THE BAILEY TRANSFORM AND D. B. SEARS

BY

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Dedicated to the memory of D. B. Sears.

ABSTRACT. In recent years, the pioneering q-series methods initiated by Rogers and extended by Bailey have found extensive applications in number theory and physics. In this paper we show how some of the work of D. B. Sears can be understood and extended once it has been related to the Bailey Transform.

1. Introduction. In a series of papers written in the late 1940's and early 1950's [6]–[10], D. B. Sears provided an extensive investigation of basic hypergeometric series. His discoveries have been used many times in application (cf. [5]).

Unfortunately his work predates the explosion of interest in q-series. This means that his slightly idiosyncratic notation has masked some of the relations of his work with more recent developments.

In this paper we propose to examine just one of his discoveries, Theorem 3 of [8; p. 167], which we restate without use of Sears' umbral notation.

Sears' Theorem. Let $\theta_r (r \geqq 0)$ denote a sequence of complex numbers. Then subject to absolute convergence conditions

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n \theta_n}{(q)_n (e)_n} = \sum_{s=0}^{\infty} \left(\frac{ab}{e}\right)^s \frac{(e/a)_s (e/b)_s}{(q)_s (e)_s} \sum_{t=0}^{\infty} \frac{\left(\frac{ab}{e}\right)_t \theta_{s+t}}{(q)_t}, \tag{1.1}$$

where

$$(a)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}). \tag{1.2}$$

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Sears' principal use of this quite general result is to prove the following identity which has turned out to be quite useful [5; p. 41].

$$(1.3)\qquad {}_4\phi_3\left(\begin{matrix} a, b, c, q^{-n}; q, q \\ e, g, h \end{matrix}\right) = \frac{(g/c)_n(eg/ab)_n}{(g)_n(eg/abc)_n}\, {}_4\phi_3\left(\begin{matrix} e/a, e/b, c, q^{-n}; q, q \\ e, cq^{1-n}/g, cq^{1-n}/h \end{matrix}\right),$$

where $abc = eghq^{n-1}$, and

$$(1.4)\qquad {}_{r+1}\phi_r\left(\begin{matrix} \alpha_0, \alpha_1, \ldots, \alpha_r; q, t \\ \beta_1, \ldots, B_r \end{matrix}\right) = \sum_{n=0}^{\infty} \frac{(\alpha_0)_n(\alpha_1)_n \ldots (\alpha_r)_n t^n}{(q)_n(\beta_1)_n \cdots (\beta_r)_n}.$$

In Section 2, we shall show that Sears' Theorem is, in fact, an instance of the Bailey transform [11; §2.4, p. 58]. Given this observation we are led to a Bailey Chain-like iteration of Sears' Theorem in Section 3. In Section 4, we examine some of the corollaries of the general results in Section 3.

2. Bailey's Transform.

In the 1940's, W. N. Bailey [4] boiled the methods of L. J. Rogers down into one simple assertion:

Bailey's Transform ([4; p. 1] [11; §2.4, p.58])

If

$$\beta_n = \sum_{r=n}^{n} \alpha_r u_{n-r} v_{n+r},$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{n-r} v_{n+r},$$

then

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n.$$

As Bailey remarks [4; p. 1], "The proof is almost trivial." From this point, Bailey then examines instances wherein

$$(2.1)\qquad \delta_n = (\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n,$$

and

$$(2.2)\qquad \gamma_n = (\rho_1)_n(\rho_2)_n \left(\frac{aq}{\cdots}\right)^n \frac{(aq^{n+1}/\rho_1)_\infty(aq^{n+1}/\rho_2)_\infty}{\cdots}.$$

Then in a Section 4 less than a half page in length, Bailey provides the recipe for what later becomes Bailey's Lemma [1; p. 270], [2; p. 25] the basis of the Bailey Chains [1; p. 278], [2; p. 28].

In each of these applications, the γ_n and δ_n are a fixed pair, and the "Bailey pair" (α_n, β_n) are then specialized in a variety of ways.

