RAMANUJAN'S SHORT UNPUBLISHED MANUSCRIPT ON INTEGRALS AND SERIES RELATED TO EULER'S CONSTANT

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0. Introduction

Included in the volume [15] featuring his lost notebook are some published and unpublished fragments of papers by Ramanujan. In particular, on pages 274 and 275 in [15], there is the beginning of a manuscript that probably was to focus on integrals related to Euler's constant γ and $\psi(s) := \Gamma'(s)/\Gamma(s)$, and to integrals and series related to Frullani's integral theorem. This fragment contains only two short sections comprising $1\frac{1}{2}$ pages. Afterwards, Ramanujan wrote "3." to indicate the beginning of a third section, but the manuscript ends abruptly at this point.

The purpose of this brief note is to present Ramanujan's work in this fragment and to relate it to other theorems in the literature. We also offer a couple new related theorems.

1. Ramanujan's Theorems on γ and $\psi(s)$

We first prove the primary theorem in Ramanujan's first section.

Proposition 1.1 (Ramanujan). *Let p, q, and r be positive. Then*

$$\int_0^1 \left(\frac{x^{p-1}}{1-x} - \frac{rx^{q-1}}{1-x^r}\right) dx = \psi(q/r) - \psi(p) + \log r. \tag{1.1}$$

Proof (due to Ramanujan). Using the continuity of the integrand on the right side below for $0 \le x, s \le 1$, a well-known integral representation for the beta function, the change of variable $t = x^r$ in the second part of the integrand, and L'Hospital's rule, we find that

$$\begin{aligned}
&\int_0^1 \left(\frac{x^{p-1}}{1-x} - \frac{rx^{q-1}}{1-x^r}\right) dx \\
&= \lim_{s\to 0+} \int_0^1 \left\{x^{p-1}(1-x)^{s-1} - r^{1-s}x^{q-1}(1-x^r)^{s-1}\right\} dx \\
&= \lim_{s\to 0} \left\{\frac{\Gamma(p)\Gamma(s)}{\Gamma(s+p)} - r^{-s}\frac{\Gamma(q/r)\Gamma(s)}{\Gamma(s+q/r)}\right\} \\
&= \lim_{s\to 0} \frac{\left\{\dfrac{\Gamma(p)}{\Gamma(s+p)} - r^{-s}\dfrac{\Gamma(q/r)}{\Gamma(s+q/r)}\right\}\Gamma(s+1)}{s} \\
&= \lim_{s\to 0} \left\{-\frac{\Gamma(p)\Gamma'(s+p)}{\Gamma^2(s+p)} + \Gamma(q/r)\left(\frac{r^{-s}\log r}{\Gamma(s+q/r)} + \frac{r^{-s}\Gamma'(s+q/r)}{\Gamma^2(s+q/r)}\right)\right\} \\
&= -\,\psi(p) + \log r + \psi(q/r),
\end{aligned}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX which completes the proof.

Theorem 1.2 (Ramanujan). *Suppose that $a, b,$ and c are positive with $b > 1$. Then*

$$\int_0^1 \left(\frac{x^{c-1}}{1-x} - \frac{bx^{bc-1}}{1-x^b}\right) \sum_{k=0}^{\infty} x^{ab^k}\, dx = \psi\left(\frac{a}{b}+c\right) - \log\frac{a}{b}.$$

Proof. By Proposition 1.1 and the facts that $b > 1$ and $\psi(x) \sim \log x$, as x tends to ∞,

$$\begin{aligned}
\int_0^1 \left(\frac{x^{c-1}}{1-x} - \frac{bx^{bc-1}}{1-x^b}\right) \sum_{k=0}^{n} x^{ab^k}\, dx &= \sum_{k=0}^{n} \int_0^1 \left(\frac{x^{c+ab^k-1}}{1-x} - \frac{bx^{bc+ab^k-1}}{1-x^b}\right) dx \\
&= \sum_{k=0}^{n} \left(\psi\left(ab^{k-1}+c\right) - \psi\left(ab^k+c\right) + \log b\right) \\
&= \psi\left(\frac{a}{b}+c\right) - \psi\left(ab^n+c\right) + (n+1)\log b \\
&= \psi\left(\frac{a}{b}+c\right) - \log\left(ab^n+c\right) + (n+1)\log b + o(1) \\
&= \psi\left(\frac{a}{b}+c\right) - n\log b - \log a + (n+1)\log b + o(1) \\
&= \psi\left(\frac{a}{b}+c\right) - \log\frac{a}{b} + o(1),
\end{aligned}$$

as n tends to ∞. Letting $n \to \infty$, we complete the proof.

