q-difference operators, orthogonal polynomials, and symmetric expansions

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Contents

Abstract

We explore ramifications and extensions of a q-difference operator method first used by L.J. Rogers for deriving relationships between special functions involving certain fundamental q-symmetric polynomials. In special cases these symmetric polynomials reduce to well-known classes of orthogonal polynomials. A number of basic properties of these polynomials follow from our approach. This leads naturally to the evaluation of the Askey-Wilson integral and generalizations. We also find expansions of certain generalized basic hypergeometric functions in terms of the symmetric polynomials. This provides us with a quick route to understanding the group structure generated by iterating the two-term transformations of these functions. We also lay some infrastructure for more general investigations in the future.

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CHAPTER 1

Introduction and preliminaries

In 1812 Gauss [**26**] introduced the hypergeometric function

$$ {}_2F_1(A,B;C;z) = 1 + \frac{AB}{C\cdot 1!}z + \frac{A(A+1)B(B+1)}{C(C+1)\cdot 2!}z^2 + \cdots $$

which unifies the treatment of many classical special functions. By specializing its parameters most of the standard transcendental functions as well as a number of orthogonal polynomials can be obtained. Many relations between special functions arise from transformations of F. By considering generalizations of F, further special functions can be treated. Most of these generalizations are obtained by just adding more parameters similar to A, B or C. However in 1846 Heine [**30**] generalized F in a different direction. His generalization is

$$ {}_2\Phi_1\left(\begin{matrix} a,\, b \\ c \end{matrix};z\right) = 1 + \frac{(1-a)(1-b)}{(1-c)(1-q)}z + \frac{(1-a)(1-aq)(1-b)(1-bq)}{(1-c)(1-cq)(1-q)(1-q^2)}z^2 + \cdots . $$

Heine used $a = q^A$, $b = q^B$ and $c = q^C$, and let $q \to 1$, to obtain Gauss' series. Therefore ${}_2\Phi_1$ is called a q-analogue of F. Today a, b, and c are taken to be independent of q, which not only makes printing easier, but also adds generality. For brevity, the standard notation $(x)_n = (x;q)_n = (1-x)(1-xq)\ldots(1-xq^{n-1})$ will be employed. In particular $(q)_n = (1-q)(1-q^2)\ldots(1-q^n)$. When we let $n\to\infty$, convergence is taken either in the formal sense (with respect to x or q) or in the topology of the complex x or q-planes. More parameters can be added as in the case of F, and the general *basic hypergeometric series* is defined by

$$ (1.1)\quad {}_r\Phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix};z\right) = \sum_{n\geq 0} \frac{(a_1)_n(a_2)_n\ldots(a_r)_n}{(b_1)_n(b_2)_n\ldots(b_s)_n(q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{s+1-r} z^n. $$

Since Heine's paper, research in this area has focused on finding transformation and summation theorems for specializations of ${}_r\Phi_s$. This was started by Heine in his 1847 paper [**31**] which showed that

$$ (1.2)\qquad {}_2\Phi_1\left(\begin{matrix} a,\, b \\ c \end{matrix};z\right) = \frac{(az)_\infty(b)_\infty}{(c)_\infty(z)_\infty}\,{}_2\Phi_1\left(\begin{matrix} c/b,\, z \\ az \end{matrix};b\right). $$

This theorem generalizes many previous results on q-series such as the q-binomial theorem

$$ \sum_{n\geq 0}\frac{(a)_n}{(q)_n}z^n = \frac{(az)_\infty}{(z)_\infty}, $$

which is obtained by putting $b = c = 0$ in (1.2).

Other transformations for specializations of ${}_r\Phi_s$ have been found; they occur somewhat sporadically. Moreover, there is no theory guaranteeing existence or non-existence of transformations for a particular ${}_r\Phi_s$. To develop such a theory, it is instructive to return to (1.2) and seek to understand what is happening in this case. Why does ${}_2\Phi_1$ have this transformation formula?

An answer to this question was found in 1893 by L.J. Rogers. To understand his solution, note that iterating (1.2), along with the obvious symmetry between a and b, leads to two further transformations. These are, after multiplying by $(c)_\infty(z)_\infty$,

$$(c)_\infty(z)_\infty\,{}_2\Phi_1\left(\begin{matrix} a,\, b \\ c \end{matrix};z\right) = \begin{cases} (bz)_\infty(c/b)_\infty\,{}_2\Phi_1\left(\begin{matrix} abz/c,\, b \\ bz \end{matrix};c/b\right) \\ (c)_\infty(abz/c)_\infty\,{}_2\Phi_1\left(\begin{matrix} c/a,\, c/b \\ c \end{matrix};abz/c\right) \end{cases}. \tag{1.3}$$

Now make the change of variables $a = \lambda_1\lambda_2$, $b = \lambda_1\lambda_3$, $c = \lambda_2\lambda_3$ and $z = \lambda\lambda_4$, and define $f(\lambda,\lambda_1;\lambda_2,\lambda_3,\lambda_4)$ to be the function on the left in (1.3). Then the first identity asserts that $f(\lambda,\lambda_1;\lambda_2,\lambda_3,\lambda_4) = f(\lambda,\lambda_1;\lambda_4,\lambda_3,\lambda_2)$, while the second states that $f(\lambda,\lambda_1;\lambda_2,\lambda_3,\lambda_4) = f(\lambda_1,\lambda;\lambda_2,\lambda_3,\lambda_4)$. The obvious symmetry between a and b translates to invariance of f under interchanging λ_2 and λ_3. It follows that f is invariant under $\mathrm{Sym}(\lambda_2,\lambda_3,\lambda_4)$ and also under $\mathrm{Sym}(\lambda,\lambda_1)$. Here and throughout $\mathrm{Sym}(x,y,\dots)$ denotes the symmetric group on the symbols $x,y,\dots$ and S_n denotes the symmetric group on n elements. We can summarize the above by saying that the Heine transformation (1.2) and the symmetry between a and b generate an automorphism group for ${}_2\Phi_1$ isomorphic to $S_2\times S_3$.

Now a question naturally arises. Can f be expanded in such a way as to make these symmetries explicit? Rogers [**41**] gives a positive solution by his Symmetric Function Expansion Theorem:

$$\begin{aligned} f(\lambda,\lambda_1;\lambda_2,\lambda_3,\lambda_4) =& (\lambda\lambda_2)_\infty(\lambda\lambda_3)_\infty(\lambda\lambda_4)_\infty(\lambda_1\lambda_2)_\infty(\lambda_1\lambda_3)_\infty(\lambda_1\lambda_4)_\infty \\ &\times\sum_{n\geq 0}\frac{h_n^{(2)}(\lambda,\lambda_1)h_n^{(3)}(\lambda_2,\lambda_3,\lambda_4)}{(q)_n}, \end{aligned} \tag{1.4}$$

where

$$h_n^{(k)} = (q)_n\sum_{i_1+i_2+\cdots+i_k=n}\frac{x_1^{i_1}x_2^{i_2}\dots x_k^{i_k}}{(q)_{i_1}(q)_{i_2}\dots(q)_{i_k}}.$$

Equating together the definition of f with the right-hand-side above gives the form we shall use later:

THEOREM 1.1.

$$\begin{aligned} &\frac{(\lambda\lambda_1\lambda_2\lambda_3)_\infty}{(\lambda_1\lambda_4)_\infty(\lambda\lambda_2)_\infty(\lambda\lambda_3)_\infty(\lambda_1\lambda_2)_\infty(\lambda_1\lambda_3)_\infty}\,{}_2\Phi_1\left(\begin{matrix}\lambda_1\lambda_2,\,\lambda_1\lambda_3\\ \lambda\lambda_1\lambda_2\lambda_3\end{matrix};\lambda\lambda_4\right) \\ &\qquad= \sum_{n\geq 0}\frac{h_n^{(3)}(\lambda_2,\lambda_3,\lambda_4)h_n^{(2)}(\lambda,\lambda_1)}{(q)_n}. \end{aligned} \tag{1.5}$$

Clearly $h_n^{(k)}$ is symmetric in $x_1,\dots,x_k$. This theorem shows that (1.2) follows from the symmetries of the polynomials $h_n^{(2)}$ and $h_n^{(3)}$. This raises the possibility that other basic hypergeometric transformations are also special cases of symmetries of the class of series in the $h_n^{(k)}$ and other polynomials. Pursuing this idea, we will obtain several new symmetric expansions for q-series, of which the following is typical:

$$\begin{aligned}
(1.6)\qquad &\frac{(\lambda\lambda_1\lambda_2\lambda_3)_\infty}{(\lambda\lambda_2)_\infty(\lambda\lambda_3)_\infty(\lambda_1\lambda_2)_\infty(\lambda_1\lambda_3)_\infty(\lambda_1\lambda_4)_\infty(\lambda_1\lambda_5)_\infty}\\
&\times\sum_{k\geq 0}\left(\sum_{m=0}^{k}\begin{bmatrix}k\\m\end{bmatrix}(\lambda_1\lambda_4)_m\lambda_4^{k-m}\lambda_5^m\right)\frac{(\lambda_1\lambda_2)_k(\lambda_1\lambda_3)_k}{(\lambda\lambda_1\lambda_2\lambda_3)_k(q)_k}\lambda^k\\
&=\sum_{n\geq 0}\frac{h_n^{(2)}(\lambda,\lambda_1)h_n^{(4)}(\lambda_2,\lambda_3,\lambda_4,\lambda_5)}{(q)_n},
\end{aligned}$$

where

$$\begin{bmatrix}k\\m\end{bmatrix}=\frac{(q)_k}{(q)_m(q)_{k-m}},\ \text{for } 0\leq m\leq k,\qquad \begin{bmatrix}k\\m\end{bmatrix}=0 \text{ for } m<0 \text{ or } m>k.$$

This result reduces to (1.2) when $\lambda_4=0$. We will also obtain a symmetric expansion for the Sears and Hall transformations for a ${}_3\Phi_2$ in five parameters. This will show that they generate a transformation group isomorphic to S_5. These and other new symmetric expansions are contained in Section 2.1 and throughout Chapter 3. In fact (1.6) is just a special case of our more general result, Theorem 3.11 contained in Chapter 3. To obtain general theorems a larger class of symmetric polynomials is used. We define them here along with some other basic notation.

