PARTIAL q-DIFFERENCE EQUATIONS FOR BASIC HYPERGEOMETRIC FUNCTIONS AND THEIR q-CONTINUED FRACTIONS

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ABSTRACT. In this paper we characterize the partial q-difference equations of arbitrary order satisfied by several families of basic hypergeometric functions. The method is based on a proposition about free abelian groups which has wider applicability. The partial q-difference equations can be used for, among other things, generating q-continued fractions. The method thus, in principle, allows one to generate all possible q-continued fractions for these functions. Discussions of methods of proving convergence are included as well as related open problems.

1. INTRODUCTION AND HISTORY

The main result of this paper is the characterization of three-term partial q-difference equations satisfied by several basic hypergeometric functions in a number of variables. All q-continued fractions which may be obtained by successive use of a sequence of these q-difference equations are thus in principle known. The method used relies on a proposition which is actually algorithmic and leads to the explicit equations, and thus the continued fractions. We begin with a brief history of q-continued fractions.

The first example of a q-continued fraction was given by Gauss in an entry of his diary [8, p. 68], dated February 16, 1797:

$$\frac{1}{1+}\,\frac{q}{1-}\,\frac{q-q^2}{1+}\,\frac{q^3}{1-}\,\frac{q^2-q^4}{1+}\,\frac{q^5}{1-\cdots}=\sum_{n\geq 0}(-1)^n q^{\binom{n+1}{2}},$$

where q is complex with $|q|<1$. Here we are using the space saving notation for continued fractions

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$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}} = b_0 + \frac{a_1}{b_1 +}\,\frac{a_2}{b_2 +}\,\frac{a_3}{b_3 + \cdots}.$$

The next instance of a q-continued fraction in the literature seems to be the q-analogue of the continued fraction of Gauss (not the above continued fraction, but rather his famous continued fraction for the quotient of the ordinary hypergeometric function,) given by Heine [12] in 1847:

$$(1-c)\frac{{}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right)}{{}_2\phi_1\left(\begin{matrix} a, bq \\ cq \end{matrix}; x\right)} = 1 - c +$$
$$\frac{(1-a)(c-b)z}{1-cq+}\,\frac{(1-bq)(cq-a)z}{1-cq^2+}\,\frac{(1-aq)(cq-b)zq}{1-cq^3+}\,\frac{(1-bq^2)(cq^2-a)zq}{1-cq^4+\cdots}. \tag{1.1}$$

The above function ${}_2\phi_1$ is the $r = 1$ case of the general basic hypergeometric series defined by

$${}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \cdots, a_{r+1} \\ b_1, b_2, \cdots, b_r \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(a_1, \cdots, a_{r+1})_n}{(b_1, \cdots, b_r, q)_n} z^n. \tag{1.2}$$

We are also using the following standard notation. Throughout this paper q is a fixed complex number with $|q| < 1$. The q-shifted factorial is defined by

$$(a)_0 := 1, \qquad (a)_n := \prod_{j=1}^{n}(1 - aq^{j-1}), \quad n = 1, 2, \cdots,$$

$$(a)_{-n} := \frac{1}{(aq^{-n})_n}, \quad n = 1, 2, \cdots,$$

and for $|q| < 1$,

$$(a)_\infty = \prod_{j=1}^{\infty}(1 - aq^{j-1}).$$

Also

$$(a_1, a_2, \cdots, a_k)_n := \prod_{j=1}^{k}(a_j)_n,$$

where n is an integer or infinity.

The most famous q-continued fraction is the Rogers-Ramanujan continued fraction. It is the continued fraction in the following identity which was first given by L. J. Rogers [24] :

$$1+\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+\cdots}=\prod_{n\geq 0}\frac{(1-q^{5n+2})(1-q^{5n+3})}{(1-q^{5n+1})(1-q^{5n+4})}.$$

Ramanujan's work on this continued fraction eventually triggered a flood of results on q-continued fractions. This mass of work began with the following cryptic words of Ramanujan in his 27 Feb. 1913 letter to G.H. Hardy (referring to a theorem about a special case of the Rogers-Ramanujan continued fraction):

"The above theorem is a particular case of a theorem on the continued fraction

$$\frac{1}{1+}\,\frac{ax}{1+}\,\frac{ax^2}{1+}\,\frac{ax^3}{1+}\,\frac{ax^4}{1+}\,\frac{ax^5}{1+\cdots},$$

which is a particular case of the continued fraction

$$\frac{1}{1+}\,\frac{ax}{1+bx+}\,\frac{ax^2}{1+bx^2+}\,\frac{ax^3}{1+bx^3+\cdots},$$

which is a particular case of a general theorem on continued fractions."

Even today with Ramanujan's notebooks in front of us, it is not clear what this general theorem was to which Ramanujan referred. But his statement has caused many people to try to discover q-continued fractions with an increasing number of parameters. The main purpose of this paper is to show that the Rogers-Ramanujan continued fraction is just one of infinitely many (indeed uncountably many!) q-continued fractions which are generated naturally from three-term partial q-difference equations connecting basic hypergeometric functions. The continued fraction of Heine (1.1) is another particularly simple case. In fact this paper will show how to generate "all" q-continued fractions arising from such q-difference equations and should thus help to clarify the place of the many special q-continued fractions which have been given and are still being given today. This paper thus subsumes the somewhat sporadic examples that have appeared in the literature.

There is also the issue of convergence, which has been neglected in much of the work on q-continued fractions. We will give correct proofs of convergence for some examples. The methods of proof can be applied to the types of continued fractions which may be found by the techniques in this paper.

Before we can begin, we need to discuss the analytic continuation of basic hypergeometric series. This topic has not been discussed much in print. Unfortunately, even in the recent reference by Gasper and Rahman [7], the fact that these series are meromorphic in their z variable is missing. As we wish the highest generality for our results, we treat this continuation briefly in the next section.

Section 3 discusses methods for proving convergence of particular q-continued fractions. Two techniques are given. Again in the literature this topic is frequently overlooked, so we thought it to be important to go through two examples in detail. The rest of that section details the connection between three-term partial q-difference equations and continued fractions. We give two examples to illustrate how the q-difference equations are used to generate continued fractions. Following this are our main theorems which classify three-term partial q-difference equations that exist for three of the most common general basic hypergeometric functions. The theorems come from a proposition which is more generally applicable (it can be applied to ordinary hypergeometric functions as well). This proposition and its proof are in the Section 5.

We do not wish to suggest that all problems concerning q-continued fractions have been solved. At the end of the paper we discuss a number of unsolved problems and directions for future research.

