Rational approximation to a new generalization of the Rogers-Ramanujan continued fraction

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1. HISTORY: IRRATIONALITY FOR q-SPECIAL FUNCTIONS

In 1915 F. Bernstein and O. Szász showed that if $r/s = q, v \in \mathbb{Q}$, then

$$\sum_{n\geq 0} q^{n^2} v^n \notin \mathbb{Q}.$$

This is the first work on the irrationality of q-special functions that we are aware of. Since then, there have been a number of papers on this general topic, see the paper of Bundschuh and Waldschmidt [1] for a brief history. More recently Osgood [3] and then Shiokawa [4] showed that the Rogers-Ramanujan continued fraction

$$1+\frac{xq}{1+}\,\frac{xq^2}{1+}\,\frac{xq^3}{1+\dots}$$

is irrational for all $x \in \mathbb{Q}$ and $q = r/s$ with $|r|^2 < |s|$. In fact Shiokawa gave a best possible measure of the irrationality. This function may be expressed as the quotient $f(x)/f(xq)$, where f is the analytic solution of the q-difference equation

$$f(x) = f(xq) + xqf(xq^2).$$

Recently Duverney [2] constructed linear approximations to powers of solutions of more general q-difference equations obtaining bounds on the number of algebraic values of given degree. In this paper, we obtain through elementary means irrationality measures for the quotient $g(x, yz^{-1}, z^2)/g(xyz, yz, z^2)$ evaluated at rational values, where g is the analytic solution of the difference equation

$$g(x, y, z) = g(xyz, yz, z) + xyzg(xy^2z^3, yz^2, z).$$

This paper thus forms the beginnings of the arithmetical investigation into solutions of difference equations under the more general family of mappings in n variables typified by:

$$\sigma : (x_1, x_2, \dots, x_n) \mapsto (x_1x_2\cdots x_n, x_2x_3\cdots x_n, x_3x_4\cdots x_n, \dots, x_n).$$

Here our methods are elementary, being extensions of those of Shiokawa.

We consider the continued fraction

$$G(x, y, z) = 1+\frac{xyz}{1+}\,\frac{xy^2z^{2^2}}{1+}\,\frac{xy^3z^{3^2}}{1+\dots}. \tag{1}$$

It is easy to show that $G(x, y, z) = g(x, yz^{-1}, z^2)/g(xyz, yz, z^2)$. For this continued fraction we obtain extensions of the results of Shiokawa and obtain irrationality measures. In future papers we will apply more sophisticated methods to functions satisfying difference equations in the operator σ and similar operators to obtain information about algebraic values.

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[1] The first author's research was partially supported by grant DMS-9705782.

2. Main theorems

Our first theorem gives a uniform best possible measure of irrationality for G. That the measure is best possible are consequences of Theorem 2. It is interesting to compare our result with that of Shiokawa which we state here for convenience.

Theorem (Shiokawa). *Let a, b, c, and d be non-zero integers with $(a,b)=1$, $(c,d)=1$. Suppose that $|d|>c^2$. Then there is a positive constant $\widetilde{D}(a,b,c,d)$ such that for all integers p, q $(q \geq q_0)$,*

$$\left|G\left(\frac{a}{b},\frac{c}{d},1\right)-\frac{p}{q}\right|>\widetilde{D}q^{-2-2\widetilde{A}-\widetilde{B}/\sqrt{\log q}},$$

where

$$\widetilde{A}=\frac{\log|c|}{\log|d/c^2|},$$

and

$$\widetilde{B}=\frac{\log|a^2d|-2\widetilde{A}\log|b/a^2|}{\sqrt{\log|d/c^2|}}.$$

Our result extending this follows. In the course of our proof we will be able to recover Shiokawa's results.