1.1. q-Symmetric polynomials and the q-multinomial

When multiplying together q-products we employ the following standard shorthand:

$$(A_1,A_2,\ldots,A_m)_n=(A_1)_n(A_2)_n\cdots(A_m)_n.$$

We also introduce the following convenient notation which we will use. Let

$$[x,y]_n=(x-y)(x-yq)\ldots(x-yq^{n-1}),\qquad n\in\mathbb{Z}^+.$$

If $x\neq 0$, then $[x,y]_n=x^n(y/x)_n$. For $x\neq 0$ we define $[x,y]_k$ for real k by the equation

$$[x,y]_k=x^k\frac{(y/x)_\infty}{(yq^k/x)_\infty}. \tag{1.7}$$

Here and above $(A)_\infty$ is defined by

$$(A)_\infty=\lim_{n\to\infty}(A)_n.$$

We note that $(y)_n=[1,y]_n$ and define $[x]_n=[x,1]_n$. Clearly

$$[x,0]_n=x^n,\qquad [0,y]_n=(-1)^nq^{\binom{n}{2}}y^n. \tag{1.8}$$

The symmetric polynomials $h_j^{(m)}$ above are special cases of a more general class of polynomials denoted by

$$h_j=h_j^{(m)}(\mathbf{a};\mathbf{b})=h_j^{(m)}(a_1,\ldots,a_m;b_1,\ldots,b_m:q).$$

We typically suppress the superscript as its value is usually clear from context. These polynomials are defined explicitly by the equation:

$$h_j=\sum_{n_1+\cdots+n_m=j}\begin{bmatrix}j\\n_1,\ldots,n_m\end{bmatrix}[b_1,a_1]_{n_1}\cdots[b_m,a_m]_{n_m}, \tag{1.9}$$

where

$$\begin{bmatrix} j \\ n_1, \dots, n_m \end{bmatrix} = \frac{(q)_j}{(q)_{n_1} \cdots (q)_{n_m}}.$$

h_j may also be defined by their generating function:

$$\frac{(a_1 z)_\infty \cdots (a_m z)_\infty}{(b_1 z)_\infty \cdots (b_m z)_\infty} = \sum_{j \geq 0} \frac{h_j}{(q)_j} z^j, \qquad |b_i z| < 1. \tag{1.10}$$

In applications such as Rogers's symmetric expansion above we abbreviate $h_j(0, \dots, 0; b_1, \dots, b_m : q)$ by $h_j(b_1, \dots, b_m)$ or $h_j(\mathbf{b})$. The explicit formula for the h_j is obtained by multiplying together m different cases of the q-binomial theorem [**7**],[**21**],[**25**]:

$$\sum_{n \geq 0} \frac{[b, a]_n}{(q)_n} z^n = \frac{(az)_\infty}{(bz)_\infty}, \qquad |bz| < 1. \tag{1.11}$$

Taken together we refer to equations (1.9) and (1.10) as the q-multinomial theorem. We used the q-multinomial theorem in [**15**] to give a multivariate generalization of Heine's transformation. We give this result in Section 3.2. Subsequently a special case of our transformation was used by J. Haglund [**28**] to give a generating function for certain rook numbers.

The polynomials h_j are fundamental in another extremely important way: they are discrete analogs of the basic hypergeometric function ${}_m\Phi_{m-1}$. This is proved in [**15**] where they are characterized as satisfying a discrete q-difference equation analogous to the continuous q-difference equation which characterizes ${}_m\Phi_{m-1}$.

1.2. q-Difference operators

The symmetric expansion theorems will be obtained by developing the properties of certain q-difference operators introduced by L.J. Rogers [**42**]. Much of our work was motivated by the study of one particular operator theorem of Rogers. Before we can give this theorem, we need to define certain operators.

Let f be a function of the complex variable λ. Then we define ηf by

$$(\eta f)(\lambda) = f(\lambda q).$$

Another operator constantly used is δ, defined by

$$(\delta f)(\lambda) = \frac{f(\lambda) - f(\lambda q)}{\lambda}.$$

Thus

$$\delta = \lambda^{-1}(1 - \eta).$$

Frequently we require η or δ to operate on different variables $\lambda_1, \dots, \lambda_n$ at different times. In this situation we indicate the variable acted on by making the convention that η_i or δ_i acts on λ_i. Thus

$$\eta_2 f(\lambda_1, \lambda_2) = f(\lambda_1, \lambda_2 q).$$

An unsubscripted η will be assumed to act on λ. Occasionally we wish η to act on some other variable such as x. In this case we use η_x. Also instead of writing $f(\lambda_1, \lambda_2, \lambda_3 q, \lambda_4)$ we will just write $f(\lambda_3 q)$, the dependence of f on the other variables being suppressed.

Throughout we (as did Rogers) make use of the following convention involving non-commuting variables in sums. Suppose that x and y are non-commuting variables. Suppose also that

$$f(z) = \sum_{n\geq 0} a_n z^n$$

is a formal power series expansion. Then when we write $f(xy)$ we mean the sum

$$\sum_{n\geq 0} a_n x^n y^n.$$

As usual operator multiplication is defined by putting $(\sigma\tau)(f) = \sigma(\tau f)$.

Rogers's paper [**42**] consists mostly of ingenious but recondite computations involving these operators and their application to obtain q-series identities. One operator theorem in particular, (Lemma IV of [**42**]), plays a key role in his work. By appealing to an analogy with the exponential map from Lie algebras to Lie groups, we obtain a much simpler and conceptual proof of this result. We give our proof here as it forms motivation for results in Chapter 3. An analogy holds in that the exponential map for Lie algebras (through its homomorphism property) yields automorphisms, while the analogue of the homomorphism property for $e(z\delta)$ (which is a q-analogue of the operator $\exp(z\frac{d}{d\lambda})$) yields automorphisms of q-series. These operators are related to classes of operators studied by Roman in the area of q-umbral calculus, [**45**],[**46**], [**47**] and [**48**].

It is interesting to note that Rogers's Lemma IV, Theorem 1.2 below, is the image under a one parameter family of representations of an algebraic identity in a q-Heisenberg algebra. In Section 2.7 we show how another operator theorem (Theorem 1.4 below) of Rogers represents the same q-Heisenberg algebra identity under a different family of representations. [**27**], [**36**], [**37**], [**38**] discuss the q-Heisenberg identity equivalent to the following theorem. Apparently the first three papers contain rediscoveries of this result of Rogers.

THEOREM 1.2 (Rogers's Lemma IV). *As an equation in operators*

$$\frac{1}{(\lambda\delta_1)_\infty}\frac{1}{(\mu\lambda_1)_\infty} = \frac{1}{(\mu\lambda_1)_\infty}\frac{1}{(\mu\lambda\eta_1)_\infty}\frac{1}{(\lambda\delta_1)_\infty}. \tag{1.12}$$

A bit of explanation is needed here. Consider the left hand side. The factor $1/(\mu\lambda_1)_\infty$ on the left side is an operator acting by multiplication on some function. The operator $1/(\lambda\delta_1)_\infty$ then acts on the product. The right hand side is to be similarly interpreted. In the sequel we will often encounter equations of this form in which both sides are functions rather than operators. To distinguish these two cases, in the first we say the equation holds as an operator equation, while in the second we say that it holds as an equation between functions. Finally $1/(A)_\infty$ is the product expression for the series

$$\sum_{n\geq 0} \frac{A^n}{(q)_n}.$$

This follows from the q-binomial theorem (1.11) by setting $b = 0$.

Rogers nowhere in his paper considers the matter of convergence of his results. In Chapter 2 we give the appropriate convergence theorems which will validate our uses of the operator theorems. In particular, Proposition 2.21 shows that when one side of the operator equation in Theorem 1.2 is applied to a function of λ_1 which is analytic in the unit circle, then the resulting function is analytic for $|\lambda|, |\lambda_1| < 1$.

Before we give the proof of Theorem 1.2, we review a Lie algebra result which is the inspiration for our proof.

Let U be a field and $D(U)$ the Lie algebra of derivations on U. It is well known that if $d \in D(U)$ is nilpotent, then $\exp(d)$ is an automorphism of U. In mimicking for q-difference operators the proof of this standard result, we obtain a much easier proof of Theorem 1.2. More importantly, we will see how to obtain similar theorems from which we will generate a number of new symmetric function expansions.