2. Analytic continuation of basic hypergeometric series

In order to give the limits of the continued fractions in their widest possible domains, we need to first consider the analytic continuation of the functions ${}_{r+1}\phi_r$. Briefly, these functions are meromorphic in all their parameters, and thus the analytic continuation is unique. In fact, from the series definition (1.2), it is obvious that the function is analytic in the a_i and meromorphic in the b_i. It is also obvious that the function is analytic for $|z| < 1$. What is less obvious is that the function is meromorphic in z. But this follows from the canonical q-difference equation satisfied by the function, first given by Thomae [26] in 1870. Put

$$f(z) = {}_{r+1}\phi_r \left(\begin{matrix} a_1, a_2, \cdots, a_{r+1} \\ b_1, b_2, \cdots, b_r \end{matrix} ; z \right).$$

Then

$$(1-z)f(z) + (\beta_1 - \alpha_1 z)f(zq^2) + \cdots + (\beta_{r+1} - \alpha_{r+1} z)f(zq^{r+1}) = 0,$$

where

$$\begin{aligned} 1 + \alpha_1 x + \cdots + \alpha_{r+1} x^{r+1} &= (1 - a_1 x) \cdots (1 - a_{r+1} x), \\ 1 + \beta_1 x + \cdots + \beta_{r+1} x^{r+1} &= (1 - x)(1 - b_1 x) \cdots (1 - b_r x). \end{aligned}$$

The right hand side of this equation provides an analytic continuation of the function $(1-z)f(z)$ to the disk $|z| < q^{-1}$. Now replace z with zq throughout the equation and substitute for $f(zq)$ in the previous equation to see that $(1-z)(1-zq)f(z)$ is analytic in $|z|<q^{-2}$. Continuing in this way shows that the function $(z)_\infty f(z)$ is an entire function and thus $f(z)$ is meromorphic. We let ${}_{r+1}\Phi_r$ denote the analytic continuation of the series ${}_{r+1}\phi_r$.

3. Generating continued fractions from q-difference equations and convergence

We begin with perhaps the simplest non-trivial q-continued fraction carried out in full generality. Surprisingly, this continued fraction is not as well known as it should be, although in principle it could have been derived quite easily by Thomae [26] in 1870. It seems to have been first evaluated by Ismail and Libis in 1989 under considerable restriction on the parameters. A more general evaluation was given in the first author's dissertation in 1993 [4]. Here we prove the even more general result:

$$1+c-(a+b)x+\frac{(abx-c)(1-xq)}{1+c-(a+b)xq+}\,\frac{(abxq-c)(1-xq^2)}{1+c-(a+b)xq^2+\cdots}=$$
$$\begin{cases} (1-x)\dfrac{{}_2\Phi_1\left(\begin{matrix}a,b\\cq\end{matrix};x\right)}{{}_2\Phi_1\left(\begin{matrix}a,b\\cq\end{matrix};xq\right)}, & \text{for } |c|<1 \\ (1-x)c\dfrac{{}_2\Phi_1\left(\begin{matrix}a/c,b/c\\q/c\end{matrix};x\right)}{{}_2\Phi_1\left(\begin{matrix}a/c,b/c\\q/c\end{matrix};xq\right)}, & \text{for } |c|>1 \end{cases}.$$

where $|q|<1$, $c\neq q^n$, for $n=0,1,2,\ldots$ and $x\neq q^{-n}$, for n non-negative.

Previously the first author [5] proved this formula under the condition $|x|<1$ with ϕ instead of Φ. To prove this we begin with the the canonical q-difference equation for the function ${}_2\phi_1$, namely,

$$(1-x)f(x)-(1+c-(a+b)x)f(xq)+(c-abx)f(xq^2)=0,$$

where

$$f(x)={}_2\phi_1\left(\begin{matrix}a,b\\cq\end{matrix};x\right).$$

To generate a continued fraction one divides through by $f(xq)$ giving

$$\frac{(1-x)f(x)}{f(xq)}=1+c-(a+b)x+\frac{(1-xq)(abx-c)}{\dfrac{(1-xq)f(xq)}{f(xq^2)}}.$$

Now one iterates this formula to obtain the continued fraction above. There are, however, two obstacles: the convergence of the continued fraction, and the validity of obtaining its value by this iteration process. Both of these obstacles are overcome by the following theorem of Pincherle [19].

Theorem 3.1. *Suppose that the sequences* $\{X_n\}$ *and* $\{Y_n\}$, $n \geq 0$, *satisfy the difference equation*

$$c_n Z_n = b_n Z_{n+1} + a_{n+1} Z_{n+2},$$

where $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *are given sequences of complex numbers and additionally*

$$\lim_{n\to\infty} \frac{X_n}{Y_n} = 0.$$

Then, if $X_1 \neq 0$, *it follows that*

$$\frac{c_0 X_0}{X_1} = b_0 + \frac{a_1 c_1}{b_1 +}\,\frac{a_2 b_2}{c_2 +}\,\frac{a_3 b_3}{c_3 + \cdots},$$

and in particular the continued fraction converges.

In Theorem 3.1, $\{X_n\}$ is called the minimal solution and $\{Y_n\}$ the dominant solution (of the recurrence.)

The above case of the recurrence is $c_n = 1 - xq^n$, $b_n = 1 + c - (a+b)xq^n$, $a_n = abxq^{n-1} - c$, and one solution is given by $X_n = f(xq^n)$. Thomae [26] showed that a second solution is given by

$$Y_n = x^{-\gamma} c^{-n} {}_2\phi_1 \left(\begin{matrix} a/c, b/c \\ c/q \end{matrix} ; xq^n \right),$$

where $q^\gamma = c$. Hence when $|c| < 1$, Y_n is dominant, while when $|c| > 1$, X_n is dominant. These give the two values of the continued fraction in the above evaluation. Finally when $|x| \geq 1$, as the analytic continuations of the series are derived by iterating the q-difference equations, it follows that the asymptotic behaviors continue to hold and so the theorem is valid in this more general case. There only remains the degenerate cases in which $c = q^N, N = 0, 1, 2, \ldots$. Here LeCaine [16] gave the second solution by the method of Frobenius and the minimal solutions are unchanged, so the evaluation is valid here as well.

The above method works when one knows a second solution of the recurrence which generates the continued fraction. In general one does not know this and a different method is used. We will give an example similar to the proof of Rogers-Ramanujan continued fraction transformation given in Lorentzen and Waadeland [17]. We actually give a more general result which is more generally applicable to the continued fractions which are generated by the techniques in this paper. Before we begin, we need some notations and theorems from the books of Jones and Thron [15] and Lorentzen and Waadeland [17]. Let $\mathbf{L}$ denote the field of all power series with zero element $l_0 = \sum 0z^n$, and define the order of L at 0, $\lambda(L)$, for $L \in \mathbf{L}$, by

$$\lambda(L) = \begin{cases} m & \text{for } L(z) = \sum_{n=m}^{\infty} c_n z^n \text{ with } c_m \neq 0, \\ \infty & \text{for } L(z) = l_0(z) = \sum 0z^n. \end{cases}$$

We will use the following three theorems [15, p. 160] and [17, p. 271, p. 35]:

