follows: if  $L$  equals 1 or doesn't exist, we don't know anything. If  $L < 1$  then the series  $\sum a_n$  is convergent. If  $L > 1$  or is infinity, then  $\sum a_n$  must be divergent.

As we said before, if the tests in this section are applied to  $\sum |a_n|$  and show that it diverges, we normally can't conclude anything from this with regard to the original  $\sum a_n$ . However the last case in the root and ratio tests is an exception: one can easily show that when  $L > 1$  (or infinity) then  $\lim_{n \rightarrow \infty} a_n$  cannot possibly be zero. So in studying absolute convergence this means that  $\lim_{n \rightarrow \infty} |a_n| \neq 0$ , which can only happen if the original  $a_n$ 's did not converge to zero – and we can use the first basic fact mentioned in these notes to make sure that then  $\sum a_n$  has to be divergent as well.