Actually Sears' Theorem is the following instance of Bailey's Transform:

$$u_n = \frac{\left(\frac{ab}{e}\right)_n}{(q)_n} \,, \qquad v_n = 1$$

$$\delta_n = \theta_n \quad , \quad \gamma_s = \sum_{t=0}^{\infty} \frac{\left(\frac{ab}{e}\right)_t \theta_{s+t}}{(q)_t} \,,$$

and

$$\beta_n = \frac{(a)_n (b)_n}{(q)_n (e)_n} \quad , \quad \alpha_n = \frac{\left(\frac{ab}{e}\right)^n \left(\frac{e}{a}\right)_n \left(\frac{e}{b}\right)_n}{(q)_n (e)_n}$$

All that is necessary to guarantee that this is a valid instance of Bailey's Transform to check the hypothesis

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r} = \sum_{r=0}^{n} \alpha_r \frac{\left(\frac{ab}{e}\right)_{n-r}}{(q)_{n-r}} \,.$$

However, this is nothing more than the q-Pfaff-Saalschutz identity [5; eq. (1.7.2), p. 13] in full generality.

Thus Sears' Theorem is an instance of the Bailey Transform in which the α_n and β_n are strictly specified while the γ_n, δ_n pair are left in terms of the free sequence θ_n.

Thus suggests that the Bailey Chain style of iteration may well have some counterpart. It turns out that there is less freedom here; however there is one possible iteration that is the counterpart of the unitary Bailey Chain [3]. This provides Theorem 1 of the next section.

3. The Iteration

Theorem 1. *For integers* $0 \leqq \ell \leqq r-1$

(3.1)
$$
\begin{aligned}
&{}_{r+1}\phi_r\left(\begin{matrix} C_0, C_1, \ldots, C_r; q, x \\ f_1, \ldots, f_r \end{matrix}\right) \\
&= \sum_{i_1,\ldots,i_\ell \geqq 0} \frac{\left(\frac{f_1}{C_1}\right)_{i_1}\left(\frac{f_2}{C_2}\right)_{i_2}\cdots\left(\frac{f_\ell}{C_\ell}\right)_{i_\ell}}{(q)_{i_1}(q)_{i_2}\cdots(q)_{i_\ell}} \\
&\times \frac{(C_2)_{i_1}(C_3)_{i_1+i_2}\cdots(C_{\ell+1})_{i_1+\cdots+i_\ell}(C_{\ell+2})_{i_1+\cdots+i_\ell}\cdots(C_r)_{i_1+\cdots+i_\ell}}{(f_1)_{i_1}(f_2)_{i_1+i_2}\cdots(f_\ell)_{i_1+\cdots+i_\ell}(f_{\ell+1})_{i_1+\cdots+i_\ell}\cdots(f_r)_{i_1+\cdots+i_\ell}} \\
&\times \left(\frac{f_1}{C_0}\right)_{i_1}\left(\frac{f_1 f_2}{C_0 C_1}\right)_{i_1+i_2}\cdots\left(\frac{f_1\cdots f_\ell}{C_0\cdots C_{\ell-1}}\right)_{i_1+\cdots+i_\ell} \\
&\times \left(\frac{C_0 C_1 x}{f_1}\right)^{i_1+\cdots+i_\ell}\left(\frac{C_2}{f_2}\right)^{i_2+\cdots+i_\ell}\left(\frac{C_3}{f_3}\right)^{i_3+\cdots+i_\ell}\cdots\left(\frac{C_\ell}{f_\ell}\right)^{i_\ell} \\
&\times {}_{r-\ell+1}\phi_{r-\ell}\left(\begin{matrix} \frac{C_0\cdots C_\ell}{f_1\cdots f_\ell}, C_{\ell+1}q^{i_1+\cdots+i_\ell}, \cdots, C_r q^{i_1+\cdots+i_\ell}; q, x \\ f_{\ell+1}q^{i_1+\cdots+i_\ell}, \cdots, f_r q^{i_1+\cdots+i_\ell} \end{matrix}\right)
\end{aligned}
$$

Proof. We proceed by mathematical induction. The case $\ell = 0$ is an immediate tautology.