Theorem 3.2. *Let $\mathbf{K}(a_n(z)/b_n(z))$ be a continued fraction, where $a_n, b_n \in \mathbf{L}$ are polynomials with all $a_n \neq l_0$, and let $\{P_n\}; P_n \in \mathbf{L}$, be a solution of*

$$P_n = b_n P_{n+1} + a_{n+1} P_{n+2}, \quad n = 0, 1, 2, \ldots$$

with $P_n \neq l_0$ for all n. If

$$\lambda(b_{n-1}) + \lambda(b_n) < \lambda(a_n), \quad n = 1, 2, 3, \ldots,$$

and

$$\lambda(P_n/P_{n+1}) + \lambda(b_{n-1}) < \lambda(a_n), \quad n = 1, 2, 3, \ldots,$$

then $\mathbf{K}(a_n(z)/b_n(z))$ corresponds to P_0/P_1. Moreover, for each $m = 0, 1, 2, \ldots,$ the continued fraction

$$b_m(z) + \frac{a_{m+1}(z)}{b_{m+1}(z)+} \frac{a_{m+2}(z)}{b_{m+2}(z)+} \frac{a_{m+3}(z)}{b_{m+3}(z) + \cdots}$$

corresponds to P_m/P_{m+1}.

Theorem 3.3. *Let $\mathbf{K}(a_n(z)/b_n(z))$ with polynomial elements $a_n(z) \not\equiv 0$ and $b_n(z)$ correspond to an $L \in \mathbf{L}$ and have holomorphic approximants in D, a deleted neighborhood of the origin. Then the following statements hold.*

(A) $\mathbf{K}(a_n(z)/b_n(z))$ converges locally uniformly in D if and only if its approximants are uniformly bounded on every compact subset of D.

(B) If $\mathbf{K}(a_n(z)/b_n(z))$ converges locally uniformly in D, then its value $f(z)$ is holomorphic in D, meromorphic at $z = 0$ and $L = \mathcal{L}(f)$, where $\mathcal{L}(f)$ denotes the Laurent series expansion of f in a neighborhood of $z = 0$.

Theorem 3.4 (Worpitzky). *Suppose $|a_n| \leq \frac{1}{4}$ for all $n \geq 1$. Then $\mathbf{K}(a_n/1)$ converges. All approximants f_n are in the disk $|w| < \frac{1}{2}$, and the value f is in the disk $|w| \leq \frac{1}{2}$.*

The following example is the continued fraction mentioned by Ramanujan [21, p. xxviii] in his second letter to Hardy. The evaluation given here is certainly the theorem Ramanujan had in mind as it is given in Chapter 16 of his second notebook [20]. See also Berndt's book [3, p. 30, Entry 15]. Here we give a careful proof of the convergence of the continued fraction.

Example 3.5. *We want to find the value of the generalized Rogers-Ramanujan continued fraction*

$$1+a+\frac{zq}{1+aq+}\,\frac{zq^2}{1+aq^2+}\,\frac{zq^3}{1+aq^3+\cdots}, \tag{3.1}$$

where $a \neq -q^{-1}, -q^{-2}, -q^{-3}, \dots$.

It is easy to check that

$$P_n(z) = \sum_{k=0}^{\infty} \frac{q^{k(k+n)}}{(q)_k(-a)_{n+k}} z^k$$

is a solution of

$$P_n(z) = (1+aq^n)P_{n+1}(z) + q^{n+1}zP_{n+2}(z), \quad n = 0, 1, 2, \dots.$$

Now using Theorem 3.2 with $a_n = q^n z$ and $b_n = 1 + aq^n$, it follows that (3.1) corresponds to $P_0(z)/P_1(z)$, since $\lambda(b_{n-1}) = \lambda(b_n) = 0, \lambda(a_n) = 1$, and $\lambda(P_n/P_{n+1}) = \lambda(P_n) - \lambda(P_{n+1}) = 0 - 0 = 0$. Note that the n-th approximants of (3.1) is the same as the n-th approximant of

$$(1+a)\left(1+\frac{\frac{zq}{(1+a)(1+aq)}}{1+}\,\frac{\frac{zq^2}{(1+aq)(1+aq^2)}}{1+}\,\frac{\frac{zq^3}{(1+aq^2)(1+aq^3)}}{1+\cdots}\right). \tag{3.2}$$

Hence convergence of (3.1) is equivalent to the convergence of (3.2). Fix $R > 0$ and let $D = \{z \in \mathbb{C}; |z| < R\}$. Then for any fixed $z \in D$, there exists an $N \in \mathbb{N}$ such that $\left|\frac{zq^n}{(1+aq^{n-1})(1+aq^n)}\right| \leq 1/4$ for all $n \geq N$. That is, the continued fraction

$$(1+aq^{N-1})\left(1+\frac{\frac{zq^N}{(1+aq^{N-1})(1+aq^N)}}{1+}\,\frac{\frac{zq^{N+1}}{(1+aq^N)(1+aq^{N+1})}}{1+}\,\frac{\frac{zq^{N+2}}{(1+aq^{N+1})(1+aq^{N+2})}}{1+\cdots}\right) \tag{3.3}$$

satisfies the conditions of Worpitzky's theorem for $z \in D$, and thus its approximants are uniformly bounded by $(1+|a|)(1+1/2)$ in D. Hence the approximants of the continued fraction

$$1+aq^{N-1}+\frac{zq^N}{1+aq^N+}\,\frac{zq^{N+1}}{1+aq^{N+1}+}\,\frac{zq^{N+2}}{1+aq^{N+2}+\cdots}, \tag{3.4}$$

which corresponds to $P_{N-1}(z)/P_N(z)$ by the second part of Theorem 3.2, are also uniformly bounded by $(1+|a|)(1+1/2)$ in D. Hence (3.4) converges to $P_{N-1}(z)/P_N(z)$ in D by virtue of Theorem 3.3. This in turn implies that (3.1) converges to $P_0(z)/P_1(z)$ in D and thus, in particular,

$$1+a+\frac{zq}{1+aq+}\,\frac{zq^2}{1+aq^2+}\,\frac{zq^3}{1+aq^3+\cdots} = \left(\sum_{k=0}^{\infty}\frac{q^{k^2}z^k}{(q)_k(-a)_k}\right) \Big/ \left(\sum_{k=0}^{\infty}\frac{q^{k(k+1)}z^k}{(q)_k(-a)_{k+1}}\right),$$

where the equality sign stands for convergence.

4. Partial difference equations

Partial q-difference equations for basic hypergeometric functions have played a major role since the earliest papers on the subject. For example, Heine's first papers on basic hypergeometric functions used three-term q-difference equation to give the q-analogue of Gauss's hypergeometric continued fraction. Thomae [26] actually approached the theory of more general basic hypergeometric series from the point of view of their q-difference equations. Next, L.J. Rogers applied q-difference equations in a sophisticated way to, among other things, prove the Rogers-Ramanujan identities. But even before this, Rogers [23] used q-difference equations to prove his symmetric expansion theorem for Heine's ${}_2\Phi_1$ basic hypergeometric function.

