A general Heine transformation and symmetric polynomials of Rogers

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1 Introduction

Our purpose here is first to give a multivariate generalization of the important transformation of Heine [6, pp. 305-306]] (see also [1, ch.2]),

$$(1) \qquad {}_2\Phi_1\left(\begin{matrix} a,\, b \\ c \end{matrix};z\right) = \frac{(az)_\infty (b)_\infty}{(c)_\infty (z)_\infty}\, {}_2\Phi_1\left(\begin{matrix} c/b,\, z \\ az \end{matrix};b\right), \qquad |b|, |z| < 1,$$

and second to characterize the symmetric polynomials introduced by L.J. Rogers [8] in 1893.

Throughout we employ the standard notation:

$$(A)_n = (A;q)_n = (1-A)(1-Aq)\ldots(1-Aq^{n-1}),$$

so that

$$(q)_n = (1-q)(1-q^2)\ldots(1-q^n).$$

We always take $|q| < 1$. Let

$$(A_1, A_2, \ldots, A_m)_n = (A_1)_n (A_2)_n \cdots (A_m)_n.$$

In (1) the ${}_2\Phi_1$ is the $m = 1$ case of the general basic hypergeometric function ${}_{m+1}\Phi_m$ defined by

$${}_{m+1}\Phi_m\left(\begin{matrix} a_1, a_2, \ldots, a_{m+1} \\ b_1, b_2, \ldots, b_m \end{matrix}; z\right) = \sum_{n\ge 0} \frac{(a_1)_n (a_2)_n \ldots (a_{m+1})_n}{(b_1)_n (b_2)_n \ldots (b_m)_n (q)_n} z^n.$$

For elementary facts about these functions we refer the reader to [5].

We introduce a convenient notation which we will use continually. Let

$$[x,y]_n = (x-y)(x-yq)\ldots(x-yq^{n-1}), \qquad n \in \mathbb{Z}^+.$$

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If $x \neq 0$, then $[x,y]_n = x^n (y/x)_n$. For $x \neq 0$ we define $[x,y]_k$ for real k by the equation

$$[x,y]_k = \frac{(y/x)_\infty}{(yq^k/x)_\infty}. \tag{2}$$

Here and above $(A)_\infty$ is defined by

$$(A)_\infty = \lim_{n\to\infty} (A)_n.$$

We note that $(y)_n = [1,y]_n$ and define $[x]_n = [x,1]_n$. Clearly

$$[x,0]_n = x^n, \qquad [0,y]_n = (-1)^n q^{\binom{n}{2}} y^n. \tag{3}$$

Rogers's polynomials

$$h_j = h_j^{(m)}(\mathbf{a};\mathbf{b}) = h_j^{(m)}(a_1,\ldots,a_m;b_1,\ldots,b_m : q)$$

are defined explicitly by

$$h_j = \sum_{n_1+\cdots+n_m=j} \begin{bmatrix} j \\ n_1,\ldots,n_m \end{bmatrix} [b_1,a_1]_{n_1} \cdots [b_m,a_m]_{n_m},$$

where

$$\begin{bmatrix} j \\ n_1,\ldots,n_m \end{bmatrix} = \frac{(q)_j}{(q)_{n_1}\cdots(q)_{n_m}}.$$

Usually the argument q is implicit and is not written in the h_j. In applications we abbreviate $h_j(0,\ldots,0;b_1,\ldots,b_m : q)$ by $h_j(b_1,\ldots,b_m)$ or $h_j(\mathbf{b})$.