So we assume the theorem true for a particular $\ell < r-1$, and apply Sears' Theorem to the ${}_{r-\ell+1}\phi_{r-\ell}$ on the right-hand side with

$$
a = \frac{C_0\cdots C_\ell}{f_1\cdots f_\ell} \quad , \quad b = C_{\ell+1}q^{i_1+\cdots+i_\ell} \quad , \quad e = f_{\ell+1}q^{i_1+\cdots+i_\ell}
$$

and

$$
\theta_n = \frac{(C_{\ell+2}q^{i_1+\cdots+i_\ell})_n\cdots(C_r q^{i_1+\cdots+i_\ell})_n x^n}{(f_{\ell+2}q^{i_1+\cdots+i_\ell})_n\cdots(f_r q^{i_1+\cdots+i_\ell})_n} \, .
$$

Hence Sears' Theorem tells us that

(3.2)

$$
\begin{aligned}
&{}_{r-\ell+1}\phi_{r-\ell}\left(\begin{matrix}\frac{C_0\cdots C_\ell}{f_1\cdots f_\ell}, C_{\ell+1}q^{i_1+\cdots+i_\ell},\cdots, C_r q^{i_1+\cdots+i_\ell}; q, x\\ f_{\ell+1}q^{i_1+\cdots+i_\ell},\cdots, f_r q^{i_1+\cdots+i_\ell}\end{matrix}\right)\\
&=\sum_{i_{\ell+1}\geqq 0}\frac{\left(\frac{C_0\cdots C_\ell C_{\ell+1}}{f_1\cdots f_\ell f_{\ell+1}}\right)^{i_{\ell+1}}\left(\frac{f_1\cdots f_\ell f_{\ell+1}q^{i_1+\cdots+i_\ell}}{C_0\cdots C_\ell}\right)_{i_{\ell+1}}\left(\frac{f_{\ell+1}}{C_{\ell+1}}\right)_{i_{\ell+1}}}{(q)_{i_{\ell+1}}\,(f_{\ell+1}q^{i_1+\cdots+i_\ell})_{i_{\ell+1}}}\\
&\quad\times\sum_{t=0}^{\infty}\frac{\left(\frac{C_0\cdots C_\ell C_{\ell+1}}{f_1\cdots f_\ell f_{\ell+1}}\right)_t (C_{\ell+2}q^{i_1+\cdots+i_\ell})_{i_{\ell+1}+t}\cdots(C_r q^{i_1+\cdots+i_\ell})_{i_{\ell+1}+t}x^{i_{\ell+1}+t}}{(q)_t(f_{\ell+2}q^{i_1+\cdots+i_\ell})_{i_{\ell+1}+t}\cdots(f_r q^{i_1+\cdots+i_\ell})_{i_{\ell+1}+t}}\\
&=\sum_{i_{\ell+1}\geqq 0}\frac{\left(\frac{C_0\cdots C_{\ell+1}}{f_1\cdots f_{\ell+1}}\right)^{i_{\ell+1}}\left(\frac{f_1\cdots f_{\ell+1}q^{i_1+\cdots+i_\ell}}{C_0\cdots C_\ell}\right)_{i_{\ell+1}}\left(\frac{f_{\ell+1}}{C_{\ell+1}}\right)_{i_{\ell+1}}}{(q)_{i_{\ell+1}}(f_{\ell+1}q^{i_1+\cdots+i_\ell})_{i_{\ell+1}}}\\
&\quad\times\frac{(C_{\ell+2}q^{i_1+\cdots+i_\ell})_{i_{\ell+1}}\cdots(C_r q^{i_1+\cdots+i_\ell})_{i_{\ell+1}}x^{i_{\ell+1}}}{(f_{\ell+2}q^{i_1+\cdots+i_\ell})_{i_{\ell+1}}\cdots(fr\, q^{i_1+\cdots+i_\ell})_{i_{\ell+1}}}\\
&{}_{r-\ell}\phi_{r-\ell-1}\left(\begin{matrix}\frac{C_0\cdots C_{\ell+1}}{f_1\cdots f_{\ell+1}}, C_{\ell+2}q^{i_1+\cdots+i_{\ell+1}},\cdots, C_r q^{i_1+\cdots+i_{\ell+1}}; q, x\\ f_{\ell+2}q^{i_1+\cdots+i_{\ell+1}},\cdots, f_r\, q^{i_1+\cdots+i_{\ell+1}}\end{matrix}\right)
\end{aligned}
$$

Inserting this latter expression for the ${}_{r-\ell+1}\phi_{r-\ell}$ in (3.1) we obtain (after simplification arising from the fact that $(A)_n(Aq^n)_m=(A)_{n+m}$) (3.1) with ℓ replaced by $\ell+1$. Consequently Theorem 1 is proved by mathematical induction. □

There are a number of natural corollaries of Theorem 1. We choose two of the most obvious.