In recent years this approach has continued to be one of the most important attacks on problems dealing with basic hypergeometric series. There are two fundamental problems associated with q-difference equations and basic hypergeometric series. The first is, given a q-difference equation, to find all basic hypergeometric solutions. This problem was recently solved by Abramov, Paule, and Petkovesek [1], building on the work of Gosper [9]. The second problem is, given a basic hypergeometric function, find all three-term partial q-difference equations (with polynomial coefficients) which the given function satisfies. This problem is attacked here.

The relevance of partial q-difference equations to q-continued fractions will be illustrated by two examples. The general situation will then be described. As a first example, we show how Heine's q-analogue of the continued fraction of Gauss arises.

Convention. *We have a notational convention for writing a function with q-shifted parameters, but no other specializations of its parameters after it's definition; we only show the shifted parameters. For examples, $f(aq^2, b, xq^{-1})$ would be written $f(aq^2, xq^{-1})$, and $f(a, bq, x)$ would be just $f(bq)$.*

Let $f = f(a,b,c,z) = {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right)$. Then from (3.2.1) ([17], p. 320), we have

$$(1-c)f = (1-c)f(bq,cq) + \frac{(1-a)(c-b)z}{(1-cq)} f(aq,bq,cq^2). \tag{4.1}$$

Interchanging a and b gives (since f is symmetric in a and b)

$$(1-c)f = (1-c)f(aq,cq) + \frac{(1-b)(c-a)z}{(1-cq)} f(aq,bq,cq^2). \tag{4.2}$$

If we divide (4.1) and (4.2) by $f(bq,cq)$ and $f(aq,cq)$ respectively, we obtain

$$\frac{(1-c)f}{f(bq,cq)} = 1-c+\frac{(1-a)(c-b)z}{\dfrac{(1-cq)f(bq,cq)}{f(aq,bq,cq^2)}} \tag{4.3}$$

and

$$\frac{(1-c)f}{f(aq,cq)} = 1-c+\cfrac{(1-b)(c-a)z}{\cfrac{(1-cq)f(aq,cq)}{f(aq,bq,cq^2)}}. \tag{4.4}$$

Observe that we obtain the fraction $\dfrac{(1-cq)f(bq,cq)}{f(aq,bq,cq^2)}$ in (4.3) by replacing $b \to bq, c \to cq$ in (4.4). So after one iteration we have

$$\frac{(1-c)f}{f(bq,cq)} = 1-c+\cfrac{(1-a)(c-b)z}{1-cq+\cfrac{(1-bq)(cq-a)z}{\cfrac{(1-cq^2)f(aq,bq,cq^2)}{f(aq,bq^2,cq^3)}}}. \tag{4.5}$$

Observe that we obtain the fraction $\dfrac{(1-cq^2)f(aq,bq,cq^2)}{f(aq,bq^2,cq^3)}$ in (4.5) by replacing $a \to aq, b \to bq$, and $c \to cq^2$ in (4.3). So after one more iteration we have

$$\frac{(1-c)f}{f(bq,cq)} = 1-c+\frac{(1-a)(c-b)z}{1-cq+}\,\cfrac{(1-bq)(cq-a)z}{1-cq^2+\cfrac{(1-aq)(cq^2-bq)z}{\cfrac{(1-cq^3)f(aq,bq^2,cq^3)}{f(aq^2,bq^2,cq^4)}}}. \tag{4.6}$$

Now use (4.4) once again after shifting $a \to aq, b \to bq^2$, and $c \to cq^3$. Hence using (4.3) and (4.4) alternatively along with the shifting of variables a, b, and c generates the continued fraction (1.1). For convergence, we can again apply Theorem 3.3. This gives (1.1) is true for $z \in \mathbb{C}$ and convergence is uniform in compact subset of $\left\{ z \in \mathbb{C} \;\middle|\; \frac{f}{f(bq,cq)} \neq \infty \right\}$.

For our second example, consider the following list of three partial q-difference equations from (A.12), (A.13), and (A.11) [14, p. 369] for the function $f = f(a,b,c,z) = {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right)$:

$$c(1-c)f = (1-c)(c-abt)f(bq) + bt(a-c)f(bq,cq), \tag{i}$$

$$a(1-c)f = (a-c)f(cq) + c(1-a)f(aq,cq), \tag{ii}$$

and

$$b(c-aq)f = [(a+b)c - ab(q+c)]f(aq) - a(1-b)(c-abtq)f(aq,bq). \tag{iii}$$

If we divide these equations by their respective second terms, then we obtain,

$$\frac{c(1-c)f}{f(bq)} = (1-c)(c-abt) + \cfrac{abt(1-c)(a-c)}{\cfrac{a(1-c)f(bq)}{f(bq,cq)}}, \tag{4.7}$$

$$\frac{a(1-c)f}{f(cq)} = a - c + \cfrac{bc(cq-aq)(1-a)}{\cfrac{b(cq-aq)f(cq)}{f(aq,cq)}}, \tag{4.8}$$

and

$$\frac{b(c-aq)f}{f(aq)} = (a+b)c - ab(q+c) - \cfrac{ac(1-b)(1-c)(c-abtq)}{\cfrac{c(1-c)f(aq)}{f(aq,bq)}}. \tag{4.9}$$

Observe that the fraction $\frac{a(1-c)f(bq)}{f(bq,cq)}$ in (4.7) may be obtained by replacing $b \to bq$ in (4.8); the fraction $\frac{b(cq-aq)f(cq)}{f(aq,cq)}$ in (4.8) may be obtained by replacing $c \to cq$ in (4.9). So after two iterations we find

$$\begin{aligned}
\frac{c(1-c)f}{f(bq)} =& (1-c)(c-abt) + \cfrac{abt(a-c)(1-c)}{\cfrac{a(1-c)f(bq)}{f(bq,cq)}} \\
=& (1-c)(c-abt) + \frac{ab(1-c)(a-c)t}{a-c+} \, \cfrac{bcq(1-a)(cq-aq)}{\cfrac{bq(cq-aq)f(bq,cq)}{f(aq,bq,cq)}} \\
=& (1-c)(c-abt) + \frac{ab(1-c)(a-c)t}{a-c+} \\
& \frac{bcq(1-a)(cq-aq)}{(a+bq)cq - abq(q+cq) -} \, \cfrac{acq(1-bq)(1-cq)(cq-abtq^2)}{\cfrac{cq(1-cq)f(aq,bq,cq)}{f(aq,bq^2,cq)}}.
\end{aligned} \tag{4.10}$$

Observe that the fraction $\frac{cq(1-cq)f(aq,bq,cq)}{f(aq,bq^2,cq)}$ in (4.10) may be obtained by replacing $a \to aq, b \to bq$, and $c \to cq$ in (4.7). Now one wishes to iterate these equations (4.7)–(4.9) to obtain the continued fraction (after cancelling some factors)

$$\begin{aligned}
\frac{cf}{f(bq)} = c - abt + \frac{abt}{1-} \, \frac{bcq(1-a)}{(a+bq)c - abq(1+c) -} \, \frac{ac(1-bq)(c-abtq)}{c-abtq+} \\
\frac{abtq}{1-} \, \frac{bcq(1-aq)}{(a+bq)c - abq(1+cq) -} \, \frac{ac(1-bq^2)(c-abtq^2)}{c-abtq^2+\cdots}.
\end{aligned} \tag{4.11}$$