Theorem 1. *Let a, b, c, d, e and f be non-zero integers with $(a,b)=1$, $(c,d)=1$, and $(e,f)=1$. Suppose that*

$$|d|>c^2 \text{ and } |f|>e^2. \tag{2}$$

Then there exists a positive constants $D(a,b,c,d,e,f)$, such that

$$\left|G\left(\frac{a}{b},\frac{c}{d},\frac{e}{f}\right)-\frac{p}{q}\right|>Dq^{-2-A-B(\log q)^{-1/3}-C(\log q)^{-2/3}},$$

for all integers p, $q(\geq q_0)$, where

$$\begin{aligned}
A&=\log|e|^2\cdot(\log|f/e^2|)^{-1},\\
B&=\frac{6^{2/3}}{2}(\log|f/e^2|)^{-2/3}\Big\{\log|cf|-(\log|f/e^2|)^{-1}\log|e|\log|d/c^2|\Big\},\\
C&=\frac{6^{1/3}}{2}(\log|f/e^2|)^{-1/3}\Big\{-\frac{A}{3}\log|b^3e^2/a^3f|+\frac{A}{2}\log|d/c^2|E\\
&\qquad-\log|cf|\log|d/c^2|(\log|f/e^2|)^{-1}+\frac{1}{3}\log|a^6d^3f/e^2|\Big\}, \quad \textit{and}\\
E&=\begin{cases}(\log|f/e^2|)^{-1}\log|d/c^2|, & |d|\leq|c^2|\\ (\log|f/e^2|)^{-1}\log|d/c^2|+2, & \textit{otherwise.}\end{cases}
\end{aligned}$$

It follows from Theorem 1 that $G(x,y,z)$ is irrational when x, y and z are rational with $z=e/f$ satisfying (2), but this can be seen by more direct methods. The point of the theorem is the quality of rational approximation possible.

Corollary 1. *Let a, b, e and f be non-zero integers with $|f| > |e|^2$. Then $G\left(\frac{a}{b}, 1, \frac{e}{f}\right)$ is an irrational number and there is a positive constant $D = D(a,b,e,f)$ such that*

$$\left|G\left(\frac{a}{b}, 1, \frac{e}{f}\right) - \frac{p}{q}\right| > Dq^{-2-A-B(\log q)^{-1/3}-C(\log q)^{-2/3}}$$

for all integers p, $q(\geq q_0)$, where

$$A = (\log|e^2|)\cdot(\log|f/e^2|)^{-1},$$

$$B = \frac{6^{2/3}}{2}\log|f|(\log|f/e^2|)^{-2/3},$$

$$C = \frac{6^{1/3}}{2}(\log|f/e^2|)^{-1/3}\Big\{-\frac{A}{3}\log|b^3e^2/a^3f| + \frac{1}{3}\log|a^6f^3/e^2|\Big\}.$$

The following corollary will be used to show that Theorem 1 is best possible in the uniform sense.

Corollary 2. *Let b, d and f be positive integers with $f \geq 2$.*
Then, for any $\epsilon > 0$, there is a positive constant $q_o(b,d,f,\epsilon)$ such that

$$\left|G\left(\frac{1}{b}, \frac{1}{d}, \frac{1}{f}\right) - \frac{p}{q}\right| > D(1-\epsilon)q^{-2-B(\log q)^{-1/3}-C(\log q)^{-2/3}}$$

for all integers p, $q(\geq q_o)$, where

$$B = \frac{6^{2/3}}{2}(\log f)^{1/3},$$

$$C = \frac{6^{1/3}}{2}(\log f)^{2/3},$$

$$D = d^{-1/4}f^{-1/3}\exp\{-1/8(\log f)^{-1}(\log d)^2\}.$$

Notice now that Theorem 1 is uniformly best possible since we have the following theorem:

Theorem 2. *Let b, d and f be positive integers with $f \geq 2$.*
Then, for any $\epsilon > 0$,

$$\left|G\left(\frac{1}{b}, \frac{1}{d}, \frac{1}{f}\right) - \frac{p}{q}\right| < D(1+\epsilon)q^{-2-B(\log q)^{-1/3}-C(\log q)^{-2/3}}$$

for infinitely many integers p, $q(\geq 0)$, where

$$B = \frac{6^{2/3}}{2}(\log f)^{1/3},$$

$$C = \frac{6^{1/3}}{2}(\log f)^{2/3},$$

$$D = d^{-1/4}f^{-1/3}\exp\{-1/8(\log f)^{-1}(\log d)^2\}.$$

Since the constants are them same in Theorem 2 and Corollary 2, it follows that Theorem 1 is best possible.