Rogers [7, 8] used his polynomials h_j for, among other things, his powerful symmetric expansion for Heine's ${}_2\Phi_1$:

$$\text{If} \quad f(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = (\lambda_1\lambda_2)_\infty(\lambda_1^{-1}\lambda_4)_\infty\, {}_2\Phi_1\left(\begin{matrix}\lambda_1\lambda_2,\ \lambda_1\lambda_3 \\ \lambda_2\lambda_3\end{matrix};\lambda_1^{-1}\lambda_4\right),$$

then

$$\begin{aligned} f(\lambda_1,\lambda_2,\lambda_3,\lambda_4) =&(\lambda_1^{-1}\lambda_2)_\infty(\lambda_1^{-1}\lambda_3)_\infty(\lambda_1^{-1}\lambda_4)_\infty(\lambda_1\lambda_2)_\infty(\lambda_1\lambda_3)_\infty(\lambda_1\lambda_4)_\infty \\ &\times \sum_{n\geq 0} \frac{h_n^{(2)}(\lambda_1^{-1},\lambda_1)h_n^{(3)}(\lambda_2,\lambda_3,\lambda_4)}{(q)_n}. \end{aligned} \tag{4}$$

Clearly $h_n^{(k)}(\lambda_1,\ldots,\lambda_k)$ is symmetric in its parameters. It is amazing how much information about special functions is contained in this one expansion; see Andrews [1, ch. 2].

In a recent book Fine [4, p. 30] gave a q-multinomial theorem. However it is easily seen to be a special case of that given by Rogers [8]. We use the Rogers q-multinomial theorem to obtain an extension of the Heine transformation to an arbitrary number of variables. As applications we give some new expansions of the general ${}_{m+1}\Phi_m$ in terms of Rogers symmetric polynomials. (These expansions are not extensions of (4), but rather more generic.) Our characterization of Rogers's symmetric polynomials will reveal them to be discrete analogues of the general ${}_{m+1}\Phi_m$ function.

2 The Rogers q-multinomial theorem and a multivariate Heine transformation

Recall Cauchy's q-binomial theorem, [3], [5, pp. 7-8]:

$$\sum_{n\geq 0}\frac{(a)_n}{(q)_n}z^n=\frac{(az)_\infty}{(z)_\infty},\qquad |z|<1. \tag{5}$$

In the notation $[x,y]$, (5) may be written in the form

$$\sum_{n\geq 0}\frac{[b,a]_n}{(q)_n}z^n=\frac{(az)_\infty}{(bz)_\infty},\qquad |bz|<1. \tag{6}$$

We are now ready to discuss the q-multinomial theorem. Although it is implicit in [8], it does not seem to have received much attention. To obtain Rogers's theorem it is only necessary to multiply (6) together m times.

Theorem 2.1 *q-Multinomial Theorem*

$$\frac{(a_1z)_\infty\cdots(a_mz)_\infty}{(b_1z)_\infty\cdots(b_mz)_\infty}=\sum_{j\geq 0}\frac{h_j}{(q)_j}z^j,\qquad |b_iz|<1. \tag{7}$$

To obtain the theorem in Fine [4] put $a_1=\cdots=a_m=0$.

We now apply (7) to obtain an extension of the Heine transformation.

Theorem 2.2 *Generalized Heine Transformation*

For $m\geq 1$, $|b_i|<1$, $|d_i|<1$ we have

$$\sum_{k\geq 0}\frac{(d_1)_k\cdots(d_m)_k}{(c_1)_k\cdots(c_m)_k}\frac{h_k(\mathbf{a};\mathbf{b})}{(q)_k}=$$

$$\frac{(d_1)_\infty\cdots(d_m)_\infty(a_1)_\infty\cdots(a_m)_\infty}{(c_1)_\infty\cdots(c_m)_\infty(b_1)_\infty\cdots(b_m)_\infty}\sum_{k\geq 0}\frac{(b_1)_k\cdots(b_m)_k}{(a_1)_k\cdots(a_m)_k}\frac{h_k(\mathbf{c};\mathbf{d})}{(q)_k}.$$

The case $\mathbf{a} = \mathbf{b} = \mathbf{c} = \mathbf{0}$ of our theorem is equation (21.31) of [4]. Our theorem first appeared in the author's dissertation [2].