Convergence can again be established with the aid of Theorem 3.3. Notice that the shifting between the second two terms of (i) is: $b \to bq$, which is also the shift of the first two terms of (ii). Next the shifting between the second two terms of (ii) equals the shifting between the first two terms of (iii), and finally the shift in the second two terms of (iii) equals the shift in the first two terms of (i). In general what is required is that the q-shifting between the second two terms of one three-term equation in a list be the same as the relation between the first two terms of the next three-term equation and so on, until in the last equation in the list, the q-shifting between the last two terms is the same as the q-shift between the first two terms of the first three-term equation in the list. We define the q-period of a q-continued fraction to be the number of equations in the list that are used. Thus the Heine continued fraction has q-period 2, while our second example has q-period 3. Notice that as $q \to 1$, the q-continued fraction tends to a periodic continued fraction of period n. Also, as $n \to \infty$, since $|q| < 1$, the continued fraction will tend to a periodic continued fraction. More generally, continued fractions satisfying this later condition are termed "limit periodic" continued fractions. There exists a considerable literature on the convergence of limit periodic continued fractions. See for example [15, p. 113]. One also desires that the q-shifting be such that upon iteration, the continued fraction converges. This is usually easy to check. Then one applies one of the methods given earlier to prove that the convergence is to the desired quotient of basic hypergeometric functions. Obviously, to make the above work, one needs a large supply of q-difference equations. The theorems in this paper guarantee that this is the case for several basic hypergeometric functions with a number of free parameters and as a consequence give infinitely many q-continued fraction evaluations.

Our method is based on a proposition about free abelian groups which can be applied to many different functions yielding characterizations of the q-difference equations they satisfy. The method is algorithmic and actually constructs their q-difference equations. We first state interesting special cases which characterize the partial q-difference equations for three basic hypergeometric functions in 4, 5, and 7 parameters, respectively. Indeed Theorems 4.1 and 4.2 show that for the first two functions any q-shifting between the terms of a three-term equation may occur, so that, in fact, uncountably many q-continued fractions may be written down, if we don't require the sequence of q-difference equations used to be periodic. Theorem 4.3 is slightly less general but still gives uncountably many q-continued fractions.

Theorem 4.1. *The set of functions* $\left\{ {}_2\phi_1 \left(\begin{matrix} aq^{n_1}, bq^{n_2} \\ cq^{n_3} \end{matrix} ; xq^{n_4} \right) \middle| \; n_i \in \mathbb{Z} \right\}$ *span a two dimensional vector space over the field* $\mathbb{Q}(a, b, c, x, q)$.

Theorem 4.2. $\left\{ {}_3\phi_2 \left(\begin{matrix} aq^{n_1}, bq^{n_2}, cq^{n_3} \\ dq^{n_4}, eq^{n_5} \end{matrix} ; \frac{de}{abc} q^{n_4+n_5-n_1-n_2-n_3} \right) \middle| \; n_i \in \mathbb{Z} \right\}$ *span a two dimensional vector space over the field* $\mathbb{Q}(a, b, c, d, e, q)$.

Theorem 4.3. *Let ϕ be the specialized basic hypergeometric function defined by:*

$$\phi(b,c,d,e,f,g,h) = {}_{10}\phi_9\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f, g, h \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq/f, aq/g, aq/h \end{matrix}; q\right)$$
$$+ \frac{(aq, b/a, c, d, e, f, g, h, bq/c, bq/d, bq/e, bq/f, bq/g, bq/h)_\infty}{(b^2q/a, q/b, aq/c, aq/d, aq/e, aq/f, aq/g, aq/h, bc/a, bd/a, be/a, bf/a, bg/a, bh/a)_\infty}$$
$$\times {}_{10}\phi_9\left(\begin{matrix} b^2/a, qb/\sqrt{a}, -qb/\sqrt{a}, b, bc/a, bd/a, be/a, bf/a, bg/a, bh/a \\ b/\sqrt{a}, -b/\sqrt{a}, bq/a, bq/c, bq/d, bq/e, bq/f, bq/g, bq/h \end{matrix}; q\right),$$

subject to the relation $a^3q^2 = bcdefgh$. Then for $k = 0, 1, \ldots, 5$, each of the sets of functions

$$S_k = \left\{ \phi(bq^{n_1}, cq^{n_2}, dq^{n_3}, eq^{n_4}, fq^{n_5}, gq^{n_6}, hq^{n_7}) \,\middle|\, \sum_{i=1}^{7} n_i \equiv k (mod\ 6), n_i \in \mathbb{Z} \right\}$$

generate a two dimensional vector spaces over the field $\mathbb{Q}(b, c, d, e, f, g, h, q)$.

Remark. *Notice from the definition of ϕ that it is symmetric in $c, d, \ldots, h$ but not in b. For this reason b is called the distinguished parameter.*

Corollary 4.4. *There exist polynomials $P, Q, R \in \mathbb{Z}[b, c, d, e, f, g, h, q]$ such that $P\phi + Q\phi(bq^6) + R\phi(bq^{12}) = 0$, where ϕ is defined in Theorem 4.3.*

The polynomials P, Q, and R in this Corollary have not been computed. Attempting to compute them with the algorithm in the next section led to an out of memory error in the computer used. They are evidently rather large.

The proofs of these theorems rely on the machinery developed in the next section.

5. The proposition

Let P be the free abelian group generated by $\{x_i\}_{0 \le i \le n}$, written multiplicatively. Let S be a set of subsets of P with three elements. Suppose S satisfies the axioms:

(a) $D \in P, \{A, B, C\} \in S \Longrightarrow \{AD, BD, CD\} \in S$

(b) $\{A, B, C\}, \{A, B, D\} \in S \Longrightarrow \{A, C, D\} \in S$.

Such a set S is called an <u>*elimination set*</u> (*e − set*).

Remarks. *The axiom (a) is called the shifting axiom and axiom (b) is the axiom of elimination.*

Convention. *When we write the set $\{A, B, C\}$ we mean that $\{A, B, C\} \in S$.*

Proposition 5.1. *Let S be an elimination set. Then necessary and sufficient conditions for S to consist of all subsets of P with three elements is*

(1) $\{x_0^{-1}, 1, x_0\} \in S$

(2) *For each* $x_i, 1 \leq i \leq n$, *one of the following four conditions holds :*
(*i*) $\{x_i, 1, x_0\} \in S$
(*ii*) $\{x_i, x_0x_i, x_0\} \in S$
(*iii*) $\{1, x_i, x_0x_i\} \in S$
(*iv*) $\{1, x_0, x_0x_i\} \in S$.

