A few more examples of Taylor series

Example. Let's see if we can get the expression for the sum of the geometric series, $\frac{1}{1-x} = 1 + x + x^2 + \ldots$ for $|\,x\,| < 1$, out of the Taylor expansion method. If $f(x) = \frac{1}{1-x}$, then $f'(x) = (1-x)^{-2}$, $f''(x) = 2(1-x)^{-3}$, $f'''(x) = 2 \cdot 3(1-x)^{-4}$, etc. In general, $f^{(n)}(x) = n!(1-x)^{-n-1}$, so $f^{(n)}(0) = n!$. Then the Taylor series around $a = 0$ looks like

$$1 + \frac{1}{1}x + \frac{2}{2}x^2 + \ldots + \frac{n!}{n!}x^n + \ldots = 1 + x + x^2 + x^3 + \ldots$$

which is what we expected. Caution: if we didn't know from our earlier considerations that this series does converge to $\frac{1}{1-x}$ for $|\,x\,| < 1$, then we would still need to analyze the remainder (error) terms $R_n(x)$ and show that they approach 0 as n increases – and in this innocent example it would be quite hard to do this directly.

Example. We know that the Maclaurin series for $f(x) = \sin x$ is $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$, which converges to $\sin x$ for all x. For any fixed x the remainder is equal to

$$R_n(x) = \frac{f^{(n+1)}(A)}{(n+1)!}x^{n+1} \ ,$$

where A is some number between 0 and x, and $f^{(n)}(A)$ is one of the four possibilities: $\pm\sin(A)$ or $\pm\cos(A)$. All of those are between -1 and 1. So for example, given any x smaller than or equal to 1 we know that the absolute value of $R_n(x)$ will be at most $\frac{1}{(n+1)!}$. So if we compute just the sum of $0.5 - \frac{0.5^3}{6} + \frac{0.5^5}{120} - \frac{0.5^7}{5040}$, we will have the value of $\sin(0.5)$ with the worst possible error of $1/8!$ or about 0.000025.
The series above also explains why for very small values of x, $\sin x$ behaves approximately like x: the terms after x in the Taylor expansion are extremely small compared to x, and their sum stays between $-\frac{x^3}{6}$ and $-\frac{x^3}{6} + \frac{x^5}{120}$ (remember, an alternating series!) For example when $x = 0.1$ then the contribution of all the terms of degree 3 or higher taken together is somewhere between $-0.001/6$ and $-0.001/6 + 0.00001/120$, and so is a couple of orders of magnitude smaller than x itself.

Example. Since $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ converges to $\sin x$ for all x, $x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \ldots$ will converge to $\sin(x^2)$ for all x. Now integrating this new series term by term we get

$$\int \sin(x^2)\,dx = C + \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \ldots$$

This, together with what we know about alternating series, can be used to find a definite integral, say, $\int_0^1 \sin(x^2)\,dx$ even though we can't find the antiderivative directly (see comments at the end of 8.5). As in example 8 in section 12.10, we get $\int_0^1 \sin(x^2)\,dx \simeq \frac{1}{3} - \frac{1}{7\cdot 3!} + \frac{1}{11\cdot 5!} - \frac{1}{15\cdot 7!}$ with an error at most as large as the next term, i.e. $\frac{1}{19\cdot 9!}$ (which is pretty small!).