First recall the proof of the Lie algebra result: We need to show

$$\exp(d)(fg) = [\exp(d)(f)][\exp(d)(g)].$$

To this end, note that

$$\begin{aligned}\exp(d)(fg) &= \sum_{n\geq 0} \frac{d^n(fg)}{n!} = \sum_{n\geq 0}\sum_{i+j=n} \binom{n}{i} \frac{(d^i f)(d^j g)}{n!} \\ &= \sum_{i,j\geq 0} \frac{d^i f}{i!}\frac{d^j g}{j!} = [\exp(d)f][\exp(d)g].\end{aligned}$$

Now we try to mimic this proof for the operator δ_1.

Proof of Theorem 1.2. The q-Leibniz rule for δ_1 is

$$\delta_1^r(fg) = \sum_{i+j=r} \begin{bmatrix} r \\ i \end{bmatrix} (\delta_1^i f)(\eta_1^i \delta_1^j g). \tag{1.13}$$

(In Chapter 3 we give a proof of a more general result.) We now try to evaluate the q-exponential of a product of two functions. Under suitable conditions allowing rearrangement of series,

$$\begin{aligned}\frac{1}{(\lambda\delta_1)_\infty}(fg) &= \sum_{r\geq 0} \frac{\delta_1^r(fg)\lambda^r}{(q)_r} = \sum_{r\geq 0}\sum_{i+j=r} \begin{bmatrix} r \\ i \end{bmatrix} (\delta_1^i f)(\eta_1^i\delta_1^j g)\frac{\lambda^r}{(q)_r} \\ &= \sum_{i+j\geq 0} \frac{(\delta_1^i f)(\eta_1^i \delta_1^j g)}{(q)_i (q)_j}\lambda^{i+j} = \sum_{i\geq 0} \frac{(\delta_1^i f)\lambda^i}{(q)_i}\eta_1^i \sum_{j\geq 0}\frac{\delta_1^j g}{(q)_j}\lambda^j.\end{aligned} \tag{1.14}$$

This is as close as we can get to the homomorphism property of the exponential map of a derivation. Writing (1.14) as an operator equation gives

$$\frac{1}{(\lambda\delta_1)_\infty} f = \sum_{i\geq 0} \frac{(\delta_1^i f)\lambda^i}{(q)_i}\eta_1^i \frac{1}{(\lambda\delta_1)_\infty}. \tag{1.15}$$

Now if f is an eigenfunction of δ_1 with eigenvalue μ, then we can pull f in front of the summation sign in the right hand side of (1.15) and we establish a commutativity relation for the product on the left. Clearly,

$$f(\lambda_1) = \frac{1}{(\mu\lambda_1)_\infty} = \sum_{n\geq 0} \frac{\mu^n}{(q)_n}\lambda_1^n$$

is such an eigenfunction. Hence (1.15) becomes

$$\frac{1}{(\lambda\delta_1)_\infty}\frac{1}{(\mu\lambda_1)_\infty} = \frac{1}{(\mu\lambda_1)_\infty}\sum_{i\geq 0}\frac{\mu^i\lambda^i}{(q)_i}\eta_1^i \frac{1}{(\lambda\delta_1)_\infty} = \frac{1}{(\mu\lambda_1)_\infty}\frac{1}{(\mu\lambda\eta_1)_\infty}\frac{1}{(\lambda\delta_1)_\infty},$$

and the proof is complete. $\square$

Rogers employed this theorem to obtain Theorem 1.1 as well as the following theorem (known as the q-Mehler formula) which we shall also use:

THEOREM 1.3 (q-Mehler formula).

$$\frac{(\lambda\lambda_1\lambda_2\lambda_3)_\infty}{(\lambda\lambda_2)_\infty(\lambda\lambda_3)_\infty(\lambda_1\lambda_2)_\infty(\lambda_1\lambda_3)_\infty} = \sum_{n\geq 0} \frac{h_n(\lambda_2,\lambda_3)h_n(\lambda,\lambda_1)}{(q)_n}. \tag{1.16}$$

A special case of this theorem was given earlier by Rogers [**41**]. It was later rediscovered by L. Carlitz [**19**] in 1956 and is a fundamental theorem in the theory of q-generalizations of orthogonal polynomials.

We finally indicate how Theorem 1.1 is derived from Theorem 1.2. Theorem 1.1 arises when the function

$$\frac{1}{(\lambda\delta_1)_\infty}\frac{1}{(\lambda_1\lambda_4)_\infty(\lambda_1\lambda_3)_\infty(\lambda_1\lambda_2)_\infty}$$

is calculated in two different ways. The left hand side of Theorem 1.1 comes about when this function is calculated by applying Theorem 1.2 iteratively. This gives

(1.17)

$$\begin{aligned}\frac{1}{(\lambda\delta_1)_\infty}&\frac{1}{(\lambda_1\lambda_4)_\infty(\lambda_1\lambda_3)_\infty(\lambda_1\lambda_2)_\infty}\\ &= \frac{(\lambda\lambda_1\lambda_2\lambda_3)_\infty}{(\lambda_1\lambda_4)_\infty(\lambda\lambda_2)_\infty(\lambda\lambda_3)_\infty(\lambda_1\lambda_2)_\infty(\lambda_1\lambda_3)_\infty}\,{}_2\Phi_1\left(\begin{matrix}\lambda_1\lambda_2,\ \lambda_1\lambda_3\\ \lambda\lambda_1\lambda_2\lambda_3\end{matrix};\lambda\lambda_4\right).\end{aligned}$$

The right hand side results from first expanding the three products by the q-multinomial theorem, and then applying Theorem 1.2. For details in the above calculations see Rogers [**42**].

Similar to the operator δ_1 is the operator σ_1 defined by

$$\sigma_1 = \lambda^{-1}(\eta_1^{-1}-1).$$

Corresponding to Theorem 1.2 is the following result of Rogers [**42**]:

THEOREM 1.4.

$$E(\lambda\sigma_1)E(\mu\lambda_1) = E(\mu\lambda_1)E(\mu\lambda\eta_1^{-1})E(\lambda\sigma_1). \tag{1.18}$$

Here $E(x)$ is the q-exponential function

$$E(x) = (-qx)_\infty = \sum_{n\geq 0}\frac{q^{\binom{n}{2}}}{(q)_n}x^n.$$

Applying this result to the constant function 1 and equating coefficients of λ_2^n leads to

$$E(\lambda\sigma_1)\lambda_1^n = \chi_n(\lambda,\lambda_1), \tag{1.19}$$

where $\chi_j(x_1,\ldots,x_m)$ is defined by

$$\chi_j(x_1,\ldots,x_m) = q^{-\binom{j+1}{2}}h_j(-x_1q,\ldots,-x_mq;0) = q^{-\binom{j}{2}}h_j(-x_1,\ldots,-x_m;0).$$

Notice that this is equal to

$$\sum_{n_1+\cdots+n_m=j}\begin{bmatrix} j\\ n_1,\ldots,n_m\end{bmatrix}q^{-(n_1n_2+n_1n_3+\cdots+n_{m-1}n_m)}x_1^{n_1}\cdots x_m^{n_m}$$

by virtue of the identity

$$\binom{n_1}{2} + \binom{n_2}{2} + \cdots + \binom{n_m}{2} + \sum_{1 \le i < j \le m} n_i n_j = \binom{n_1 + n_2 + \cdots + n_m}{2}.$$

Finally by the same identity it is clear that

$$\chi_j(x_1, \ldots, x_m) = h_j(x_1, \ldots, x_m : q^{-1}). \tag{1.20}$$

For convergence of series involving the polynomials χ_j note that

$$|\chi_j(x_1, \ldots, x_m)| < |q|^{-\frac{(m-1)j^2}{2m}} h_j(x_1, \ldots, x_m) < |q|^{-\frac{(m-1)j^2}{2m}} (|x_1| + \cdots + |x_m|)^j.$$

Using Theorem 1.4 Rogers obtained the following dual of Theorem 1.3 which we shall refer to in the sequel:

THEOREM 1.5.

$$\begin{aligned} &E(\lambda\sigma_1)E(\lambda_1\lambda_3)E(\lambda_1\lambda_2) \\ &= \frac{E(\lambda\lambda_3)E(\lambda_1\lambda_3)E(\lambda\lambda_2)E(\lambda_1\lambda_2)}{E(-\lambda\lambda_1\lambda_2\lambda_3)} = \sum_{n\ge 0} \frac{q^{\binom{n+1}{2}}}{(q)_n} \chi_n(\lambda, \lambda_1)\chi_n(\lambda_2, \lambda_3). \end{aligned} \tag{1.21}$$

We note that although Rogers did not explicitly state this theorem in [**42**], he indicated its existence and derivation in his paper. (He did give the case $\lambda = xe^{i\theta}$, $\lambda_1 = xe^{-i\theta}$, $\lambda_2 = e^{i\phi}$, $\lambda_3 = e^{-i\phi}$, with typos, however.) The first direct statement of the general theorem is given in Carlitz [**20**].