3. PROOFS

This section contains proofs of our theorems. We will also obtain Shiokawa's results along the way with little extra effort.

The following lemma is a variant of one used by Shiokawa in [4]. As a proof was not given there and requires a little work, we present it here.

Lemma. *Let* $a_1, a_2, a_3, \cdots$ *be a sequence of real numbers such that* $|a_n a_{n+1}| > 4$ *and* $\sum_{n=1}^{\infty} |a_n a_{n+1}|^{-1} = \sigma < \infty$.

Define as usual $p_n = a_n p_{n-1} + p_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$ $(n \geq 1)$ *with* $p_o = q_{-1} = 0$, $p_{-1} = q_o = 1$. *Then* $p_n/(a_2 a_3 \ldots a_n)$ *and* $q_n/(a_1 a_2 \ldots a_n)$ *converge to finite non-zero limits and they satisfy*

$$\exp(-\sigma) < |p_n/(a_2 a_3 \ldots a_n)| < \exp(3\sigma), \tag{3}$$

$$\exp(-\sigma) < |q_n/(a_1 a_2 \ldots a_n)| < \exp(3\sigma) \tag{4}$$

so that the continued fraction

$$\frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+\ldots} = \lim_{n\to\infty} \frac{p_n}{q_n}$$

is convergent.

Proof of Lemma. Define a sequence V_n by $q_n = a_1 a_2 \cdots a_n \prod_{j=1}^{n-1} \left(1 + \dfrac{V_j}{a_j a_{j+1}}\right)$.

Then, from the recurrence relation for q_n,

$$V_{n+1} = \frac{1}{1 + \dfrac{V_n}{a_n a_{n+1}}},$$

where $V_1 = 1$, $a_o = 1$, $n \geq 1$. If $\left|\dfrac{V_j}{a_j a_{j+1}}\right| < 1$ for $j \geq 1$, then

$$\left|\frac{q_n}{a_1 a_2 \cdots a_n}\right| = \prod_{j=1}^{n-1} \left(1 + \frac{V_j}{a_j a_{j+1}}\right) = \frac{q_n}{a_1 a_2 \cdots a_n}.$$

Now

$$Q = \log\left(\frac{q_n}{a_1 a_2 \cdots a_n}\right) = \sum_{j=1}^{n-1} \log\left(1 + \frac{V_j}{a_j a_{j+1}}\right).$$

Let $\overline{V} = \overline{\lim}_{n\to\infty} |V_n|$. It is easy to show by induction that if $|a_j a_{j+1}| > 4$, then $\dfrac{2}{3} < V_j < 2$, $j \geq 1$. So $\overline{V} \leq 2$ and $\left|\dfrac{V_j}{a_j a_{j+1}}\right| < \dfrac{1}{2}$. Hence

$$Q = \sum_{j=1}^{n-1} \log\left(1 + \frac{V_j}{a_j a_{j+1}}\right) < \frac{3}{2}\sum_{j=1}^{n-1} \frac{V_j}{a_j a_{j+1}} \leq \frac{3}{2}\sum_{j=1}^{n-1} \left|\frac{V_j}{a_j a_{j+1}}\right|$$

$$\leq \frac{3}{2}\sum_{j=1}^{\infty} \left|\frac{V_j}{a_j a_{j+1}}\right| \leq \frac{3}{2}\overline{V}\sum_{j=1}^{\infty} \frac{1}{|a_j a_{j+1}|} = \frac{3}{2}\overline{V}\sigma.$$

On the other hand,

$$Q = \sum_{j=1}^{n-1} \log\left(1 + \frac{V_j}{a_j a_{j+1}}\right) > \frac{1}{2}\sum_{j=1}^{n-1} \frac{V_j}{a_j a_{j+1}} \geq \frac{1}{2}\sum_{j=1}^{n-1} \frac{-|V_j|}{|a_j a_{j+1}|}$$

$$\geq -\frac{1}{2}\sum_{j=1}^{\infty} \left|\frac{V_j}{a_j a_{j+1}}\right| \geq -\frac{1}{2}\overline{V}\sigma.$$

Thus $-\sigma < Q < 3\sigma$ and the result follows.