Corollary 1.3 (Ramanujan). *We have*

(a) $$\int_0^1 \frac{1}{1+x} \sum_{k=1}^{\infty} x^{2^k}\, dx = 1-\gamma,$$

(b) $$\int_0^1 \frac{1+2x}{1+x+x^2} \sum_{k=1}^{\infty} x^{3^k}\, dx = 1-\gamma,$$

and

(c) $$\int_0^1 \frac{1+\frac{1}{2}\sqrt{x}}{(1+\sqrt{x})(1+\sqrt{x}+x)} \sum_{k=1}^{\infty} x^{(3/2)^k}\, dx = 1-\gamma.$$

Proof. In Theorem 1.2, set, respectively, $c = 1, a = b = 2; c = 1, a = b = 3;$ and $c = 1, a = b = 3/2$. Use the fact that [8, p. 954]

$$\psi(2) = 1 - \gamma \tag{1.2}$$

to complete the proof.

According to Bromwich [5, p. 526], (a) is due to Catalan. Parts (b) and (c) may be new.

Before discussing the very brief second section of Ramanujan's fragment, we offer some alternative proofs, references, and connections with further work of Ramanujan, as well as others.

Lemma 1.4. *For $x>0$ and any integer $n>1$,*
(1.3)
$$\frac{1}{\log x}+\frac{1}{1-x}=\sum_{k=1}^{\infty}\frac{(n-1)+(n-2)x^{1/n^k}+(n-3)x^{2/n^k}+\cdots+x^{(n-2)/n^k}}{n^k(1+x^{1/n^k}+x^{2/n^k}+\cdots+x^{(n-1)/n^k})}.$$

Proof. It is easy to verify that

$$\frac{1}{1-x^n}=\frac{1}{n}\left(\frac{(n-1)+(n-2)x+(n-3)x^2+\cdots+x^{n-2}}{1+x+x^2+\cdots+x^{n-1}}+\frac{1}{1-x}\right). \tag{1.4}$$

Replacing x by $x^{1/n}$ and iterating m times, we find that

$$\frac{1}{1-x}=\sum_{k=1}^{m}\frac{(n-1)+(n-2)x^{1/n^k}+(n-3)x^{2/n^k}+\cdots+x^{(n-2)/n^k}}{n^k(1+x^{1/n^k}+x^{2/n^k}+\cdots+x^{(n-1)/n^k})}+\frac{1}{n^m(1-x^{1/n^m})}.$$

If we now let m tend to ∞ and apply L'Hospital's rule, we complete the proof.

The special cases $n=2,3$ of Lemma 1.4 can be found in Ramanujan's third notebook [13, p. 364], and proofs can be found in Berndt's book [4, pp. 399–400]. Our proof generalizes these proofs.

Lemma 1.5. *For every integer $n>1$,*

$$\gamma=\int_0^1\left(\frac{n}{1-x^n}-\frac{1}{1-x}\right)\sum_{k=1}^{\infty}x^{n^k-1}dx. \tag{1.5}$$

Proof. Integrate (1.3) over $0\le x\le 1$ and employ the well-known integral representation [5, p. 507], [8, p. 955]

$$\gamma=\int_0^1\left(\frac{1}{\log x}+\frac{1}{1-x}\right)dx.$$

Accordingly, replacing x by x^{n^k}, we find that

$$\begin{aligned}\gamma&=\int_0^1\sum_{k=1}^{\infty}\frac{(n-1)+(n-2)x^{1/n^k}+(n-3)x^{2/n^k}+\cdots+x^{(n-2)/n^k}}{n^k(1+x^{1/n^k}+x^{2/n^k}+\cdots+x^{(n-1)/n^k})}dx\\&=\sum_{k=1}^{\infty}\int_0^1\frac{1}{n^k}\frac{(n-1)+(n-2)x^{1/n^k}+(n-3)x^{2/n^k}+\cdots+x^{(n-2)/n^k}}{1+x^{1/n^k}+x^{2/n^k}+\cdots+x^{(n-1)/n^k}}dx\\&=\sum_{k=1}^{\infty}\int_0^1\frac{(n-1)+(n-2)x+(n-3)x^2+\cdots+x^{n-2}}{1+x+x^2+\cdots+x^{n-1}}x^{n^k-1}dx\\&=\int_0^1\left(\frac{n}{1-x^n}-\frac{1}{1-x}\right)\sum_{k=1}^{\infty}x^{n^k-1}dx,\end{aligned} \tag{1.6}$$

by (1.4). This completes the proof.

