

Nine Point Circle and Feuerbach's theorem

First, a few terms to refresh our memory. Given a triangle $\triangle ABC$, its *orthocenter* is the point where the three altitudes intersect. For example, the orthocenter of a right triangle is its vertex at the right angle. Where is the orthocenter of an acute triangle? and an obtuse one? We will label the orthocenter with H . [I might add some illustrations later when I have more time.]

The *circumcenter*, the center of a circumscribed circle, is the common intersection of perpendicular bisectors of sides. This is easy to see if we start with any circle, then draw any triangle with vertices on the circle, and look at the relative position of the center of the circle and any two vertices. If you have a compass and know how to use it with a ruler to find a perpendicular bisector of a segment, you can now easily find the circumcenter of any triangle. This allows you to construct the circle circumscribed around the triangle. Let's label the circumcenter O .

Finally, the *centroid* of $\triangle ABC$ is the intersection of the medians, i.e. the segments connecting each vertex with the midpoint of the opposite side. We will call it G .

Now construct a unique circle C passing through the midpoints L , M and N of the three sides (can those midpoints ever be *collinear*, i.e. lie on a single straight line? when they are, they do not define a circle with finite radius!)

This is the so-called *nine point circle*. The reason for the name is that it passes through 9 points closely related to the triangle we started with. For starters, it passes through the midpoints of the three sides (i.e. through the feet of the medians) – that's how it was constructed. Let's look for six more points.

Observations (Poncelet, Feuerbach, others?)

It turns out that C also passes through the three feet D , E , F of the altitudes of $\triangle ABC$ (so we could have started by constructing the unique circle through these three points, and then our first claim would be that it has to pass through the feet of the medians). We have three more points.

The next three are a little stranger. Remember H , the orthocenter? Draw segments connecting it with the vertices of the triangle, i.e. AH , BH , CH . Their midpoints are the so-called *Euler points* of $\triangle ABC$, named after the famous mathematician Leonhard Euler. It turns out that C also passes through those points, and that makes 9.

There are more quite surprising properties. For example, as Euler noticed, the three points H , O and G lie on a line (called the *Euler line*). The center of the nine-point circle must lie on this line also.

Further, if one draws the tangents to the circle C at points L , M , N , they will form a triangle whose sides are all parallel to the sides of $\triangle DEF$, the triangle formed by the feet of the altitudes, often called the *orthic triangle*. So in particular the triangle formed by the tangents to C at L , M , N is similar to the $\triangle DEF$.

Feuerbach's Theorem

Finally, Karl Feuerbach proved two more statements about C . Recall that the *incircle* is the inscribed circle of $\triangle ABC$. You can find it by locating the common intersection of the bisectors of the interior angles of

$\triangle ABC$ – this will be the *incenter* of the triangle, i.e. the center of the unique inscribed circle. Feuerbach showed that our C is always tangent to the incircle, i.e. they touch, and at one point only.

Now for the last step. Extend the sides of the triangle so that you will have three “nooks”, regions outside the triangle each bounded by one side of $\triangle ABC$, and two extensions of the other two sides. In each of these “nooks” you can draw a unique circle tangent to these three “walls”. These circles (called the *excircles* of $\triangle ABC$) will typically be quite large, and will all live on the outside of the triangle. It turns out that each of them is again tangent to C .

All this is relatively basic geometry, dating back to the early 19th century.

You can probably find some more information by searching groups.google.com for words such as “Euler line”, “Feuerbach’s theorem”, etc.