Corollary 1.

(3.3)

$$
\begin{aligned}
&{}_{r+1}\phi_r\left(\begin{matrix}C_0, C_1,\ldots,C_r; q, \frac{f_1\cdots f_r}{C_0C_1\cdots C_r}\\ f_1,\ldots,f_r\end{matrix}\right)\\
&=\frac{\left(\frac{f_r}{C_r}\right)_\infty\left(\frac{f_1\cdots f_r}{C_0\cdots C_{r-1}}\right)_\infty}{(f_r)_\infty\left(\frac{f_1\cdots f_r}{C_0\cdots C_r}\right)_\infty}\sum_{i_1,\ldots,i_{r-1}\geqq 0}\frac{\left(\frac{f_2}{C_2}\right)^{i_1}\left(\frac{f_3}{C_3}\right)^{i_1+i_2}\cdots\left(\frac{f_r}{C_r}\right)^{i_1+\cdots+i_{r-1}}}{(q)_{i_1}(q)_{i_2}\cdots(q)_{i_{r-1}}}\\
&\quad\times\frac{(C_2)_{i_1}(C_3)_{i_1+i_2}\cdots(C_r)_{i_1+\cdots+i_{r-1}}}{(f_1)_{i_1}(f_2)_{i_1+i_2}\cdots(f_{r-1})_{i_1+\cdots+i_{r-1}}}\\
&\quad\times\frac{\left(\frac{f_1}{C_1}\right)_{i_1}\cdots\left(\frac{f_{r-1}}{C_{r-1}}\right)_{i_{r-1}}\left(\frac{f_1}{C_0}\right)_{i_1}\left(\frac{f_1f_2q^{i_1}}{C_0C_1}\right)_{i_2}\cdots\left(\frac{f_1\cdots f_{r-1}q^{i_1+\cdots+i_{r-2}}}{C_0C_1\cdots C_{r-2}}\right)_{i_{r-1}}}{\left(\frac{f_1\cdots f_r}{C_0\cdots C_{r-1}}\right)_{i_1+\cdots+i_{r-1}}}
\end{aligned}
$$

Proof. In Theorem 1, set $\ell = r-1, x = \frac{f_1\cdots f_r}{C_0\cdots C_r}$ and note that

$$
{}_2\phi_1\left(\begin{matrix}\frac{C_0\cdots C_{r-1}}{f_1\cdots f_{r-1}}, C_r q^{i_1+\cdots+i_{r-1}}; q, \frac{f_1\cdots f_r}{C_0\cdots C_r}\\ f_r q^{i_1+\cdots+i_{r-1}}\end{matrix}\right) = \frac{\left(\frac{f_1\cdots f_r q^{i_1+\cdots+i_{r-1}}}{C_0\cdots C_{r-1}}\right)_\infty \left(\frac{f_r}{C_r}\right)_\infty}{(f_r q^{i_1+\cdots+i_{r-1}})_\infty \left(\frac{f_1\cdots f_r}{C_0 C_1\cdots C_r}\right)_\infty}, \tag{3.4}
$$

by the q-analog of Gauss's theorem [5; eq. (1.5.1), p. 10]. Simplification then yields Corollary 1. □

Corollary 2.