The proof will be constructive and algorithmic.

Proof. We first show that any one of the conditions (i) - (iv) together with (1) implies the rest:

(i) with (1) $\Rightarrow$ (ii)

If we eliminate x_0 from (i) and (1) we have $\{1, x_0^{-1}, x_i\}$ which is equivalent to $\{x_0, 1, x_ix_0\}$ by axiom (a). Now if we eliminate 1 from this and (i), we get $\{x_0, x_i, x_0x_i\}$.

(ii) with (1) $\Rightarrow$ (iii)

(1) is equivalent to $\{x_i, x_0x_i, x_0^{-1}x_i\}$ and (ii) is equivalent to $\{x_0^{-1}x_i, 1, x_i\}$. If we use these two sets we get $\{1, x_i, x_0x_i\}$ after eliminating $x_0^{-1}x_i$.

(iii) with (1) $\Rightarrow$ (iv)

(1) is equivalent to $\{x_i, x_0x_i, x_0^{-1}x_i\}$. From this set and (iii) we can eliminate x_0x_i giving $\{1, x_i, x_0^{-1}x_i\}$. But this set is equivalent to $\{x_0, x_i, x_0x_i\}$. With this and (iii) we have $\{1, x_0, x_0x_i\}$ after eliminating x_i.

(iv) with (1) $\Rightarrow$ (i)

(iv) is equivalent to $\{x_0^{-1}, 1, x_i\}$ which together with (1) and eliminating x_0^{-1}, gives $\{1, x_0, x_i\}$.

Lemma 5.2. *If* $\{1, x, x^n\}$ *and* $\{1, x^{n-1}, x^n\}$ *are in an e-set for some* $n \neq 0, 1$, *then* $\{1, x, x^m\}$ *and* $\{1, x^{m-1}, x^m\}$ *are in the e-set for all* $m \neq 0, 1$.

Proof. First we will prove that $\{1, x, x^n\}$ and $\{1, x^{n-1}, x^n\}$ are in an e-set iff $\{1, x, x^{n+1}\}$ and $\{1, x^n, x^{n+1}\}$ are in the same e-set. Let $A = \{1, x, x^n\}$ and $B = \{x, x^n, x^{n+1}\}$ which is equivalent to $\{1, x^{n-1}, x^n\}$. From A and B after eliminating x^n we get $\{1, x, x^{n+1}\}$. From A and B after eliminating x we get $\{1, x^n, x^{n+1}\}$. Conversely, let $C = \{1, x, x^{n+1}\}$ and $D = \{1, x^n, x^{n+1}\}$. After eliminating 1 from C and D we have $\{x^n, x, x^{n+1}\}$ which is equivalent to $\{1, x^{n-1}, x^n\}$. After eliminating x^{n+1} from C and D we get $\{1, x, x^n\}$. Now note that $\{1, x, x^2\} \Leftrightarrow \{x^{-1}, 1, x\} \Leftrightarrow \{1, x^{-1}, x^{-2}\}$. Hence the case $n = 2$ is equivalent to the $n = -1$ case and the induction to both ∞ and $-\infty$ can continue. Thus, if $\{1, x, x^n\}$ and $\{1, x^{n-1}, x^n\}$ are in an e-set for some integer n, then they are both in the e-set for all integers n except $n = 0, 1$. $\square$

<u>*Proof of the Proposition* 5.1 *continued*</u>

Step 1 It suffices to show for any $u \in P, \{u, 1, x_0\} \in S$ implies $\{u, v, w\} \in S$. Suppose $\{u, 1, x_0\}$,$\{v, 1, x_0\}$,$\{w, 1, x_0\} \in S$. From the first two, we get $\{u, 1, v\}$; from the last two we get $\{v, 1, w\}$. Finally we get $\{u, v, w\}$ after eliminating 1 from the above two sets.

Step 2 To show $\{1, x_i, x_i^n\}, \{1, x_i^{n-1}, x_i^n\} \in S$, for $1 \le i \le n$.

(i) $= \{x_i, 1, x_0\}$ is equivalent to $\{x_0 x_i, x_i, x_i^2\}$ and (iii) $= \{1, x_i, x_0 x_i\}$. Eliminating $x_0 x_i$ gives $\{1, x_i, x_i^2\}$ for $1 \le i \le n$. Hence by Lemma 5.2, we have $\{1, x_i, x_i^n\}$ and $\{1, x_i^{n-1}, x_i^n\} \in S$ for $n \neq 0, 1$.

Step 3 Put $x = x_i$, for a fixed i satisfying $1 \le i \le n$ and put $t = x_0$.

For $n \neq 0, 1$, $\{1, x, x^n\} \in S$ from Step 2. From (i), $\{1, t, x\}$. If we eliminate x then we get $A = \{1, t, x^n\}, n \neq 0$. (the case $n = 1$ is covered by (i)). From Step 2, we have $\{1, x, x^n\}, n \neq 0, 1$, which is the same as $\{t, tx, tx^n\}, n \neq 0, 1$. Combine this with (iv) (which is $\{1, t, tx\}$) and eliminate tx to get $B = \{1, t, tx^n\}, n \neq 0$. (the case $n = 1$ is covered by (iv)). From A and B we have $C = \{1, x^n, tx^n\}, n \neq 0$ after eliminating t. With $x = t$ in Lemma 5.2 we get $\{1, t, t^m\} \in S, (m \neq 0, 1)$ which is equivalent to $D = \{x^n, tx^n, t^m x^n\}$. With C and D eliminate x^n giving $E = \{1, tx^n, t^m x^n\}, m \neq 1, (m, n) \neq (0, 0)$. After eliminating tx^n from B and E we get $\{1, t, t^m x^n\}$. Hence $\{x_0^{n_0} x_i^{n_i}, 1, x_0\} \in S$, $n_0, n_i \in \mathbb{Z}, (n_0, n_i) \neq (0, 0), (1, 0)$.

Step 4 Induction

To prove $\{x_0^{n_0} x_1^{n_1} \cdots x_k^{n_k}, 1, x_0\} \in S$ we use the induction on k. For $k = 0$, this is $\{x_0^{n_0}, 1, x_0\} \in S$, which we know. For $k = 1$, this is $\{x_0^{n_0} x_1^{n_1}, 1, x_0\} \in S$, which we just showed. Assume the claim is true for $k - 1$. Then we have $\{x_0^{n_0} x_1^{n_1} \cdots x_{k-1}^{n_{k-1}}, 1, x_0\} \in S$ which is equivalent to $F = \{x_0^{n_0} x_1^{n_1} \cdots x_{k-1}^{n_{k-1}} x_k^{n_k}, x_k^{n_k}, x_0 x_k^{n_k}\}$. From Step 3, we know $\{x_0 x_k^{n_k}, 1, x_0\}$ and $G = \{x_k^{n_k}, 1, x_0\} \in S$. After eliminating x_0, we get $H = \{x_0 x_k^{n_k}, x_k^{n_k}, 1\}$. Eliminate $x_0 x_k^{n_k}$ from F and H to obtain $I = \{1, x_k^{n_k}, x_0^{n_0} \cdots x_k^{n_k}\}$. From I and G eliminate $x_k^{n_k}$ to get $\{x_0^{n_0} x_1^{n_1} \cdots x_k^{n_k}, 1, x_0\}$. By Step 1 the proof is complete. $\square$

6. Proofs of Theorems

We now proceed with proofs of the theorems.