The estimate for $|p_n/(a_2 a_3 \cdots a_n)|$ follows *mutatis mutandis.*

$\square$

To apply the lemma, we transform the continued fraction (1) into the regular continued fraction

$$G(x, y, z) = 1 + \frac{1}{a_1 +}\,\frac{1}{a_2 +}\,\frac{1}{a_3 + \ldots},$$

where

$$a_{2n-1} = x^{-1} y^{-n} z^{-n(2n-1)}, \quad a_{2n} = y^{-n} z^{-n(2n+1)} \qquad \text{for } n \geq 1. \tag{5}$$

We note here that for $n \geq 1$,

$$a_n a_{n+1} = x^{-1} y^{-(n+1)} z^{-(n+1)^2}, \tag{6}$$

$$a_1 a_2 \cdots a_{2n-1} = x^{-n} y^{-n^2} z^{\frac{-n(2n+1)(2n-1)}{3}}, \tag{7-1}$$

$$a_1 a_2 \cdots a_{2n} = x^{-n} y^{-n^2-n} z^{\frac{-2n(n+1)(2n+1)}{3}} \tag{7-2}$$

and hence

$$\log|a_1 a_2 \cdots a_n| = \frac{-n(n+1)(n+2)}{6}\log|z| - \frac{1}{4}n^2 \log|y| - \frac{1}{2}n \log|xy| + C_1, \tag{8}$$

where $C_1 = \begin{cases} -\frac{1}{4}\log\left|x^2 y\right|, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even.} \end{cases}$

Proof of Theorem 1. Let $x = \frac{a}{b}$, $y = \frac{c}{d}$, $z = \frac{e}{f}$, $d_{2n-1} = |a^n c^{n^2} e^{\frac{n(2n-1)(2n+1)}{3}}|$ and $d_{2n} = |a^n c^{n^2+n} e^{\frac{2n(n+1)(2n+1)}{3}}|$. Then $d_n p_n$, $d_n q_n$ are integers and

$$\log d_n = \frac{n^3}{6}\log|e| + \frac{n^2}{4}\log|ce^2| + \frac{n}{6}\log|a^3 c^3 e^2| + C_2, \tag{9}$$

where $$C_2 = \begin{cases} \dfrac{1}{4}\log|a^2c|, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

Since the series $\sum_{n=1}^{\infty} |a_n a_{n+1}|^{-1}$ converges absolutely with $|d| > c^2,\ |f| > e^2$, there is an integer $N \geq 1$ such that $|a_n a_{n+1}| > 4\ (n \geq N)$. So we can apply the lemma to the continued fraction

$$\theta_n := \frac{1}{a_{n+1}+}\,\frac{1}{a_{n+2}+}\,\frac{1}{a_{n+3}+\ldots}, \qquad n \geq N.$$

So,

$$\left|\theta_n - \frac{p_{n,k}}{q_{n,k}}\right| = \left|\frac{p_{n,k}\theta_{n+k+1} + p_{n,k+1}}{q_{n,k}\theta_{n+k+1} + q_{n,k+1}} - \frac{p_{n,k}}{q_{n,k}}\right| < \frac{2}{|{q_{n,k}}^2 a_{n+k+1}|}$$

for sufficiently large n, where $p_{n,k}/q_{n,k}$ are k-th convergents of the continued fraction θ_n. But from (4), (5) and (9), we have, for a given $\epsilon > 0$,

$$\frac{\log|{q_{n,k}}^2 a_{n+k+1}|}{\log|d_{n,k} q_{n,k}|} > \begin{cases} \dfrac{-2\log|e/f|}{\log|f|} - \epsilon, & |e/f| \neq 1 \\ 2 - \dfrac{2\log|c|}{\log|d|} - \epsilon, & |e/f| = 1. \end{cases}$$

This establishes the irrationality of θ_n, since $d_{n+k}p_{n,k}$, $d_{n+k}q_{n,k}$ are integers and $\dfrac{-2\log|e/f|}{\log|f|} > 1$, $\quad 2 - \dfrac{2\log|c|}{\log|d|} > 1$ by assumptions.