$$
\begin{aligned}
&{}_{r+1}\phi_r\left(\begin{matrix}C_0, C_1, \ldots, C_r & ; q, -\frac{f_1\cdots f_{r-1}q}{C_0\cdots C_{r-1}}\\ f_1, \ldots, f_{r-1}, \frac{f_1\cdots f_{r-1}C_r q}{C_0\cdots C_{r-1}} & \end{matrix}\right)\\
&= \frac{(-q)_\infty}{\left(-\frac{f_1\cdots f_{r-1}q}{C_0\cdots C_{r-1}}\right)_\infty \left(\frac{f_1\cdots f_{r-1}C_r q}{C_0\cdots C_{r-1}}\right)_\infty}\\
&\times \sum_{i_1,\ldots,i_{r-1}\geqq 0} \frac{\left(\frac{f_1}{C_1}\right)_{i_1}\cdots\left(\frac{f_{r-1}}{C_{r-1}}\right)_{i_{r-1}} (C_2)_{i_1}(C_3)_{i_1+i_2}\cdots(C_r)_{i_1+\cdots+i_{r-1}}}{(q)_{i_1}\cdots(q)_{i_{r-1}}(f_1)_{i_1}(f_2)_{i_1+i_2}\cdots(f_{r-1})_{i_1+\cdots+i_{r-1}}}\\
&\times \frac{\left(\frac{f_1}{C_1}\right)_{i_1}\left(\frac{f_1 f_2}{C_0 C_1}\right)_{i_1+i_2}\cdots\left(\frac{f_1\cdots f_{r-1}}{C_0\cdots C_{r-2}}\right)_{i_1+\cdots+i_{r-1}}}{\left(\frac{f_1 f_2}{C_0 C_1}\right)_{i_1}\left(\frac{f_1 f_2 f_3}{C_0 C_1 C_2}\right)_{i_1+i_2}\cdots\left(\frac{f_1\cdots f_{r-1}}{C_0\cdots C_{r-2}}\right)_{i_1+\cdots+i_{r-2}}}\\
&\times \left(\frac{f_2}{C_2}\right)^{i_1}\left(\frac{f_3}{C_3}\right)^{i_1+i_2}\cdots\left(\frac{f_{r-1}}{C_{r-1}}\right)^{i_1+\cdots+i_{r-1}} (-q)^{i_1+\cdots+i_{r-1}}\\
&\times (C_r q^{1+i_1+\cdots+i_{r-1}}; q^2)_\infty \left(\frac{f_1^2\cdots f_{r-1}^2 C_r q^{2+i_1+\cdots+i_{r-1}}}{C_0^2\cdots C_{r-1}^2}; q^2\right)_\infty .
\end{aligned} \tag{3.5}
$$

Proof. In Theorem 1, set $\ell = r-1$, $f_r = \frac{f_1\cdots f_{r-1}C_r q}{C_0\cdots C_{r-1}}$, $x = -\frac{f_1\cdots f_{r-1}q}{C_0\cdots C_{r-1}}$ and note that

$$
\begin{aligned}
&{}_2\phi_1\left(\begin{matrix}\frac{C_0\cdots C_{r-1}}{f_1\cdots f_{r-1}}, C_r q^{i_1+\cdots+i_{r-1}}; q, -\frac{f_1\cdots f_{r-1}q}{C_0\cdots C_{r-1}}\\ \frac{f_1\cdots f_{r-1}C_r q^{1+i_1+\cdots+i_{r-1}}}{C_0\cdots C_{r-1}}\end{matrix}\right)\\
&= \frac{(-q)_\infty (C_r q^{1+i_1+\cdots+i_{r-1}}; q^2)_\infty \left(\frac{f_1^2\cdots f_{r-1}^2 C_r q^{2+i_1+\cdots+i_{r-1}}}{C_0^2\cdots C_{r-1}^2}; q^2\right)_\infty}{\left(\frac{f_1\cdots f_{r-1}C_r q^{i_1+\cdots+i_{r-1}+1}}{C_0\cdots C_{r-1}}\right)_\infty \left(-\frac{f_1\cdots f_{r-1}q}{C_0\cdots C_{r-1}}\right)_\infty},
\end{aligned} \tag{3.6}
$$

by the Bailey-Daum summation [5; eq. (1.8.1), p. 14]. Simplification then yields Corollary 2.

We are unaware of any instances of Corollary 2 appearing before; so we conclude this section by exhibiting the $r = 2$ case of Corollary 2. Namely,

(3.7)