Proof

Observe that the sum

$$\sum_{k,l\geq 0} \frac{h_k(\mathbf{a};\mathbf{b})q^{kl}h_l(\mathbf{c};\mathbf{d})}{(q)_k(q)_l}$$

is invariant under the interchange of $\mathbf{a}$ with $\mathbf{c}$, $\mathbf{b}$ with $\mathbf{d}$ and converges absolutely for $|b_i|, |d_i| < 1$. Summing with respect to l and applying the q-multinomial theorem, Theorem 2.1, and (2), we find that

$$\frac{(c_1)_\infty \cdots (c_m)_\infty}{(d_1)_\infty \cdots (d_m)_\infty} \sum_{k\geq 0} \frac{(d_1)_k \cdots (d_m)_k}{(c_1)_k \cdots (c_m)_k} \frac{h_k(\mathbf{a};\mathbf{b})}{(q)_k}$$

is also invariant. This is equivalent to the statement of the theorem.

Letting $m = 1$ in the theorem gives

$$\sum_{k\geq 0} \frac{(d)_k}{(c)_k} \frac{h_k^{(1)}(a;b)}{(q)_k} = \frac{(d)_\infty (a)_\infty}{(c)_\infty (b)_\infty} \sum_{k\geq 0} \frac{(b)_k}{(a)_k} \frac{h_k^{(1)}(c;d)}{(q)_k}.$$

Using the fact that $h_k^{(1)}(x;y) = [y,x]_k = (x/y)_k y^k$ and replacing a by ab throughout, we obtain

$$\sum_{k\geq 0} \frac{(d)_k (a)_k}{(c)_k (q)_k} b^k = \frac{(d)_\infty (ab)_\infty}{(c)_\infty (b)_\infty} \sum_{k\geq 0} \frac{(b)_k (c/d)_k}{(ab)_k (q)_k} d^k,$$

which is the Heine transformation. Notice that in this case an additional symmetry was introduced due to the fact that h_k reduces to a q-factorial. This is the symmetry between a and d. For $m > 1$, h_k doesn't reduce to a factor of this type and the transformation group of substitutions of parameters induced by our theorem is isomorphic to $S_m^4 \times S_2$. Thus when $m = 1$ the group is $S_2 \times S_3$, Rogers [7], while for $m = 2$, the group is S_2^5. Before giving our characterization of $h_k^{(m)}$ we give some applications of our Heine extension. Note these results are valid when $|b|, |d_i| < 1$ for $1 \leq i \leq m$.

Corollary 2.1

$${}_{m+1}\Phi_m \left(\begin{matrix} a, d_1, d_2, \ldots, d_m \\ c_1, c_2, \ldots, c_m \end{matrix}; b \right) = \frac{(d_1, \ldots, d_m, ab)_\infty}{(c_1, \ldots, c_m, b)_\infty} \sum_{k\geq 0} \frac{(b)_k}{(ab)_k (q)_k} h_k^{(m)}(\mathbf{c};\mathbf{d}).$$

Proof

In Theorem 2.2 put $\mathbf{a} = (ab, 0, \ldots, 0)$ and $\mathbf{b} = (b, 0, \ldots, 0)$ and the corollary follows.

Putting $c_m = a$ in the last result gives:

Corollary 2.2

$$ {}_m\Phi_{m-1}\left(\begin{matrix} d_1, d_2, \ldots, d_m \\ c_1, c_2, \ldots, c_{m-1} \end{matrix}; b\right) = \frac{(d_1, \ldots, d_m, ab)_\infty}{(c_1, \ldots, c_{m-1}, a, b)_\infty} \sum_{k\geq 0} \frac{(b)_k}{(ab)_k (q)_k} h_k^{(m)}(a, \mathbf{c}; \mathbf{d}). $$

Notice that the parameter a appears only on the right side. Putting $a = 0$ yields:

Corollary 2.3

$$ {}_m\Phi_{m-1}\left(\begin{matrix} d_1, d_2, \ldots, d_m \\ c_1, c_2, \ldots, c_{m-1} \end{matrix}; b\right) = \frac{(d_1, \ldots, d_m)_\infty}{(c_1, \ldots, c_{m-1}, b)_\infty} \sum_{k\geq 0} \frac{(b)_k}{(q)_k} h_k^{(m)}(\mathbf{c}; \mathbf{d}). $$

(In this last result notice that $\mathbf{c}$ has only $m-1$ elements; the first element of the first vector argument of h may be set equal to 0.)