Proof of Theorem 4.1. Let $f = f(a, b, c, x) = {}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; x \right)$. From (A.1) ([14], p. 369) we know

$$(1 - x)f - (1 + cq^{-1} - (a + b)x)f(xq) + (cq^{-1} - abx)f(xq^2) = 0. \tag{6.1}$$

By using the property $(a)_n = (1 - a)(aq)_n + a(a)_n q^n$, we get

$$f = (1 - a)f(aq) + af(xq). \tag{6.2}$$

Interchanging a and b gives

$$f=(1-b)f(bq)+bf(xq). \tag{6.3}$$

Now using the equation $\dfrac{1-c}{(c)_n}=\dfrac{1}{(cq)_n}-\dfrac{cq^n}{(cq)_n}$ gives

$$(1-c)f=f(cq)+cf(cq,xq). \tag{6.4}$$

Let x_0 represent shifting x to xq, x_1 the shift from a to aq, x_2 the shift from b to bq, and x_3 the shift from c to cq. Thus, x_0^{-1} represents the shift from x to x/q . After replacing x by xq^{-1} throughout (6.1), we see that $\{1,x_0,x_0^{-1}\}\in S$. From (6.2) and (6.3), we know $\{1,x_0,x_1\}$ and $\{1,x_0,x_2\}\in S$. From (6.4), we know $\{1,x_3,x_0x_3\}\in S$. Then Proposition 5.1 gives an e-set consisting of triples of elements of the form

$$\begin{aligned}&(x_0)^{n_1}(x_1)^{n_2}(x_2)^{n_3}(x_3)^{n_4}(x_0x_3)^{n_5}\\=&(x_0)^{n_1+n_5}(x_1)^{n_2}(x_2)^{n_3}(x_3)^{n_4+n_5}\\=&(x_0)^{m_1}(x_1)^{m_2}(x_2)^{m_3}(x_3)^{m_4}.\end{aligned}$$

So after solving the system of equations $m_1=n_1+n_5, m_2=n_2, m_3=n_3$, and $m_4=n_4+n_5$ we can easily derive that there's no restriction for n_1,n_2,n_3, and n_4. Thus Proposition 5.1 implies Theorem 4.1. □

Lemma 6.1. *For the specialized hypergeometric function* $f=f(a,b,c,d,e)$ $={}_3\phi_2\left(\begin{matrix}a,b,c\\d,e\end{matrix};\dfrac{de}{abc}\right)$

$$(a-\frac{d}{q})f=(1-\frac{d}{q})f(d/q)+(a-1)f(aq) \tag{6.5}$$

and

$$(a-\frac{e}{q})f=(1-\frac{e}{q})f(e/q)+(a-1)f(aq). \tag{6.6}$$

Proof. By using $(aq)_n=(a)_n\dfrac{1-aq^n}{1-a}$ and $(d/q)_n=(d)_n\dfrac{1-d/q}{1-dq^{n-1}}$ we can easily verify (6.5). Equation (6.6) results by interchanging d and e in (6.5). □

Proof of Theorem 4.2. Let $f=f(a,b,c,d,e)={}_3\phi_2\left(\begin{matrix}a,b,c\\d,e\end{matrix};\dfrac{de}{abc}\right)$. From (2.7) and (2.8) ([10], p. 480) we have

$$\begin{aligned}&[de(a-b-c)+abc(d+e+q-a-aq)]f\\&\qquad+(1-a)(de-abcq)f(aq)+bc(d-a)(e-a)f(a/q)=0\end{aligned} \tag{6.7}$$

and

$$(a-b)f+(1-a)f(aq)-(1-b)f(bq)=0. \tag{6.8}$$

By interchanging b and c in (6.8), we also get

$$(a-c)f+(1-a)f(aq)-(1-c)f(cq)=0. \tag{6.9}$$

Say x_0 is shifting a to aq, x_1 is shifting b to bq, x_2 is shifting c to cq, x_3 is shifting d to dq, and x_4 is shifting e to eq. Similarly, say x_0^{-1} is shifting a to a/q, x_1^{-1} is shifting b to b/q, and so on. From (6.7) we know $\{1, x_0, x_0^{-1}\} \in S$. From (6.8) and (6.9) we know $\{1, x_0, x_1\}, \{1, x_0, x_2\} \in S$. From Lemma 6.1, we know $\{1, x_0, x_3^{-1}\}$ and $\{1, x_0, x_4^{-1}\}$, which is equivalent to $\{1, x_3, x_0x_3\}$ and $\{1, x_4, x_0x_4\}$, respectively, are in S. By the same argument as in the proof of Theorem 4.1, we get the result. □

Proof of Theorem 4.3. Consider $\phi = \phi(b,c,d,e,f,g,h)$. From (2.8) and (2.10) ([11] pp. 774–775) we have

$$\begin{aligned}
&\frac{g^2(1-h)(1-\frac{aq}{gb})(1-\frac{aq}{gc})(1-\frac{aq}{gd})(1-\frac{aq}{ge})(1-\frac{aq}{gf})}{(1-\frac{aq}{g})(1-\frac{aq^2}{g})}\phi_+(g-)\\
&-\frac{h^2(1-g)(1-\frac{aq}{hb})(1-\frac{aq}{hc})(1-\frac{aq}{hd})(1-\frac{aq}{he})(1-\frac{aq}{hf})}{(1-\frac{aq}{h})(1-\frac{aq^2}{h})}\phi_+(h-)\\
&-\frac{g(1-\frac{h}{g})(1-\frac{aq}{b})(1-\frac{aq}{c})(1-\frac{aq}{d})(1-\frac{aq}{e})(1-\frac{aq}{f})}{(1-aq)(1-aq^2)}\phi=0
\end{aligned} \tag{6.10}$$