Lemma 1.5 is equivalent to Theorem 1.2 in the case $c = 1, a = b = n$. To see this, first make these substitutions in Theorem 1.2 and use (1.2) to deduce that

$$1 - \gamma = \int_0^1 \left(\frac{1}{1-x} - \frac{nx^{n-1}}{1-x^n} \right) \sum_{k=1}^{\infty} x^{n^k} dx. \tag{1.7}$$

Adding (1.5) and (1.7) and simplifying, we readily find that

$$1 = (n-1) \int_0^1 \sum_{k=1}^{\infty} x^{n^k - 1} dx,$$

which is trivially verified by termwise integration.

The arguments in the proof of Lemma 1.5 lead to another formula for γ.

Theorem 1.6. *If b is an integer exceeding 1, let*

$$\epsilon_r = \begin{cases} b-1, & \text{if } b|r, \\ -1, & \text{if } b \nmid r. \end{cases} \tag{1.8}$$

Then

$$\gamma = \sum_{r=1}^{\infty} \frac{\epsilon_r}{r} \left[\frac{\log r}{\log b} \right],$$

where $[x]$ denotes the greatest integer $\leq x$.

Proof. Return to the last equality in (1.6), with $n = b$, and rewrite it in the form

$$\gamma = \int_0^1 \sum_{k=1}^{\infty} \frac{b - 1 - x - x^2 - \cdots - x^{b-1}}{1 - x^b} x^{b^k - 1} dx.$$

Now expand $1/(1-x^b)$ in a geometric series and integrate termwise to deduce that

$$\begin{aligned} \gamma &= \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \int_0^1 (b - 1 - x - x^2 - \cdots - x^{b-1}) x^{b^k + bj - 1} dx \\ &= \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \left(\frac{b-1}{b^k + bj} - \frac{1}{b^k + bj + 1} - \frac{1}{b^k + bj + 2} - \cdots - \frac{1}{b^k + bj + (b-1)} \right). \end{aligned}$$

Now count the number of appearances of each harmonic term $1/r, 1 \leq r < \infty$. If r is a multiple of b, then the coefficient $b - 1$ appears $[\log r / \log b]$ times, while if r is not a multiple of b, the coefficient -1 also appears $[\log r / \log b]$ times, and so the proof is complete.

Corollary 1.7. *We have*

$$\gamma = \sum_{r=1}^{\infty} \frac{(-1)^r}{r} \left[\frac{\log r}{\log 2} \right]. \tag{1.9}$$

Proof. Let $b=2$ in Theorem 1.6.

The representation for γ given in (1.9) was discovered in 1909 by G. Vacca [17] and is known as Dr. Vacca's series for γ. Corollary 1.7 was rediscovered by H. F. Sandham, who submitted it as a problem [16]. M. Koecher [11] obtained a generalization of (1.9) that includes a formula for γ submitted by Ramanujan as a problem [12], [14, p. 325] to the *Journal of the Indian Mathematical Society*, and found in his notebooks [13], [3, p. 196]. Further series in the spirit of those of Ramanujan and Koecher were found by F. L. Bauer [2]. A result similar to that of Bauer was found by A. W. Addison [1], with a simpler version later established by I. Gerst [6].

J. W. L. Glaisher [7] generalized Theorem 1.6. We give below a proof of a theorem which is apparently equivalent to his theorem. We again use Theorem 1.2, but we use a different approach from that in the proof of Theorem 1.6.