1.3. Orthogonal polynomials

Specialization of the polynomials h_j lead to orthogonal polynomials. In Rogers's original work [**42**] he exploited this fact (in an old fashioned way) to derive several results. In Chapter 2 we describe in detail the connection between orthogonal polynomials and Rogers's work in [**42**]. As a consequence of this connection we derive a number of new integral evaluations. An example of one of our new evaluations is the following identity. For $|r_1|, |r_2| < 1$ and $\max(|q/r_1|, |q/r_2|) < |\lambda|^2 < 1$,

$$\begin{aligned} &\int_0^\pi \frac{P(r_1\lambda)P(q/r_1\lambda)P(r_2\lambda)P(q/r_2\lambda)(e^{2i\theta}, e^{-2i\theta})_\infty}{P(r_1)P(q/r_1\lambda^2)P(r_2)P(q/r_2\lambda^2)(\lambda e^{2i\theta}, \lambda e^{-2i\theta})_\infty} d\theta \\ &= \frac{2\pi(q)_\infty (q/\lambda)_\infty^2 (1-\lambda)}{(\lambda^2)_\infty (q/\lambda^2)_\infty^2} \sum_{j\ge 0} \frac{(\lambda^2)_j (1-\lambda q^j)}{(q)_j^3} \\ &\quad \times h_j(r_1\lambda, q/r_1\lambda; r_1, q/r_1\lambda^2) h_j(r_2\lambda, q/r_2\lambda; r_2, q/r_2\lambda^2). \end{aligned} \tag{1.22}$$

This is the first part of Corollary 2.20.

In dealing with orthogonal polynomials, we make a slight change from the notation used by Rogers. Put $A_n = A_n(u) = h_n(u, u^{-1})$, with $|u| = 1$. Since $|u| = 1$, it makes sense to put $u = e^{i\theta}$, θ real, and consider the A_n as functions of θ and indeed Rogers writes $A_n(\theta)$ with the above change of variables so that Rogers's $A_n(\theta)$ is equal to our $A_n(e^{i\theta})$. However, we prefer our notation as we will not be using the exponential change of variables. Today A_n are known as the continuous q-Hermite polynomials (or sometimes the Rogers-Szegö polynomials, although Szëgo didn't work with them) and are denoted by $H_n(x; q)$ where $x = (u + u^{-1})/2 = \cos(\theta)$. Note that the polynomials H_n are of degree n in x and thus form a basis for the ring of formal power series in x over $\mathbb{C}$. Clearly from the q-multinomial theorem the sequence A_n has the generating function

$$\sum_{n\geq 0} \frac{A_n}{(q)_n} z^n = \sum_{n\geq 0} \frac{h_n(u,u^{-1})}{(q)_n} z^n = \frac{1}{(uz,u^{-1}z)_\infty}.$$

Now put $P(z) = (uz, u^{-1}z)_\infty$, so that

$$\frac{1}{P(z)} = \sum_{n\geq 0} \frac{A_n}{(q)_n} z^n.$$

To further set the stage for our work in Chapter 2, we now give a summary in more modern language of some of the results and ideas contained in Rogers [**42**].

Consider $\mathbb{C}[[A_n]]$ which is defined to be the set of series of the form $\sum_{n\geq 0} K_n A_n$, where K_n are in $\mathbb{C}$ and the series converge absolutely and uniformly in u by the Weierstrass M-test. It follows under this assumption that the series converge to continuous functions of u. In particular, if such a function is analytic at $x = 0$, then by the basis property mentioned above, the expansion of the function must be unique.

The first theorem we state was not given by Rogers, but he certainly was aware of it in some form.

THEOREM 1.6. *$\mathbb{C}[[A_n]]$ is a $\mathbb{C}$-algebra.*

We give our proof here (which is just a softer version of an argument of Rogers's) as it is short and we will later make use of one of the intermediate results.

PROOF. Theorem 1.3 states that

$$\frac{(\lambda\lambda_1\lambda_2\lambda_3)_\infty}{(\lambda\lambda_2)_\infty(\lambda\lambda_3)_\infty(\lambda_1\lambda_2)_\infty(\lambda_1\lambda_3)_\infty} = \sum_{n\geq 0} \frac{h_n(\lambda_2,\lambda_3)h_n(\lambda,\lambda_1)}{(q)_n}.$$

Put $\lambda = \lambda_1^{-1} = u$ and divide both sides by $(\lambda_2\lambda_3)_\infty$:

$$\frac{1}{P(\lambda_2)P(\lambda_3)} = \frac{1}{(\lambda_2\lambda_3)_\infty} \sum_{n\geq 0} \frac{A_n}{(q)_n} h_n(\lambda_2,\lambda_3).$$

Using the q-binomial theorem this is

$$\sum_{j\geq 0} \frac{A_j}{(q)_j} {\lambda_2}^j \sum_{k\geq 0} \frac{A_k}{(q)_k} {\lambda_3}^k = \sum_{m\geq 0} \frac{\lambda_2^m\lambda_3^m}{(q)_m} \sum_{n\geq 0} \frac{A_n}{(q)_n} h_n(\lambda_2,\lambda_3). \tag{1.23}$$

The coefficient of $\lambda_2^j\lambda_3^k$ on the left is $\frac{A_j}{(q)_j}\frac{A_k}{(q)_k}$. On the right there are finitely many terms involving $\lambda_2^j\lambda_3^k$ and these are each multiplied by a linear combination of finitely many A_n and complex constants independent of u. Hence if we equate coefficients of $\lambda_2^j\lambda_3^k$, we get A_jA_k as a $\mathbb{C}$-linear combination of certain A_n. Thus we have structure constants for a $\mathbb{C}$-algebra. □

Indeed the content of this theorem is the recognition that there exist structure constants for the polynomials A_n. Rogers gave these constants explicitly in the following linearization formula.

$$A_jA_k = (q)_j(q)_k \sum_{\max(j-k,0)\leq l\leq j} \begin{bmatrix} k+2l-j \\ l \end{bmatrix} \frac{A_{k+2l-j}}{(q)_{j-l}(q)_{k+2l-j}}. \tag{1.24}$$

Rogers derived this formula from identity (1.23) above.

The next definition is a concept which occurs in [**42**] for which we give a convenient notation.

DEFINITION 1.7 (Taking the constant term). If $y = \sum a_n A_n \in \mathbb{C}[[A_n]]$, put $C(y) = a_0$.

What follows is our own theorem which is a generalization of an idea in [**42**]. It is actually a form of Parseval's Theorem written as a constant term identity.

THEOREM 1.8. *When the sequences K_i and L_j are independent of u,*

$$C\left(\left(\sum_{i\geq 0}\frac{A_i}{(q)_i}K_i\right)\left(\sum_{j\geq 0}\frac{A_j}{(q)_j}L_j\right)\right) = \sum_{i\geq 0}\frac{K_iL_i}{(q)_i}.$$

PROOF. Taking C of both sides of (1.23) gives

$$\sum_{j,k\geq 0}\frac{\lambda_2^j\lambda_3^k}{(q)_j(q)_k}C(A_jA_k) = \sum_{m\geq 0}\frac{\lambda_2^m\lambda_3^n}{(q)_m}.$$

This implies that

$$C(A_jA_k) = \begin{cases} 0 & \text{if} \quad j \neq k \\ (q)_j & \text{if} \quad j = k. \end{cases} \tag{1.25}$$

Then

$$\begin{aligned} C\left(\left(\sum_{i\geq 0}\frac{A_i}{(q)_i}K_i\right)\left(\sum_{j\geq 0}\frac{A_j}{(q)_j}L_j\right)\right) &= \sum_{i,j\geq 0}\frac{K_iL_j}{(q)_i(q)_j}C(A_iA_j) \\ &= \sum_{j\geq 0}\frac{K_jL_j}{(q)_j}. \end{aligned}$$ □

The special case given by Rogers is the following result of which we will make immediate use.

COROLLARY 1.9 (Rogers). *When the sequence K_j is independent of u,*

$$C\left(\frac{1}{P(z)}\sum_{j\geq 0}A_jK_j\right) = \sum_{j\geq 0}K_jz^j.$$

PROOF. In the left hand side of the assertion of Theorem 1.8, substitute

$$\frac{1}{P(z)} = \sum_{i\geq 0}\frac{A_i}{(q)_i}z^i.$$ □

In Chapter 2 we will obtain new expressions for certain functions defined by Rogers in [**42**]. To define them we must follow a little of Rogers's development. The idea is to iterate Corollary 1.9. Specifically, suppose we are given a sequence of complex numbers J_i which is independent of λ and λ_1. Then starting with the series $\sum J_iA_i$ we can obtain a new series $\sum K_iA_i$ by multiplying by $1/P(\lambda_1)$, where now the K_i depend on λ_1. Multiplying now by $1/P(\lambda)$ we get a new series $\sum L_iA_i$, with the L_i dependent on both λ and λ_1. Diagrammatically we write:

$$\sum J_iA_i \xrightarrow{\frac{1}{P(\lambda_1)}} \sum K_iA_i \xrightarrow{\frac{1}{P(\lambda)}} \sum L_iA_i. \tag{1.26}$$

(Here the arrows denote the appropriate multiplication maps.) From Corollary 1.9 $C(\sum K_iA_i) = \sum J_i\lambda_1^i$ and $C(\sum L_iA_i) = \sum K_i\lambda^i$. Adding this information to the diagram gives

$$\begin{tikzcd} \sum J_iA_i \arrow[r, "\frac{1}{P(\lambda_1)}"] & \sum K_iA_i \arrow[r, "\frac{1}{P(\lambda)}"] \arrow[d, "C"'] & \sum L_iA_i \arrow[d, "C"'] \\ & \sum J_i\lambda_1^i \arrow[r, "?"] & \sum K_i\lambda^i. \end{tikzcd} \tag{1.27}$$

We have used the symbol ? to denote the unknown map which makes this diagram commute. Rogers solved the problem of determining ? in [**42**]. The solution is contained in the corollary to the following theorem.