Hence $G\left(\dfrac{a}{b}, \dfrac{c}{d}, \dfrac{e}{f}\right)$ is also irrational.

So using (4), (8), we have

$$\log|q_n| = \frac{n(n+1)(n+2)}{6}\log\left|\frac{f}{e}\right| + \frac{n^2}{4}\log\left|\frac{d}{c}\right| + \frac{n}{2}\log\left|\frac{bd}{ac}\right| + O(1) \tag{10}$$

and

$$\log\left|\frac{q_{n+1}}{d_{n+1}}\right| - \log\left|\frac{q_n}{d_n}\right| = \frac{n(n+1)}{2}\log\left|\frac{f}{e^2}\right| + \frac{n}{2}\log\left|\frac{d}{c^2}\right| + O(1). \tag{11}$$

Here $O(1)$ depends on a, b, c, d, e and f.

Hence we can choose $N_o = N_o(a,b,c,d,e,f)$ such that

$$|\theta_n| < 1/2, \qquad |q_{n-1}| < |q_n|, \qquad |q_{n-1}/d_{n-1}| < |q_n/d_n| \qquad (n \geq N_o). \tag{12}$$

Let r, s_i and F_i $(i = 1,2)$ be defined by:

$$r = -1 + 3(\log|f/e^2|)^{-1}\log|b/a^2| - \frac{3}{4}(\log|f/e^2|)^{-2}(\log|d/c^2|)^2,$$

$$F_1 = 6(\log|f/e^2|)^{-1}\Big\{\frac{1}{12}(\log|f/e^2|)^{-1}\log|d/c^2|\log|b^3e^2/a^3f| - \frac{1}{24}(\log|f/e^2|)^{-2}(\log|d/c^2|)^3 + C_4\Big\},$$

$$F_2 = -6(\log|f/e^2|)^{-1}\Big\{\frac{1}{4}(\log|f/e^2|)^{-1}\log|d/c^2|\log|bd/a^2c^2| - \frac{1}{48}(\log|f/e^2|)^{-2}(\log|f^2d/e^4c^2|)^3 + \frac{1}{6}\log|b^3d^4f^{12}/a^6c^8e^{24}| - C_5\Big\},$$

$$-s_1 = 6(\log|f/e^2|)^{-1}\log q + F_1,$$

$$-s_2 = 6(\log|f/e^2|)^{-1}\log q - F_2.$$

Now let p, q be given non-zero integers such that $q \geq \widetilde{q}_o$, $|q_{N_o}/d_{N_o}| < 4\widetilde{q}_o$,

$$-s_i > 0 \quad \text{and } \Delta_i = \left(\frac{r}{3}\right)^3 + \left(\frac{s_i}{2}\right)^2 > 0 \quad \text{for } i = 1, 2.$$

Then by (2) and (11), there is an integer $M = M(q) \geq N_o$ such that

$$|q_{M-1}/d_{M-1}| \leq 4q < |q_M/d_M|. \tag{13}$$

By the formula $p_M q_{M-1} - p_{M-1} q_M = \pm 1$, at least one of $p_{M-1}q - q_{M-1}p$, $p_M q - q_M p$ is different from zero. Assume first that $p_M q - q_M p \neq 0$.