We conclude this section stating the case $m = 2$ of these last three corollaries.

$$ (8) \qquad {}_3\Phi_2\left(\begin{matrix} a, d_1, d_2 \\ c_1, c_2 \end{matrix}; b\right) = \frac{(d_1, d_2, ab)_\infty}{(c_1, c_2, b)_\infty} \sum_{k\geq 0} \frac{(b)_k}{(ab)_k (q)_k} h_k^{(2)}(c_1, c_2; d_1, d_2). $$

$$ (9) \qquad {}_2\Phi_1\left(\begin{matrix} d_1, d_2 \\ c \end{matrix}; b\right) = \frac{(d_1, d_2, ab)_\infty}{(c, a, b)_\infty} \sum_{k\geq 0} \frac{(b)_k}{(ab)_k (q)_k} h_k^{(2)}(a, c; d_1, d_2). $$

$$ (10) \qquad {}_2\Phi_1\left(\begin{matrix} d_1, d_2 \\ c \end{matrix}; b\right) = \frac{(d_1, d_2)_\infty}{(c, b)_\infty} \sum_{k\geq 0} \frac{(b)_k}{(q)_k} h_k^{(2)}(c; d_1, d_2). $$

3 The discrete linear n'th order q-difference equation and the polynomials $h_k^{(m)}$

In this section we prove a fundamental property of the polynomials $h_k^{(m)}$. We show that they are in fact discrete analogues of the basic hypergeometric functions ${}_m\Phi_{m-1}$. It is interesting that the symmetric expansions for basic series are in terms of these polynomials and certain operators. This has been explored somewhat in the author's dissertation [2] and will be the subject of a future long paper or memoir.

It is well known that the analytic solution at the origin of the q-difference equation

$$ (1-z)f(z) + (\beta_1 - \alpha_1 z)f(zq) + \cdots + (\beta_m - \alpha_m z)f(zq^m) = 0. $$

with initial condition $f(0) = 1$ is given by ${}_m\Phi_{m-1}$ (Thomae [9, pp. 108-112]). Here we study an analogous difference equation with a polynomial solution. This equation is

$$(11)\ (1-q^n)H_n + (\beta_1 - \alpha_1 q^{n-1})H_{n-1} + \cdots + (\beta_m - \alpha_m q^{n-m})H_{n-m} = 0.$$

We specify the initial conditions $H_n = 0$ for $n < 0$ and $H_0 = 1$. It is easy to see that these conditions uniquely determine the polynomials H_n. We proceed to solve (11). We assume throughout that $\beta_0 = \alpha_0 = 1$. We refer to (11) as the discrete q-difference equation because of the dependence on the discrete variable n leading to a solution which is a sequence.

Equation (11) can be written in the form

$$\sum_{n-m\le j\le n} (\beta_{n-j} - \alpha_{n-j}q^j)H_j = 0,$$

or

$$\sum_{n-m\le j\le n} \beta_{n-j}H_j = \sum_{n-m\le j\le n} \alpha_{n-j}q^j H_j.$$

Multiplying both sides by z^n and summing over $n \ge 0$ gives

$$\sum_{n\ge 0}\sum_{n-m\le j\le n} \beta_{n-j}H_j z^n = \sum_{n\ge 0}\sum_{n-m\le j\le n} \alpha_{n-j}q^j H_j z^n.$$

Using the initial conditions, we get

$$\sum_{\substack{j\ge 0\\ 0\le i\le m}} \alpha_i H_j q^j z^{i+j} = \sum_{\substack{j\ge 0\\ 0\le i\le m}} \beta_i H_j z^{i+j},$$

or

$$P(z)\sum_{j\ge 0} H_j q^j z^j = Q(z)\sum_{j\ge 0} H_j z^j,$$

where

$$\alpha_0 + \cdots + \alpha_m x^m = (1-a_1x)\cdots(1-a_mx),$$

and

$$\beta_0 + \cdots + \beta_m x^m = (1-b_1x)\cdots(1-b_mx).$$

If $B(z) = \sum_{j\ge 0} H_j z^j$ is the generating function for the sequence H_j, then

$$B(z) = \frac{P(z)}{Q(z)}B(zq).$$

Iterating gives

$$B(z) = \frac{(a_1z)_\infty \cdots (a_mz)_\infty}{(b_1z)_\infty \cdots (b_mz)_\infty}.$$