THEOREM 1.10. *Let the sequence L_i be defined by (1.26). Then*

$$L_i = \frac{1}{(\lambda\lambda_1)_\infty(\lambda\delta_1)_\infty}K_i. \tag{1.28}$$

COROLLARY 1.11. *In the above commutative diagram ? is the operator*

$$\frac{1}{(\lambda\lambda_1)_\infty(\lambda\delta_1)_\infty}.$$

In the next section we define the functions of interest.

1.4. The symmetric functions Φ_m

In Chapter 3 we will make intimate use of certain symmetric functions defined by Rogers. Here we define them as well as state the basic results proved by Rogers.

The function $\Phi_m(\lambda_1, \ldots, \lambda_{m+1})$ is defined by the following equation

$$\Phi_m(\lambda_1, \ldots, \lambda_{m+1}) = C\left(\frac{1}{P(\lambda_1)P(\lambda_2)\cdots P(\lambda_{m+1})}\right).$$

Notice that Φ_m is symmetric in the variables $\lambda_1, \ldots, \lambda_{m+1}$, and thus has automorphism group S_{m+1}. In the next chapter we shall show how Φ_4 can be related to a basic hypergeometric function thereby giving a symmetric expansion similar to Rogers's for the ${}_2\Phi_1$.

First of all notice that by Theorem 1.6, we may put

$$\frac{1}{P(\lambda_1)\cdots P(\lambda_{m+1})} = \sum_{n\geq 0} K_{n,m}A_n, \tag{1.29}$$

thus defining a sequence of functions $K_{n,m} = K_{n,m}(\lambda_1, \ldots, \lambda_{m+1})$. By definition,

$$\Phi_m = K_{0,m}.$$

Now all this fits perfectly in the picture (1.27). We have the following commutative diagram (the vertical parallel lines denote equality):

$$
\begin{tikzcd}
\frac{1}{P(\lambda_1)\cdots P(\lambda_{m-1})} \arrow[d, equal] & \frac{1}{P(\lambda_1)\cdots P(\lambda_m)} \arrow[d, equal] & \frac{1}{P(\lambda_1)\cdots P(\lambda_{m+1})} \arrow[d, equal] \\
\sum_{n\geq 0} K_{n,m-2}A_n \arrow[r, "\frac{1}{P(\lambda_m)}"] & \sum K_{n,m-1}A_n \arrow[r, "\frac{1}{P(\lambda_{m+1})}"] \arrow[d, "C"'] & \sum K_{n,m}A_n \arrow[d, "C"'] \\
 & \sum K_{n,m-2}\lambda_m^n \arrow[r, "?"] \arrow[d, equal] & \sum K_{n,m-1}\lambda_{m+1}^n \arrow[d, equal] \\
 & \Phi_{m-1} & \Phi_m.
\end{tikzcd}
$$

Corollary 1.11 gives that

$$\Phi_m = \frac{1}{(\lambda_m\lambda_{m+1})_\infty(\lambda_{m+1}\delta_m)_\infty}\Phi_{m-1}, \tag{1.30}$$

while from Corollary 1.9 we see

$$\Phi_m = \sum_{n\geq 0} K_{n,m-1}\lambda_{m+1}^n. \tag{1.31}$$

In [**42**] Rogers computed Φ_m for $m = 1, 2, 3, 4$, and 5. His proofs made use of Theorem 1.2 and Corollary 1.11. For future reference we give his evaluations here.

$$\Phi_1(\lambda_1, \lambda_2) = \frac{1}{(\lambda_1\lambda_2)_\infty}.$$

$$\Phi_2(\lambda_1, \lambda_2, \lambda_3) = \frac{1}{(\lambda_1\lambda_2)_\infty(\lambda_1\lambda_3)_\infty(\lambda_2\lambda_3)_\infty}.$$

$$\Phi_3(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \frac{(\lambda_1\lambda_2\lambda_3\lambda_4)_\infty}{(\lambda_1\lambda_2)_\infty(\lambda_1\lambda_3)_\infty(\lambda_1\lambda_4)_\infty(\lambda_2\lambda_3)_\infty(\lambda_2\lambda_4)_\infty(\lambda_3\lambda_4)_\infty}. \tag{1.32}$$

Once one knows the integral form of the orthogonality of the polynomials A_n, it is easy to show that (1.32) is equivalent to the famous Askey-Wilson integral (2.23). We will discuss this in Section 2.3. Here Rogers anticipated a later development in mathematics. To actually get the Askey-Wilson integral from this evaluation of Φ_3 one needs to know the weight function for the polynomials A_n. This was first given by W.A. Allaway [**4**] in 1972.

Expanding the products in the above evaluation of Φ_3 using the q-multinomial theorem, Rogers obtained the following theorem which we record here for future reference.

THEOREM 1.12.
$$\frac{(\lambda_1\lambda_2, \lambda_1\lambda_3, \lambda_2\lambda_3)_\infty}{P(\lambda_1)P(\lambda_2)P(\lambda_3)} = \sum_{n\geq 0}\frac{h_n(\lambda_1\lambda_2\lambda_3; \lambda_1, \lambda_2, \lambda_3)}{(q)_n}A_n.$$

The evaluations of Φ_4 and Φ_5 are contained in the following two theorems.

THEOREM 1.13.
$$\Phi_4 = \frac{1}{(\lambda_1\lambda_2, \lambda_3\lambda_4, \lambda_3\lambda_5, \lambda_4\lambda_5)_\infty}\sum_{n\geq 0}\frac{h_n(\lambda_1, \lambda_2)h_n(\lambda_3\lambda_4\lambda_5; \lambda_3, \lambda_4, \lambda_5)}{(q)_n}.$$

THEOREM 1.14.

$$\Phi_5 = \frac{1}{(\lambda_1\lambda_2, \lambda_1\lambda_3, \lambda_2\lambda_3, \lambda_4\lambda_5, \lambda_4\lambda_6, \lambda_5\lambda_6)_\infty} \times \sum_{n\geq 0} \frac{h_n(\lambda_1\lambda_2\lambda_3; \lambda_1, \lambda_2, \lambda_3) h_n(\lambda_4\lambda_5\lambda_6; \lambda_4, \lambda_5, \lambda_6)}{(q)_n}.$$

By virtue of our Theorem 1.8 we are able to give a transparent and unified proof of Theorems 1.13 and 1.14.

PROOF OF THEOREMS 1.13 AND 1.14. By definition

$$\Phi_5 = C\left(\left(\frac{1}{P(\lambda_1)P(\lambda_2)P(\lambda_3)}\right)\left(\frac{1}{P(\lambda_4)P(\lambda_5)P(\lambda_6)}\right)\right)$$

and by Theorem 1.12

$$\frac{1}{P(\lambda_a)P(\lambda_b)P(\lambda_c)} = \sum_{n\geq 0} \frac{h_n(\lambda_a\lambda_b\lambda_c; \lambda_a, \lambda_b, \lambda_c)}{(\lambda_a\lambda_b, \lambda_a\lambda_c, \lambda_b\lambda_c)_\infty (q)_n} A_n.$$

Hence Theorem 1.14 follows from applying Theorem 1.8. Theorem 1.13 follows from Theorem 1.14 by setting $\lambda_6 = 0$. □

Finally in Chapter 2 we will make use of another expression for Φ_4 due to Rogers [**42**] which we reproduce below.

THEOREM 1.15.

$$\Phi_4 = \frac{1}{(\lambda_1\lambda_2, \lambda_1\lambda_3, \lambda_2\lambda_3, \lambda_1\lambda_4, \lambda_1\lambda_5, \lambda_4\lambda_5)_\infty} \times \sum_{n\geq 0} \frac{h_n(\lambda_1\lambda_2\lambda_3; \lambda_2, \lambda_3) h_n(\lambda_1\lambda_4\lambda_5; \lambda_4, \lambda_5)}{(q)_n}. \tag{1.33}$$

1.5. Further notation, conventions and base inversion

Besides the general basic hypergeometric series ${}_r\Phi_s$, we will also use the series ${}_r\varphi_s$, the difference being that in the latter series the terms do not have the factor

$$\left[(-1^n) q^{\binom{n}{2}}\right]^{s+1-r}.$$

Thus

$$ {}_r\varphi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; z\right) = \sum_{n\geq 0} \frac{(a_1)_n (a_2)_n \ldots (a_r)_n}{(b_1)_n (b_2)_n \ldots (b_s)_n (q)_n} z^n. \tag{1.34}$$

This series converges if it terminates, if $|q| < 1$, $|z| < 1$, or if $|q| > 1$ and either $s+1 > r$ or $s+1 = r$ and $|z| < |b_1 \cdots b_s / a_1 \cdots a_r|$. The series diverges if the inequalities involving r, s, z are replaced by their opposites. The above follows easily from the ratio test.