Then we have

$$d_M q_M \left\{ G\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right) - \frac{p}{q} \right\} = \frac{d_M(p_M q - q_M p)}{q} + d_M \left\{ q_M G\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right) - p_M \right\},$$

where $|d_M(p_M q - q_M p)| \geq 1$ and

$$\begin{aligned} d_M \left| q_M G\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right) - p_M \right| &= d_M \left| q_M \frac{p_M \theta_{M+1} + p_{M+1}}{q_M \theta_{M+1} + q_{M+1}} - p_M \right| \\ &= \frac{d_M}{|q_M \theta_{M+1} + q_{M+1}|}, \quad \text{since } p_M q_{M+1} - q_M p_{M+1} = \pm 1 \\ &\leq \frac{2d_M}{|q_M|}, \quad \text{by (12)} \\ &< \frac{1}{2q}, \quad \text{by (13)}. \end{aligned}$$

Hence

$$\left| G\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right) - \frac{p}{q} \right| > \frac{1}{2} q^{-1} |d_M q_M|^{-1} = \frac{1}{2} q^{-1-(\log|d_M q_M|)/\log q}. \tag{14}$$

The same inequality will be obtained in the case of $p_{M-1}q - q_{M-1}p \neq 0$.

Now it remains to estimate $|d_M q_M|$ in terms of q. Combining (9), (10) and (13), we get

(15)

$$\begin{aligned} \log|d_M q_M| &= \log|d_{M-1} d_M| + \log\left|\frac{q_M}{q_{M-1}}\right| + \log\left|\frac{q_{M-1}}{d_{M-1}}\right| \\ &\leq \log q + \frac{M^3}{3} \log|e| + \frac{M^2}{2} \log|cf| + + \frac{M}{6} \log\left|\frac{a^6 d^3 f^3}{e^2}\right| + C_3 \end{aligned}$$

and

(16)

$$\begin{aligned} &\frac{M^3}{6} \log\left|\frac{f}{e^2}\right| + \frac{M^2}{4} \log\left|\frac{d}{c^2}\right| + \frac{M}{6} \log\left|\frac{b^3 e^2}{a^6 f}\right| - C_4 < \log q \\ &\qquad < \frac{M^3}{6} \log\left|\frac{f}{e}^2\right| + \frac{M^2}{4} \log\left|\frac{df^2}{c^2 e^4}\right| + \frac{M}{6} \log\left|\frac{b^3 d^3 f^2}{a^6 c^6 e^4}\right| + C_5. \end{aligned}$$

Hence since $|e| \neq |f|$, $t_1^3 + rt_1 + s_1 < 0$ and $t_2^3 + rt_2 + s_2 > 0$, where

$$\begin{aligned} t_1 &= M + \frac{1}{2}(\log|f/e^2|)^{-1}\log|d/c^2|, \\ t_2 &= M + \frac{1}{2}(\log|f/e^2|)^{-1}\log|d/c^2| + 1. \end{aligned}$$

For $i = 1, 2$, let

$$u_i = \Big(-\frac{s_i}{2} - \sqrt{\Delta_i}\,\Big)^{1/3} + \Big(-\frac{s_i}{2} + \sqrt{\Delta_i}\,\Big)^{1/3}$$

and

$$v_i = \Big(-\frac{s_i}{2} - \sqrt{\Delta_i}\,\Big)^{2/3} + \Big(-\frac{s_i}{2} + \sqrt{\Delta_i}\,\Big)^{2/3}.$$

Then by solving the cubic inequalities, we have

$$u_2 - \frac{1}{2}(\log|f/e^2|)^{-1}\log|d/c^2| - 1 < M < u_1 - \frac{1}{2}(\log|f/e^2|)^{-1}\log|d/c^2| \tag{17}$$

and hence

(18)

$$\begin{aligned} &v_2 - u_2\{\ (\log|f/e^2|)^{-1}\log|d/c^2| + 2\ \} + \frac{1}{4}\{\ (\log|f/e^2|)^{-1}\log|d/c^2| + 2\ \}^2 + \frac{2r}{3} < M^2 \\ &\qquad < v_1 - u_1(\log|f/e^2|)^{-1}\log|d/c^2| + \frac{1}{4}(\log|f/e^2|)^{-2}(\log|d/c^2|)^2 + \frac{2r}{3}. \end{aligned}$$