Theorem 1.8. *Let a and b be positive integers with $b>1$, and let ϵ_r be defined by (1.8). Then*

$$\log a+\gamma=\sum_{r=a}^{\infty}\frac{\epsilon_r}{r}\left[\frac{\log(r/a)}{\log b}\right].$$

Proof. From Theorem 1.2,

$$\int_0^1\left\{b\left(1+\frac{x^{bc}}{1-x^b}\right)-1-\frac{x^c}{1-x}\right\}\sum_{k=0}^{\infty}x^{ab^k-1}dx=\frac{b}{a}+\log\frac{a}{b}-\psi\left(\frac{a}{b}+c\right). \tag{1.10}$$

(That (1.10) is equivalent to Theorem 1.2 can be easily demonstrated by adding (1.10) to the equality of Theorem 1.2.) Setting $c=1$ in (1.10) and recalling that ϵ_j is defined by (1.8), we find that

$$\begin{aligned}\frac{b}{a}+\log\frac{a}{b}-\psi\left(\frac{a}{b}+1\right)&=\int_0^1\left(\sum_{k=0}^{\infty}x^{ab^k-1}\right)\left(\sum_{j=0}^{\infty}\epsilon_jx^j\right)dx\\&=\int_0^1\sum_{n=a-1}^{\infty}\left(\sum_{\substack{ab^k-1+j=n\\k,j\geq 0}}\epsilon_j\right)x^n dx\\&=\int_0^1\sum_{n=a}^{\infty}\left(\sum_{k=0}^{\left[\frac{\log(n/a)}{\log b}\right]}\epsilon_{n-ab^k}\right)x^{n-1}dx\\&=\int_0^1\sum_{n=a}^{\infty}\left(\epsilon_{n-a}+\sum_{k=1}^{\left[\frac{\log(n/a)}{\log b}\right]}\epsilon_n\right)x^{n-1}dx\\&=\sum_{n=a}^{\infty}\frac{1}{n}\left(\epsilon_{n-a}+\left[\frac{\log(n/a)}{\log b}\right]\epsilon_n\right),\end{aligned} \tag{1.11}$$

where in the penultimate step we used the fact that $\epsilon_i=\epsilon_j$ if $i\equiv j\pmod b$. It remains to evaluate the sum $\sum_{n\geq a}\epsilon_{n-a}/n$. In the evaluation below, we use three familiar properties of $\psi(z)$ [8, formulas 8.362, no. 1, p. 952; 8.365, no. 6, p. 954; 8.366, no. 1, p. 954]. Accordingly, we find that

$$\begin{aligned}
-\sum_{n=a}^{\infty}\frac{\epsilon_{n-a}}{n} &= \sum_{k=0}^{\infty}\sum_{j=1}^{b}\left(\frac{1}{kb+j}-\frac{1}{kb+a}\right)\\
&= \frac{1}{b}\sum_{j=1}^{b}\sum_{k=0}^{\infty}\left\{\left(\frac{1}{k+\frac{j}{b}}-\frac{1}{k+1}\right)+\left(\frac{1}{k+1}-\frac{1}{k+\frac{a}{b}}\right)\right\}\\
&= \frac{1}{b}\sum_{j=1}^{b}\left\{\left(-\psi\left(\frac{j}{b}\right)-\gamma\right)+\left(\psi\left(\frac{a}{b}\right)+\gamma\right)\right\}\\
&= \frac{1}{b}\sum_{j=1}^{b}\left(\psi\left(\frac{a}{b}\right)-\psi\left(\frac{j}{b}\right)\right)\\
&= \psi\left(\frac{a}{b}\right)-\frac{1}{b}\sum_{j=0}^{b-1}\psi\left(\frac{j+1}{b}\right)\\
&= \psi\left(\frac{a}{b}\right)-\psi(1)+\log b\\
&= \psi\left(\frac{a}{b}\right)+\gamma+\log b. \qquad (1.12)
\end{aligned}$$

Substituting (1.12) into (1.11) and simplifying slightly, we complete the proof.

We have found an analogue of Theorem 1.8 in the case that $c > 1$. Since the proof again uses Theorem 1.2, and since our result is not as elegant as Theorem 1.8, we state the theorem but do not prove it.