Similarly applying the ratio test to the ${}_r\Phi_s$ series, it is found to converge in the following cases:

(1) $0 < |q| < 1$ and $r \leq s$,
(2) $0 < |q| < 1$, $r = s+1$ and $|z| < 1$,
(3) $|q| > 1$ and $|z| < |b_1 \cdots b_s / a_1 \cdots a_r|$.

Similarly the series diverges (unless it terminates) when

(1) $0 < |q| < 1$, $r > s+1$ and $z \neq 0$,

(2) $|q| > 1$ and $z > |b_1 \cdots b_s / a_1 \cdots a_r|$.

When stating convergence regions we exclude values of the parameters which make the denominators of the series vanish, such as $b_i = q^{-k}$, $k \in \mathbb{N}$.

Throughout the variable u will be a complex number on the unit circle and θ is defined by $u = e^{i\theta}$.

The symbol $\Longrightarrow$ denotes logical implication.

The following notation (q-binomial coefficients) will be used:

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{(q;q)_m}{(q;q)_n (q;q)_{m-n}}.$$

When the base is q the subscript on the coefficient is omitted.

We conclude this chapter with a proposition and a corollary which we will find occasion to use.

PROPOSITION 1.16 (Inversion of base). *When both sides converge,*

$$ {}_r\Phi_s \left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; q, z \right) = {}_r\varphi_s \left(\begin{matrix} a_1^{-1}, \ldots, a_r^{-1} \\ b_1^{-1}, \ldots, b_s^{-1} \end{matrix} ; q^{-1}, \frac{a_1 \cdots a_r}{b_1 \cdots b_s q} z \right)$$

PROOF. The result follows immediately from the identity

$$(a;q)_n = (-1)^n q^{\binom{n}{2}} a^n (a^{-1}; q^{-1})_n. \qquad \square$$

When $r = s + 1$ the same Φ function is used on both sides and convergence holds for both functions when $|z| < 1$. We isolate this corollary which will find application in Chapter 2.

COROLLARY 1.17. *For* $|z| < 1$,

$$ {}_{r+1}\Phi_r \left(\begin{matrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix} ; q, z \right) = {}_{r+1}\Phi_r \left(\begin{matrix} a_1^{-1}, \ldots, a_{r+1}^{-1} \\ b_1^{-1}, \ldots, b_r^{-1} \end{matrix} ; q^{-1}, \frac{a_1 \cdots a_{r+1}}{b_1 \cdots b_r q} z \right).$$

CHAPTER 2

New results and connections with current research

In the first two sections of this chapter we continue the development of the ideas begun in the last chapter, with the overall theme being the role played by q-difference operators. We find that with just a little work we are able to identify the symmetric function Φ_4 as a specialized basic hypergeometric function. Although we previously made this identification [**14**], our methods were ad hoc. Here the proof falls completely within the scope of our operator theory. After this we relate the theorems to modern work on orthogonal polynomials. Once this connection has been made, we will derive several new Askey-Wilson type integrals. We also take time to analyze the convergence of the operator methods, look at a case of interesting convergence behavior as well as discuss connections with q-Heisenberg algebras.

2.1. Identification of Φ_4 as q-hypergeometric

The goal of this section is to identify Φ_4 from the last chapter as a specialized ${}_3\Phi_2$. We will make use of the following facts which are special cases of general results proved in the next section. As they are of a canonical nature, we defer the proofs until then.

$$\frac{1}{(\mu\lambda_x)_\infty}(\mu\lambda_x\eta_1)_\infty\lambda_1^n = \frac{\lambda_1^n}{(\mu\lambda_x)_n}, \tag{2.1}$$

$$(\mu\delta_1)_\infty\lambda_1^n = [\lambda_1,\mu]_n, \tag{2.2}$$

$$(\lambda_6\delta_5)_\infty h_n(\lambda_3,\lambda_4,\lambda_5) = h_n(\lambda_6;\lambda_3,\lambda_4,\lambda_5), \tag{2.3}$$

(2.1) is the case $k=0$, $l=1$, $b=1$, and $x=\mu\lambda_x$ of (2.12). (2.2) and (2.3) are the cases $k=0$, $l=1$, $b=1$, $x=\mu$ and $k=0$, $l=3$, $b=1$, respectively of (2.14).

The identification of Φ_4 is contained in the following theorem.

THEOREM 2.1.

$$\begin{aligned}\Phi_4&(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5)\\ &= \frac{(\lambda_1\lambda_2\lambda_3\lambda_4,\lambda_2\lambda_3\lambda_4\lambda_5)_\infty}{(\lambda_1\lambda_2,\lambda_1\lambda_3,\lambda_1\lambda_4,\lambda_2\lambda_3,\lambda_2\lambda_4,\lambda_2\lambda_5,\lambda_3\lambda_4,\lambda_3\lambda_5,\lambda_4\lambda_5)_\infty}\\ &\qquad\times {}_3\Phi_2\left(\begin{matrix}\lambda_2\lambda_3,\ \lambda_2\lambda_4,\ \lambda_3\lambda_4\\ \lambda_1\lambda_2\lambda_3\lambda_4,\ \lambda_2\lambda_3\lambda_4\lambda_5\end{matrix};\lambda_1\lambda_5\right).\end{aligned} \tag{2.4}$$

The key to our proof is the following proposition which follows from Theorem 1.2. The significance of our proposition is that it shows how the operator $(\lambda\delta_1)_\infty$ can be used to introduce a factor of the form $(a)_n/(b)_n$ into any power series. These factors are precisely the building blocks of basic hypergeometric series.

PROPOSITION 2.2. *For $n \geq 0$ as equations between functions*

$$(\mu\delta_1)_\infty \frac{1}{(\lambda_1\lambda_x)_\infty}\lambda_1^n = \frac{(\mu\lambda_x)_\infty}{(\lambda_1\lambda_x)_\infty}\frac{[\lambda_1,\mu]_n}{(\mu\lambda_x)_n}, \tag{2.5}$$

and

$$(\mu\delta_1)_\infty \frac{1}{(\lambda_1\lambda_x)_\infty}\sum_{n\geq 0} a_n\lambda_1^n = \frac{(\mu\lambda_x)_\infty}{(\lambda_1\lambda_x)_\infty}\sum_{n\geq 0} a_n\frac{[\lambda_1,\mu]_n}{(\mu\lambda_x)_n}. \tag{2.6}$$

PROOF . Start with Theorem 1.2, change μ to λ_x, take the inverse of both sides, and multiply on the right by $\frac{1}{(\lambda_1\lambda_x)_\infty}$ to get

$$(\lambda_1\lambda_x)_\infty(\lambda\delta_1)_\infty\frac{1}{(\lambda_1\lambda_x)_\infty} = (\lambda\delta_1)_\infty(\lambda\lambda_x\eta_1)_\infty.$$

Change λ to μ and multiply both sides by $1/(\mu\lambda_x)_\infty$ to obtain

$$\frac{(\mu\lambda_1)_\infty}{(\mu\lambda_x)_\infty}(\mu\delta_1)_\infty\frac{1}{(\mu\lambda_1)_\infty} = (\mu\delta_1)_\infty\frac{(\mu\lambda_x\eta_1)_\infty}{(\mu\lambda_x)_\infty}.$$

Apply these operators to λ_1^n to get

$$\frac{(\mu\lambda_1)_\infty}{(\mu\lambda_x)_\infty}(\mu\delta_1)_\infty\frac{1}{(\mu\lambda_1)_\infty}\lambda_1^n = (\mu\delta_1)_\infty\frac{(\mu\lambda_x\eta_1)_\infty}{(\mu\lambda_x)_\infty}\lambda_1^n = (\mu\delta_1)_\infty\frac{\lambda_1^n}{(\mu\lambda_x)_n} = \frac{[\lambda_1,\mu]_n}{(\mu\lambda_x)_n}.$$

where the last two equality follow from (2.1) and (2.2), respectively. □

The proof of Theorem 2.1 is now easy.

PROOF OF THEOREM. We start with Theorem 1.1:

$$\frac{(\lambda_1\lambda_2\lambda_3\lambda_4)_\infty}{(\lambda_1\lambda_3,\lambda_1\lambda_4,\lambda_2\lambda_3,\lambda_2\lambda_4,\lambda_2\lambda_5)_\infty}\sum_{n\geq 0}\frac{(\lambda_2\lambda_3)_n(\lambda_2\lambda_4)_n}{(\lambda_1\lambda_2\lambda_3\lambda_4)_n(q)_n}\lambda_1^n\lambda_5^n = \sum_{n\geq 0}\frac{h_n(\lambda_1,\lambda_2)h_n(\lambda_3,\lambda_4,\lambda_5)}{(q)_n}.$$

We now apply the operator $(\lambda_6\delta_5)_\infty$ to both sides of this identity, using Proposition 2.2 (with the variables $\mu = \lambda_6$ and $\lambda_x = \lambda_2$,) on the left side and (2.3) on the right. This gives immediately (with no simplification necessary):

$$\frac{(\lambda_1\lambda_2\lambda_3\lambda_4)_\infty}{(\lambda_1\lambda_3,\lambda_1\lambda_4,\lambda_2\lambda_3,\lambda_2\lambda_4)_\infty}\sum_{n\geq 0}\frac{(\lambda_2\lambda_3)_n(\lambda_2\lambda_4)_n}{(\lambda_1\lambda_2\lambda_3\lambda_4)_n(q)_n}\lambda_1^n\frac{(\lambda_2\lambda_6)_\infty}{(\lambda_2\lambda_5)_\infty}\frac{[\lambda_5,\lambda_6]_n}{(\lambda_2\lambda_6)_n} = \sum_{n\geq 0}\frac{h_n(\lambda_1,\lambda_2)h_n(\lambda_6;\lambda_3,\lambda_4,\lambda_5)}{(q)_n}.$$