and

$$\begin{aligned}
&\frac{agq}{h}\frac{(1-aq)(1-aq^2)(1-\frac{aq}{gb})(1-\frac{aq}{gc})(1-\frac{aq}{gd})(1-\frac{aq}{ge})(1-\frac{aq}{gf})}{(1-\frac{aq}{g})(1-\frac{aq^2}{g})(1-\frac{aq}{b})(1-\frac{aq}{c})(1-\frac{aq}{d})(1-\frac{aq}{e})(1-\frac{aq}{f})}\\
&\times(1-gh/aq)(1-h)(1-b)(1-c)(1-d)(1-e)(1-f)\phi_+(g-)\\
&-\frac{q(1-\frac{a}{g})(1-\frac{aq}{g})(1-\frac{a}{h})(1-\frac{aq}{h})(1-\frac{a}{b})(1-\frac{a}{c})(1-\frac{a}{d})(1-\frac{a}{e})(1-\frac{a}{f})}{(1-\frac{a}{q})(1-a)}\phi_-(g+)\\
&-\left[\frac{aq}{h}\left(1-\frac{h}{g}\right)\left(1-\frac{gh}{aq}\right)(1-b)(1-c)(1-d)(1-e)(1-f)\right.\\
&+\frac{g^2q^2}{h}\frac{(1-\frac{aq}{g})(1-\frac{a}{h})(1-\frac{aq}{h})(1-\frac{h}{q})(1-\frac{a}{gb})(1-\frac{a}{gc})(1-\frac{a}{gd})(1-\frac{a}{ge})(1-\frac{a}{gf})}{(1-\frac{gq}{h})(1-\frac{a}{gq})}\\
&\left.-\frac{h(1-g)(1-\frac{a}{g})(1-\frac{aq}{g})(1-\frac{aq}{hb})(1-\frac{aq}{hc})(1-\frac{aq}{hd})(1-\frac{aq}{he})(1-\frac{aq}{hf})}{(1-\frac{gq}{h})}\right]\phi=0,
\end{aligned} \tag{6.11}$$

where $\phi_\pm(g\mp)$ denotes the expression which would be obtained by the replacements

$$(b,c,d,e,f,g,h)\to(bq^{\pm1},cq^{\pm1},dq^{\pm1},eq^{\pm1},fq^{\pm1},g,hq^{\pm1}).$$

Now with $x_0 = x_2 \cdots x_7$ and $x'_j = x_1 x_j^{-1}$, where x_1 means shifting g to gq, x_2 means shifting b to bq, x_3, x_4, x_5, and x_6 mean shifting c to cq, d to dq, e to eq, and f to fq, respectively, and x_7 means shifting h to hq, x_1^{-1} means shifting g to g/q, etc., we have $\{1, x_0, x_0 x'_7\} \in S$ and $\{1, x_0, x_0^{-1}\} \in S$ from (6.10) and (6.11), respectively. By interchanging h and x in (6.10) where $x = b, c, d, e$, or f, we'll have five more equations relating to ϕ, $\phi_+(g-)$, and $\phi_+(x-)$ where $x = b, c, d, e, f$. Since b is the distinguished parameter, the case $x = b$ is not obvious. But this is also true as the equations (6.10) and (6.11) hold regardless of the choice of distinguished parameter [11], which can thus be taken to be g. Interchanging the labeling of g and b then gives the case $x = b$. By using these five relations we have $\{1, x_0, x_0 x'_j\} \in S, 2 \leq j \leq 6$. Then Proposition 5.1 shows that S consists of triples of elements of the form

$$\begin{aligned} &(x_2 x_3 \cdots x_7)^{n_1} (x_1 x_2^{-1})^{n_2} (x_1 x_3^{-1})^{n_3} \cdots (x_1 x_7^{-1})^{n_7} \\ =& x_1^{n_2 + \cdots + n_7} x_2^{n_1 - n_2} x_3^{n_1 - n_3} \cdots x_7^{n_1 - n_7} \\ =& x_1^{m_1} x_2^{m_2} \cdots x_7^{m_7}. \end{aligned}$$

After solving the relations between m_i and n_j we have

$$n_1 = \frac{m_1 + m_2 + m_3 + m_4 + m_5 + m_6 + m_7}{6},$$

$$n_i = n_1 - m_i, \ 2 \leq i \leq 7.$$

Therefore, the set S_0 satisfies the congruence condition in Theorem 4.3. Finally observe that any relation in the vector space generated by S_k, $k \neq 0$, may be transformed into a relation in the vector space generated by S_0 by simply shifting any parameter and vice versa. Hence the theorem follows. □

Corollary 4.4 follows immediately from Theorem 4.3.

Our proposition also can be applied to ordinary hypergeometric functions, we shall discuss this in a future publication.

7. Open problems

We will conclude by mentioning some open problems concerning basic hypergeometric continued fractions.

Convergence of R-R when $|q| = 1$, $q^n \neq 1$. The convergence of the Rogers-Ramanujan continued fraction and for $|q| < 1$ is easy to show. When $|q| = 1$, however less is known. When q is a root of unity, the problem was solved essentially by Ramanujan [22, p. 46] and Schur [25]. Recently more work has been done by Huang [13]. However, when q on the unit circle is not a root of unity, nothing is known. Does the Rogers-Ramanujan continued fraction converge when $|q| = 1$, q not a root of unity? This is not known for even one such q. Similarly one may ask this question for other q-continued fractions with boundaries on the unit circle.

Summing q-continued fraction in q only. All the continued fractions in this paper have parameters other than q. When one is given a q-continued fraction having only q present, find its limit as quotients of basic hypergeometric functions if this is possible or show that this is impossible.

Representation of a given modular form as a q-continued fraction in a non-canonical way. Given a modular form, find a q-continued fraction expression for it which is non-canonical. Non-canonical here is a little vague. Roughly it means that if the form already has an infinite product or series representation, the continued fraction does not result simply from transforming these into continued fractions using the transformations of Euler or Stern, [6, p. 512, 517]. Also give a better definition to "non-canonical" as we use it here. We roughly mean that the continued fraction results by iterating q-difference equations with extra variables which are specialized in the end to obtain the modular form.

The ${}_8\phi_7$ level. We have given theorems classifying partial q-difference equations at three levels in the basic hypergeometric hierarchy. Conspicuously missing is the very well poised ${}_8\phi_7$ level. In a previous paper [2] the first author gave a three-term relation and its associated continued fraction. Other papers giving continued fractions at this level have appeared. The reason that we have not treated this level is a shortage of known recurrences to give us a two dimensional vector space as in the other levels. One problem we intend to attack in the future is the calculation of such recurrences to settle the issue of which partial q-differences of ${}_8\phi_7$ functions are related in three terms.

Other relations at the ${}_{10}\phi_9$ level. Are there other partial q-difference equations for the ϕ function used in this paper which are not in the vector space constructed here? If so, conditions on the n_i may be weakened or possibly removed in Theorem 4.3.

Finding dominant solutions of difference equations. The second method of proving convergence indicated in this paper works fine, but it lacks some of the elegance of the first technique. In order to apply the first technique, one needs second solutions of the difference equations determined by the tail sequence of the continued fraction. We do not know these second solutions in general.

Finding initial set of relations. More generally, given a specialized basic hypergeometric function, find all three-term relations satisfying the conditions of Proposition 5.1. This would effectively classify all three-term partial q-difference equations with polynomial coefficients satisfied by the given basic hypergeometric function. Theorem 4.3 suggests that the shifting involved may be a $\mathbb{Z}$-linear transformation of the shifts to which the proposition is directly applicable with leading conditions on the shifts allowed.

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