Note that for $\alpha = 2$ or 3 or $3/2$,

$$(w_1 + w_2)^{1/\alpha} \leq w_1^{1/\alpha} + w_2^{1/\alpha}, \quad \text{if } w_1,\ w_2 \text{ are nonnegative.} \tag{19}$$

$$(w_1 - w_2)^{1/\alpha} \geq w_1^{1/\alpha} - w_2^{1/\alpha}, \quad \text{if } w_1 \geq w_2 \geq 0. \tag{20}$$

By making use of (19), (20), we obtain for $i = 1, 2$,

$$\begin{aligned} u_i &= (-s_i)^{1/3} + O(1) = 6^{1/3}(\log|f/e^2|)^{-1/3}(\log q)^{1/3} + O(1), \\ v_i &= (-s_i)^{2/3} + O(1) = 6^{2/3}(\log|f/e^2|)^{-2/3}(\log q)^{2/3} + O(1). \end{aligned}$$

Hence it follows from (17) and (18) that

$$M = 6^{1/3}(\log|f/e^2|)^{-1/3}(\log q)^{1/3} + O(1)$$

and

$$\begin{aligned} &6^{2/3}(\log|f/e^2|)^{-2/3}(\log q)^{2/3} \\ &\quad - 6^{1/3}(\log|f/e^2|)^{-1/3}\{\ (\log|f/e^2|)^{-1}\log|d/c^2| + 2\ \}(\log q)^{1/3} + C_6 \leq M^2 \\ &\quad \leq 6^{2/3}(\log|f/e^2|)^{-2/3}(\log q)^{2/3} - 6^{1/3}(\log|f/e^2|)^{-4/3}\log|d/c^2|(\log q)^{1/3} + C_7. \end{aligned}$$

Hence from (16),

$$\begin{aligned} M^3 \le\ & 6(\log|f/e^2|)^{-1}\log q - \frac{3\cdot 6^{2/3}}{2}(\log|f/e^2|)^{-5/3}\log|d/c^2|(\log q)^{2/3} \\ & - \frac{6^{1/3}}{2}(\log|f/e^2|)^{-4/3}\{\ 2\log|b^3e^2/a^3f| - 3E\log|d/c^2|\ \}(\log q)^{1/3} + C_8. \end{aligned}$$

In Shiokawa's theorem we have $|e| = |f|$, in which case

$$\begin{aligned} M &= \frac{2\sqrt{\log q}}{\sqrt{\log|d/c^2|}} + O(1), \\ M^2 &\le \frac{4\log q}{\log|c/d^2|} - \frac{4\sqrt{\log q}\log|b/a^2|}{\sqrt{\log|c/d^2|}\log|c/d^2|} + C_9. \end{aligned}$$

Therefore, we obtain from (15),

$$\log|d_M q_M| < \begin{cases} (1+A)\log q + B(\log q)^{2/3} + C(\log q)^{1/3} + C_{10}, & |e| \ne |f| \\ (1+\widetilde{A})\log q + \widetilde{B}(\log q)^{1/2} + C_{11}, & |e| = |f|, \end{cases}$$

which together with (14) leads to Shiokawa's result as well as Theorem 1. □

We will use the same notation as in Theorem 1 in the sequel.

Proof of Corollary 2. Choose an even integer M from the proof of Theorem 1 so that $C_3 = 4\sigma + \log 4$. Here we note that for $\alpha > 1$ and $|w| < 1$,

$$(1+w)^{1/\alpha} = 1 + O(|w|). \tag{21}$$

Using (21), we can see that for $\alpha = 1/3,\ 2/3$ and $i = 1,\ 2$,

$$\begin{aligned} \Big(-\frac{s_i}{2} + \sqrt{\Delta_i}\,\Big)^{1/\alpha} &= (-s_i)^{1/\alpha} + o(1) \\ &= 6^{1/3}(\log f)^{-1/3}\log q + o(1) \end{aligned}$$

and

$$\Big(-\frac{s_i}{2} - \sqrt{\Delta_i}\,\Big)^{1/\alpha} = o(1), \quad \text{as } q \to \infty.$$