Now make the change of variables $\lambda_6 \mapsto \lambda_3\lambda_4\lambda_5$ and this equation becomes

$$\frac{(\lambda_1\lambda_2\lambda_3\lambda_4,\lambda_2\lambda_3\lambda_4\lambda_5)_\infty}{(\lambda_1\lambda_3,\lambda_1\lambda_4,\lambda_2\lambda_3,\lambda_2\lambda_4,\lambda_2\lambda_5)_\infty}\,{}_3\Phi_2\left(\begin{matrix}\lambda_2\lambda_3,\ \lambda_2\lambda_4,\ \lambda_3\lambda_4\\ \lambda_1\lambda_2\lambda_3\lambda_4,\ \lambda_2\lambda_3\lambda_4\lambda_5\end{matrix};\lambda_1\lambda_5\right) = \sum_{n\geq 0}\frac{h_n(\lambda_1,\lambda_2)h_n(\lambda_3\lambda_4\lambda_5;\lambda_3,\lambda_4,\lambda_5)}{(q)_n},$$

which by Theorem 1.13 is equal to $(\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_3\lambda_5,\lambda_4\lambda_5)_\infty\Phi_4$ and our theorem is proved. □

This theorem was given in the author's thesis in 1993 [**14**], however a result equivalent to Theorem 2.1 (assuming one knows the orthogonality and weight function of the polynomials A_n) was given by Ismail, Stanton and Viennot in 1987 as Theorem 3.5 of [**34**]. This will be discussed in Section 2.3. We thank M. Ismail for pointing out this connection. The proof given in [**14**] was not in the spirit of q-difference operator methods used here but rather made use of an identity of Al-Salam and Carlitz [**2**] connecting the right hand side of equation (1.33) with the above ${}_3\Phi_2$. For now we consider some applications of Theorem 2.1.

It is well known that the function

$$ {}_3\Phi_2\left(\begin{matrix} a,\, b,\, c \\ d,\, e \end{matrix}; \frac{de}{abc}\right) $$

has two transformations analogous to the Heine transformation. These are

$$ {}_3\Phi_2\left(\begin{matrix} a,\, b,\, c \\ d,\, e \end{matrix}; \frac{de}{abc}\right) = \begin{cases} \dfrac{(e/a)_\infty (de/bc)_\infty}{(e)_\infty (de/abc)_\infty} {}_3\Phi_2\left(\begin{matrix} a,\, d/b,\, d/c \\ d,\, de/bc \end{matrix}; e/a\right) \\ \dfrac{(b)_\infty (de/ab)_\infty (de/bc)_\infty}{(d)_\infty (e)_\infty (de/abc)_\infty} {}_3\Phi_2\left(\begin{matrix} d/b,\, e/b,\, de/abc \\ de/ab,\, de/bc \end{matrix}; b\right) \end{cases} $$

due to Sears [**50**] and Hall [**29**], respectively. In addition, the left side is clearly symmetric in a, b, c, as well as in d and e, and these symmetries, along with those of Sears and Hall, generate a transformation group for this ${}_3\Phi_2$. Theorem 2.1 can be used to identify this group as S_5. All we have to do is put $a = \lambda_2\lambda_3$, $b = \lambda_2\lambda_4$, $c = \lambda_3\lambda_4$, $d = \lambda_1\lambda_2\lambda_3\lambda_4$, and $e = \lambda_2\lambda_3\lambda_4\lambda_5$. Then it follows that $de/abc = \lambda_1\lambda_5$ and thus the function in the transformations of Sears and Hall is the ${}_3\Phi_2$ above. Given a function $f(\lambda_1, \lambda_2, \dots)$ we define the action of the element $\sigma \in S_n$ on f via $\sigma f = f(\lambda_1\sigma, \lambda_2\sigma, \dots)$, where σ acts on the indices of the λ. We use cycle notation for elements of the symmetric group, thus $(1,2)f(\lambda_1, \lambda_2) = f(\lambda_2, \lambda_1)$ for any function f. Then it is easy to verify that the symmetry $(2,3)(1,4)$ of Φ_4 yields the transformation of Sears, while the symmetry $(1,2,3)(4,5)$ yields the transformation of Hall. Finally, the (obvious) symmetry of a, b, and c is generated by the elements $(2,3)$ and $(2,4)$, while the symmetry of d and e is clearly $(1,5)$. It is not hard to show that these elements of S_5 generate the whole of S_5 and thus we have our assertion.

It is useful to write the basic hypergeometric function in terms of Φ_4. Solving for a, b, etc in terms of the λ's gives

$$ \begin{aligned} &{}_3\Phi_2\left(\begin{matrix} a,\, b,\, c \\ d,\, e \end{matrix}; \frac{de}{abc}\right) \\ &\quad = \frac{(a)_\infty (b)_\infty (c)_\infty (d/a)_\infty (d/b)_\infty (d/c)_\infty (e/a)_\infty (e/b)_\infty (e/c)_\infty}{(d)_\infty (e)_\infty} \\ &\quad\quad \times \Phi_4(a^{\frac12} b^{\frac12} c^{-\frac12}, a^{-\frac12} b^{-\frac12} c^{-\frac12} d, a^{\frac12} b^{-\frac12} c^{\frac12}, a^{-\frac12} b^{-\frac12} c^{-\frac12} e, a^{-\frac12} b^{\frac12} c^{\frac12}). \end{aligned} \tag{2.7} $$

We conclude this section by applying this relation to obtain two three-term q-difference equations for the Sears-Hall ${}_3\Phi_2$. These are consequences of the following

q-difference equation for Φ_4 given in [**42**]:

$$
\begin{aligned}
&(1-\lambda_1\lambda_2)(1-\lambda_1\lambda_3)(1-\lambda_1\lambda_4)(1-\lambda_1\lambda_5)\Phi_4(\lambda_1)\\
&-\{1+\lambda_2\lambda_3\lambda_4\lambda_5 q^{-1}-\lambda_1(\lambda_2\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_5+\lambda_2\lambda_4\lambda_5+\lambda_3\lambda_4\lambda_5)\\
&\quad+\lambda_1^2(1+q)\lambda_2\lambda_3\lambda_4\lambda_5\}\Phi_4(\lambda_1 q)\\
&+\lambda_2\lambda_3\lambda_4\lambda_5 q^{-1}\Phi_4(\lambda_1 q^2)=0.
\end{aligned}
\tag{2.8}
$$

Now interchange λ_1 and λ_2 in (2.8), substitute (2.4) into the result, and finally change variables to a, b, c, d, e as before. Denote the Sears-Hall ${}_3\Phi_2$ by $f(a,b,c,d,e)$. We get

$$
\begin{aligned}
&(abc-de)(1-d)(1-dq)f(d)\\
&-\{abc+abceq^{-1}-(abc+e(a+b+c))d+e(1+q)d^2\}(1-dq)f(dq)\\
&+eq^{-1}(a-dq)(b-dq)(c-dq)f(dq^2)=0.
\end{aligned}
\tag{2.9}
$$

Similarly substituting (2.4) into (2.8) and making the same variable changes yields:

$$
\begin{aligned}
&abc^2(1-d)(1-dq)(1-e)(1-eq)f(a,b,d,e)\\
&-c\{abc+cdeq^{-1}-(ade+abcd+bde+abce)\\
&\quad+(1+q)abde\}(1-d)(1-e)f(aq,bq,dq,eq)\\
&+deq^{-1}(c-dq)(c-eq)(1-aq)(1-bq)f(aq^2,bq^2,dq^2,eq^2)=0.
\end{aligned}
\tag{2.10}
$$

Applying any of the other symmetries first to Φ_4 before substituting in (2.4) gives results equivalent to one of these after an obvious symmetry in the ${}_3\Phi_2$. Equations (2.9) and (2.10) do not seem to have appeared previously in the literature, but according to Askey [**10**] they have been known to J.A. Wilson for some time. This gives another example of how symmetric expansions are able to condense information about special functions. In this case the two q-difference equations (2.9) and (2.10) coalesce into the single equation (2.8).

2.2. Symmetry creating operators

In this section we examine the actions of the operators on the polynomials h_n. To facilitate understanding of the statements in the next two propositions please recall the convention involving non-commuting variables given in Section 1.2. In the following proposition the possible dependence of x on λ gives some useful freedom in applications.