$$
\begin{aligned}
&{}_3\phi_2\left(\begin{matrix} C_0, C_1, C_2; -\frac{qf_1}{C_0C_1} \\ f_1, \frac{C_2f_1q}{C_0C_1} \end{matrix}\right) \\
&= \frac{(-q)_\infty}{\left(-\frac{qf_1}{C_0C_1}\right)_\infty \left(\frac{f_1C_2q}{C_0C_1}\right)_\infty} \\
&\quad \times \sum_{i\geqq 0} \frac{\left(\frac{f_1}{C_0}\right)_i \left(\frac{f_1}{C_1}\right)_i (C_2)_i}{(q)_i(f_1)_i} (C_2q^{i+1};q^2)_\infty \left(\frac{C_2f_1^2q^{i+2}}{C_0^2C_1^2};q^2\right)_\infty (-q)^i \\
&= \frac{(-q)_\infty (C_2q;q^2)_\infty \left(\frac{f_1^2C_2q^2}{C_0^2C_1^2};q^2\right)_\infty}{\left(-\frac{f_1q}{C_0C_1}\right)_\infty \left(\frac{f_1C_2q}{C_0C_1}\right)_\infty} \sum_{i\geqq 0} \frac{\left(\frac{f_1}{C_0}\right)_{2i} \left(\frac{f_1}{C_1}\right)_{2i} (C_2)_{2i} q^{2i}}{(q)_{2i}(f_1)_{2i}(C_2q;q^2)_i \left(\frac{f_1^2C_2q^2}{C_0^2C_1^2};q^2\right)_i} \\
&- \frac{(-q)_\infty (C_2q^2;q^2)_\infty \left(\frac{f_1^2C_2q^3}{C_0^2C_1^2};q^2\right)_\infty}{\left(-\frac{f_1q}{C_0C_1}\right)_\infty \left(-\frac{f_1C_2q}{C_0C_1}\right)_\infty} \sum_{i\geqq 0} \frac{\left(\frac{f_1}{C_0}\right)_{2i+1} \left(\frac{f_1}{C_1}\right)_{2i+1} (C_2)_{2i+1} q^{2i+1}}{(q)_{2i+1}(f_1)_{2i+1}(C_2q^2;q^2)_i \left(\frac{f_1^2C_2q^3}{C_0^2C_1^2};q^2\right)_i}.
\end{aligned}
$$

4. Applications.

The object here is to illustrate how these results may be applied. We note in passing that Corollary 1 with $r = 3$ may be applied to Bailey's identities (6.1) and (6.3) of [4] while Corollary 1 with $r = 4$ may be applied to Bailey's identities (6.2) and (6.4) of [4]. Thus we may obtain a variety of summations of the Rogers-Ramanujan type for double and triple q-series.

The novelty of Corollary 2 suggests we examine at least one special case. Let C_0 and $C_1 \to \infty$, and $C_2 = q$ in (3.7).

(4.1)

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} f_1^n}{(f_1)_n} \\
&= (-q)_\infty (q^2;q^2)_\infty \sum_{i=0}^{\infty} \frac{q^{2i}}{(f_1)_{2i}(q^2;q^2)_i} \\
&- \sum_{i=0}^{\infty} \frac{q^{2i+1}}{(f_1)_{2i+1}(q;q^2)_{i+1}}.
\end{aligned}
$$

If we set $f_1 = -q$, then the left-hand side of (4.1) is one of the fifth-order mock theta functions defined by Ramanujan, and the resulting identity is equivalent to one of those discovered by G. N. Watson [12].

References

1. Andrews, G. E., *Multiple series Rogers-Ramanujan type identities*, Pacific J. Math. **114** (1984), 267–283.
2. Andrews, G. E., *On proofs of the Rogers-Ramanujan identities*, From q-Series and Partitions, IMA Volumes in Math. and Its Appl., D. Stanton ed., Springer, NY, 18 (1989), 1–14.
3. Andrews, G. E., *q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra*, CBMS Regional Conference Lecture Series **66** (1986), Amer. Math. Soc., Providence, RI.
4. Bailey, W. N., *Identities of the Rogers-Ramanujan type*, Proc. London Math. Soc. (2) **50** (1949), 1–10.
5. Gasper, G. and Rahman, M., *Basic Hypergeometric Series*, Encycl. Math and Its Appl., Vol. 35, G.-C. Rota ed., Cambridge University Press, Cambridge, 1990.
6. Sears, D. B., *Transformations of basic hypergeometric functions of special type*, Proc. London Math. Soc. (2) **52** (1951), 467–483.
7. Sears, D. B., *On the transformation theory of hypergeometric functions and cognate trigonometric series*, Proc. London Math. Soc. (2) **53** (1951), 138–157.
8. Sears, D. B., *On the transformation theory of basic hypergeometric functions*, Proc. London Math. Soc. (2) **53** (1951), 158–180.
9. Sears, D. B., *Transformations of basic hypergeometric functions of any order*, Proc. London Math. Soc. (2) **53** (1951), 181–191.
10. Sears, D. B., *Two identities of Bailey*, J. London Math. Soc. **27** (1952), 510-511.
11. Slater, L. J., *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, (1966).
12. Watson, G. N., *The mock theta functions (2)*, Proc. London Math. Soc. (2) **42** (1937), 274–304.