Finally by Theorem 2.1,

$$\sum_{j\geq 0} H_j z^j = B(z) = \sum_{j\geq 0} \frac{h_j}{(q)_j} z^j.$$

Equating coefficients now gives

$$H_j = \frac{h_j}{(q)_j}. \tag{12}$$

This conclusion is summarized in the following theorem.

Theorem 3.1 *Let $\alpha_0 = \beta_0 = 1$ and let $m > 0$ be a fixed integer. Then the solution of the equation*

$$(\beta_0 - \alpha_0 q^n)H_n + (\beta_1 - \alpha_1 q^{n-1})H_{n-1} + \cdots + (\beta_m - \alpha_m q^{n-m})H_{n-m} = 0$$

with $H_n = 0$ for $-m < n < 0$ and $H_0 = 1$ is given by

$$H_j = \frac{h_j}{(q)_j} = \frac{1}{(q)_j} \sum_{n_1+\cdots+n_m=j} \begin{bmatrix} j \\ n_1, \ldots, n_m \end{bmatrix} [b_1, a_1]_{n_1} \cdots [b_m, a_m]_{n_m},$$

where

$$\alpha_0 + \cdots + \alpha_m x^m = (1 - a_1 x) \cdots (1 - a_m x),$$

and

$$\beta_0 + \cdots + \beta_m x^m = (1 - b_1 x) \cdots (1 - b_m x).$$

Notice that when $q = 1$, $h_j = (b_1 + \cdots + b_m - a_1 - \cdots - a_m)^j$. For future reference we mention the easily proved relation $h_j(\mathbf{a}; \mathbf{b} : q) = (-1)^j q^{\binom{j}{2}} h_j(\mathbf{b}; \mathbf{a} : q^{-1})$.

In the case $m = 1$ equation (11) reduces to $(1 - q^n)H_n = (b - aq^{n-1})H_{n-1}$ from which it follows immediately that $H_j = [b, a]_j/(q)_j$.

The case $m = 2$ of this theorem is important in that it gives a general class of polynomials which satisfy three-term recurrence relations. So for example, $h_n^{(2)}(\beta e^{i\theta}, \beta e^{-i\theta}; e^{i\theta}, e^{-i\theta})/(q)_j$ is the continuous q-ultraspherical polynomial $C_n(x; \beta|q)$, where $x = cos\theta$ (see [5, pp. 168-172]). The q-difference equation of our theorem is precisely the three-term recurrence for this orthogonal polynomial system.

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$$1+\frac{(1-q^{\alpha})(1-q^{\beta})}{(1-q)(1-q^{\gamma})}x+\frac{(1-q^{\alpha})(1-q^{\alpha+1})(1-q^{\beta})(1-q^{\beta+1})}{(1-q)(1-q^{2})(1-q^{\gamma})(1-q^{\gamma+1})}x^{2}+\cdots,$$

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$$1+\sum_{n=1}^{\infty}x^{n}\frac{1-q^{a}}{1-q}\cdot\frac{1-q^{a+1}}{1-q^{2}}\cdots\frac{1-q^{a+n-1}}{1-q^{n}}\cdot\frac{1-q^{a'}}{1-q^{b'}}\cdot\frac{1-q^{a'+1}}{1-q^{b'+1}}\cdots$$
$$\frac{1-q^{a'+n-1}}{1-q^{b'+n-1}}\cdots\frac{1-q^{a^{(h)}}}{1-q^{b^{(h)}}}\cdot\frac{1-q^{a^{(h)}+1}}{1-q^{b^{(h)}+1}}\cdots\frac{1-q^{a^{(h)}+n-1}}{1-q^{b^{(h)}+n-1}},$$

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