Theorem 1.9. *Let a, b, and c be positive integers with $b > 1$. Define*

$$\epsilon_{r,c} = \begin{cases} 1-b, & \text{if } r = b\ell, \quad \ell \geq c,\\ 1, & \text{otherwise.}\end{cases}$$

Then

$$\psi(a+c)-\log a - \sum_{n=a+c+2}^{ab+cb-1}\frac{\left[\dfrac{\log((n-c)/a)}{\log b}\right]}{n} = \sum_{n=ab+cb}^{\infty}\frac{\left[\dfrac{\log((n-cb)/a)}{\log b}\right](\epsilon_{n,c}-1)+\left[\dfrac{\log((n-c)/a)}{\log b}\right]}{n}.$$

We complete this section with a remark about Proposition 1.1. After replacing x by e^{-x} in (1.1), we obtain an integral of Frullani-type. In his third quarterly report, Ramanujan found a beautiful generalization of Frullani's theorem. In particular, the formula

$$\int_0^{\infty}\frac{(1+ax)^{-p}-(1+bx)^{-q}}{x}dx = \psi(q)-\psi(p)+\log\frac{b}{a}, \qquad (1.13)$$

where $a, b, p, q > 0$, is a special instance of Ramanujan's theorem [3, p. 314]. In view of the right sides of (1.1) and (1.13), one might surmise that (1.1) can be derived from (1.13), or Ramanujan's generalization of Frullani's theorem, but we have been unable to do this.

2. Integral Representations of $\log x$

Section 2 in Ramanujan's unpublished fragment is devoted solely to the statements of the following theorem and (2.1) below.

Theorem 2.1 (Ramanujan). *If a, b, and c are positive with $b > 1$, then*

$$\int_0^1 \frac{x^{c-1} - x^{bc-1}}{\log x} \sum_{k=0}^{\infty} x^{ab^k} dx = \log\left(1 + \frac{bc}{a}\right).$$

Proof. As indicated by Ramanujan, we begin with the equality [8, p. 575]

$$\int_0^1 \frac{x^{p-1} - x^{q-1}}{\log x} dx = \log \frac{q}{p}, \tag{2.1}$$

where $p, q > 0$. Thus, since $b > 1$,

$$\begin{aligned}
\int_0^1 \frac{x^{c-1} - x^{bc-1}}{\log x} \sum_{k=0}^{n} x^{ab^k} dx &= \sum_{k=0}^{n} \int_0^1 \frac{x^{c+ab^k-1} - x^{bc+ab^k-1}}{\log x} dx \\
&= \sum_{k=0}^{n} \log \frac{bc + ab^k}{c + ab^k} \\
&= \sum_{k=0}^{n} \left(\log b + \log(c + ab^{k-1}) - \log(c + ab^k)\right) \\
&= (n+1) \log b + \log(c + a/b) - \log(c + ab^n) \\
&= (n+1) \log b + \log(c + a/b) - n \log b - \log a + o(1) \\
&= \log(1 + bc/a) + o(1),
\end{aligned}$$

as n tends to ∞. Letting n tend to ∞, we complete the proof.

Corollary 2.2. *We have*

$$\int_0^1 \frac{1-x}{\log x} \sum_{k=1}^{\infty} x^{2^k} = \log 2.$$

Proof. Set $c = 1$ and $a = b = 2$ in Theorem 2.1.

Observe that if x is replaced by e^{-x} in (2.1), we obtain an example of Frullani's integral theorem. Ramanujan's ideas can be extended to other examples of Frullani-type integrals found by, among others, Ramanujan in his Quarterly Reports [3] and Hardy [9], [10, pp. 195–226]. For example, consider the integral [9, eq. (29)], [10, p. 200]

$$\int_0^{\infty} \frac{e^{-ax}\cos(\alpha x) - e^{-bx}\cos(\beta x)}{x} dx = -\frac{1}{2} \log \frac{a^2 + \alpha^2}{b^2 + \beta^2}, \tag{2.2}$$

where $a, b, \alpha, \beta > 0$. We conclude with a brief application of (2.2) which we do not prove, as the proof is similar to those above.

Theorem 2.3. *If* $a, b, c, \alpha > 0$ *with* $b > 1$, *then*

$$\int_0^1 \frac{x^{c-1}\cos(\alpha \log x) - x^{bc-1}\cos(\alpha b \log x)}{\log x} \sum_{k=0}^{\infty} x^{ab^k}\, dx$$

$$= -\frac{1}{2}\log\left\{\left(1+\frac{bc}{a}\right)^2 + \frac{\alpha^2 b^2}{a^2}\right\}.$$

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