Thus, it follows from (17), (18) that as $q \to \infty$,

$$M \le 6^{1/3}(\log f)^{-1/3}(\log q)^{1/3} - \frac{1}{2}(\log f)^{-1}\log d + o(1),$$

and

$$\begin{aligned} M^2 \le\ & 6^{2/3}(\log f)^{-2/3}(\log q)^{2/3} - 6^{1/3}(\log f)^{-4/3}\log d(\log q)^{1/3} \\ & + \frac{1}{4}(\log f)^{-2}(\log d)^2 + \frac{2r}{3} + o(1). \end{aligned}$$

So, by (15),

$$\begin{aligned}\log q_M &\le \log q + \frac{M^2}{2}\log f + \frac{M}{2}\log df + 4\sigma + \log 4\\ &< \log q + B(\log q)^{2/3} + C(\log q)^{1/3} + D_1,\end{aligned}$$

where

$$D_1 = -\log 2D + o(1) \quad \text{as } q \to \infty.$$

Hence for any $\epsilon(> 0)$,

$$\frac{1}{2}q^{-\log q_M/\log q} > D(1-\epsilon)q^{-1-B(\log q)^{-1/3}-C(\log q)^{-2/3}},$$

which gives Corollary 2. □

Proof of Theorem 2.

$$\left|G\left(\frac{1}{b},\frac{1}{d},\frac{1}{f}\right) - \frac{p_n}{q_n}\right| < \frac{1}{a_{n+1}q_n^2}, \qquad \text{if } a_n > 0 \text{ for all } n. \tag{22}$$

Now it is enough to estimate a_{n+1} in terms of q_n. If $n = 2k$, then

$$\log a_{2k+1} = 2k^2\log f + k\log df^3 + \log bdf \qquad \text{for } k \ge 1 \tag{23}$$

and

$$\begin{aligned}\log q_{2k} &\le \log(a_1a_2\cdots a_{2k}) + 3\sigma\\ &= \frac{4k^3}{3}\log f + k^2\log df^2 + k\log bdf^{2/3} + 3\sigma.\end{aligned} \tag{24}$$

Let

$$\begin{aligned}-s &= 6(\log f)^{-1}\log q_{2k} - F_2,\\ \Delta &= \left(\frac{r}{3}\right)^3 + \left(\frac{s}{2}\right)^2,\\ u &= \left(-\frac{s}{2} - \sqrt{\Delta}\right)^{1/3} + \left(-\frac{s}{2} + \sqrt{\Delta}\right)^{1/3},\\ v &= \left(-\frac{s}{2} - \sqrt{\Delta}\right)^{2/3} + \left(-\frac{s}{2} + \sqrt{\Delta}\right)^{2/3}.\end{aligned}$$

By the similar argument as in Corollary 2 , it follows from (24) that

$$\begin{aligned}k &\ge \frac{1}{2}u - \frac{1}{4}(\log f)^{-1}\log d - \frac{1}{2}\\ &\ge \frac{6^{1/3}}{2}(\log f)^{-1/3} - \frac{1}{4}(\log f)^{-1}\log d - \frac{1}{2} + o(1)\end{aligned}$$

and

$$\begin{aligned} k^2 &\geq \frac{v}{4} - \frac{u}{4}\big\{(\log f)^{-1}\log d + 2\big\} + \frac{1}{16}\big\{(\log f)^{-1}\log d + 2\big\}^2 \\ &\geq \frac{6^{2/3}}{4}(\log f)^{-2/3}(\log q)^{2/3} - \frac{6^{1/3}}{4}\big\{(\log f)^{-1}\log d + 2\big\}(\log f)^{-1/3}(\log q)^{1/3} \\ &\qquad - \frac{r}{6} + \frac{1}{16}\big\{(\log f)^{-1}\log d + 2\big\}^2 + o(1), \qquad \text{as } q_{2k} \to \infty. \end{aligned}$$

Therefore from (23),

$$\log a_{2k+1} \geq B(\log q_{2k})^{2/3} + C(\log q_{2k})^{1/3} - \log D + o(1). \tag{25}$$

(22) together with (25) gives Theorem 2.

□

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