PROPOSITION 2.3. *Let x be a variable possibly depending on λ. Then*

$$
(x\delta)_\infty\lambda^r=[\lambda,x]_r=h_r(x;\lambda),
$$
$$
\frac{1}{(x\delta)_\infty}\lambda^r=h_r(\lambda,x).
$$

PROOF.

$$
\begin{aligned}
(x\delta)_\infty\lambda^r&=\sum_{n\ge0}\frac{(-1)^n q^{\binom{n}{2}}x^n}{(q)_n}(\delta^n\lambda^r)=\sum_{n=0}^{r}\frac{(-1)^n q^{\binom{n}{2}}x^n}{(q)_n}\lambda^{r-n}\frac{(q)_r}{(q)_{r-n}}\\
&=\sum_{n=0}^{r}\begin{bmatrix}r\\n\end{bmatrix}(-1)^n q^{\binom{n}{2}}x^n\lambda^{r-n}=[\lambda,x]_r=h_r(x;\lambda).
\end{aligned}
$$

Similarly,

$$\frac{1}{(x\delta)_\infty}\lambda^r = \sum_{n\geq 0}\frac{x^n}{(q)_n}\delta^n\lambda^r = \sum_{n=0}^{r}\begin{bmatrix} r \\ n\end{bmatrix} x^n\lambda^{r-n} = h_r(\lambda, x).$$

Thus the operator $e(x\delta) = 1/(x\delta)_\infty$ takes the basis λ_1^n the basis $h_n(\lambda, x)$. Similarly the operator $(x\delta)_\infty$ takes the basis λ_1^n to the basis $[\lambda, x]_n$.

The main result of this section is the following proposition.

PROPOSITION 2.4. *Let x be a variable possibly depending on λ_1, and let x_i and λ_j, $j > 1$ be independent of λ_1. Then as equations between polynomials,*

$$\begin{aligned}(x)_\infty\frac{1}{(x\eta_1)_\infty}&h_n(x_1,\ldots,x_k;b\lambda_1,\lambda_2,\ldots,\lambda_l)\\ &= h_n(xb\lambda_1,x_1,\ldots,x_k;b\lambda_1,\lambda_2,\ldots,\lambda_l),\end{aligned}\tag{2.11}$$

$$\begin{aligned}(x\eta_1)_\infty\frac{1}{(x)_\infty}&h_n(xb\lambda_1,x_1,\ldots,x_k;b\lambda_1,\lambda_2,\ldots,\lambda_l)\\ &= h_n(x_1,\ldots,x_k;b\lambda_1,\lambda_2,\ldots,\lambda_l),\end{aligned}\tag{2.12}$$

$$\frac{1}{(x\delta_1)_\infty}h_n(x_1,\ldots,x_k;b\lambda_1,\lambda_2,\ldots,\lambda_l) = h_n(x_1,\ldots,x_k;xb,b\lambda_1,\lambda_2,\ldots,\lambda_l),\tag{2.13}$$

$$(x\delta_1)_\infty h_n(x_1,\ldots,x_k;b\lambda_1,\lambda_2,\ldots,\lambda_l) = h_n(bx,x_1,\ldots,x_k;b\lambda_1,\lambda_2,\ldots,\lambda_l).\tag{2.14}$$

PROOF.

$$\begin{aligned}
\frac{1}{(x\eta_1)_\infty}&\sum_{n\geq 0}\frac{h_n(x_1,\ldots,x_k;b\lambda_1,\lambda_2,\ldots,\lambda_l)}{(q)_n}z^n\\
&= \frac{1}{(x\eta_1)_\infty}\frac{(x_1z)_\infty\cdots(x_kz)_\infty}{(b\lambda_1z)_\infty\cdots(\lambda_lz)_\infty} = \frac{(x_1z)_\infty\cdots(x_kz)_\infty}{(\lambda_2z)_\infty\cdots(\lambda_lz)_\infty}\frac{1}{(x\eta_1)_\infty}\frac{1}{(b\lambda_1z)_\infty}\\
&= \frac{(x_1z)_\infty\cdots(x_kz)_\infty}{(\lambda_2z)_\infty\cdots(\lambda_lz)_\infty}\sum_{n\geq 0}\frac{x^n\eta_1^n}{(q)_n}\frac{1}{(b\lambda_1z)_\infty}\\
&= \frac{(x_1z)_\infty\cdots(x_kz)_\infty}{(\lambda_2z)_\infty\cdots(\lambda_lz)_\infty}\sum_{n\geq 0}\frac{x^n}{(q)_n(b\lambda_1zq^n)_\infty}\\
&= \frac{(x_1z)_\infty\cdots(x_kz)_\infty}{(\lambda_2z)_\infty\cdots(\lambda_lz)_\infty}\frac{(xb\lambda_1z)_\infty}{(b\lambda_1z)_\infty(x)_\infty}\\
&= \sum_{n\geq 0}\frac{h_n(xb\lambda_1,x_1,\ldots,x_k;b\lambda_1,\lambda_2,\ldots,\lambda_l)}{(x)_\infty(q)_n}z^n,
\end{aligned}$$

Equating coefficients of z^n and multiplying by $(x)_\infty$ gives (2.11). (2.12) follows immediately from (2.11) by applying the inverse operator to both sides. For (2.13), note that

$$\frac{1}{(x\delta_1)_\infty}\sum_{n\geq 0}\frac{h_n(x_1,\dots,x_k;b\lambda_1,\lambda_2,\dots,\lambda_l)}{(q)_n}z^n$$
$$=\frac{1}{(x\delta_1)_\infty}\frac{(x_1z)_\infty\cdots(x_kz)_\infty}{(b\lambda_1z)_\infty\cdots(\lambda_lz)_\infty}=\frac{(x_1z)_\infty\cdots(x_kz)_\infty}{(\lambda_2z)_\infty\cdots(\lambda_lz)_\infty}\frac{1}{(x\delta_1)_\infty}\frac{1}{(b\lambda_1z)_\infty}$$
$$=\frac{(x_1z)_\infty\cdots(x_kz)_\infty}{(\lambda_2z)_\infty\cdots(\lambda_lz)_\infty}\frac{1}{(x\delta_1)_\infty}\sum_{r\geq 0}\frac{b^rz^r}{(q)_r}\lambda_1^r$$
$$=\frac{(x_1z)_\infty\cdots(x_kz)_\infty}{(\lambda_2z)_\infty\cdots(\lambda_lz)_\infty}\sum_{r\geq 0}\frac{b^rz^r}{(q)_r}h_r(x,\lambda_1)=\frac{(x_1z)_\infty\cdots(x_kz)_\infty}{(\lambda_2z)_\infty\cdots(\lambda_lz)_\infty}\frac{1}{(b\lambda_1z)_\infty(xbz)_\infty},$$

where the second to the last equation follows from Proposition 2.3. Now expanding the last formula by the q-multinomial theorem and equating coefficients completes the proof. The proof of (2.14) is similar. □

The significance of this proposition is that it shows how the application of the q-difference operators adds new variables to the symmetric polynomials. This means that when a function has been expanded in terms of the symmetric polynomials, and it is then hit with one of these operators, the function produced has greater symmetry. These symmetries underlie the transformations of certain basic hypergeometric functions. In the next chapter we well see more examples of transformations obtained in this way.

We conclude with a statement of our theorem characterizing the polynomials h_n. This theorem is from our paper [**15**] but was inspired by Rogers's paper and as it sheds light on the importance of the polynomials h_n, we state it here.

THEOREM 2.5. *Let $\alpha_0=\beta_0=1$ and let $m>0$ be a fixed integer. Then the solution of the difference equation*

$$(\beta_0-\alpha_0q^n)H_n+(\beta_1-\alpha_1q^{n-1})H_{n-1}+\cdots+(\beta_m-\alpha_mq^{n-m})H_{n-m}=0$$

with $H_n=0$ for $-m<n<0$ and $H_0=1$ is given by

$$H_j=\frac{h_j}{(q)_j}=\frac{1}{(q)_j}\sum_{n_1+\cdots+n_m=j}\begin{bmatrix} j \\ n_1,\dots,n_m\end{bmatrix}[b_1,a_1]_{n_1}\cdots[b_m,a_m]_{n_m},$$

where

$$\alpha_0+\cdots+\alpha_mx^m=(1-a_1x)\cdots(1-a_mx),$$

and

$$\beta_0+\cdots+\beta_mx^m=(1-b_1x)\cdots(1-b_mx).$$

2.3. Rogers's paper and orthogonal polynomials

In this section we provide some of the connections between Rogers's paper [**42**] and modern work on orthogonal polynomials. In the spirit of staying as self contained as possible we first fix notation and some basic notions about orthogonal polynomials. Next we look at how the Askey-Wilson integral can be derived from Rogers's work on Φ_3 and finally we consider applications of the results of Sections 2.1 as well as other results from [**42**] in this light.

Let a function w be given which is non-negative and integrable on an interval (a, b) and satisfies

$$\int_a^b w(x)\,dx > 0.$$

In the event that (a, b) is unbounded impose the additional requirement that the moments

$$\mu_n = \int_a^b x^n w(x)\,dx \qquad n = 0, 1, 3, \ldots$$

are finite. If there is a sequence of polynomials $\{P_n\}$, P_n of degree n, such that

$$\int_a^b P_m(x) P_n(x) w(x)\,dx = 0 \qquad m \neq n,$$

then the polynomials P are called an orthogonal polynomial sequence (OPS) with respect to the weight function $w(x)$ on (a, b).

Define an operator $\mathcal{L}$ on polynomials $f(x)$ by

$$\mathcal{L}[f] = \int_a^b f(x) w(x)\,dx. \tag{2.15}$$

Then clearly

$$\mathcal{L}[x^n] = \mu_n \qquad n = 0, 1, 2, \ldots, \tag{2.16}$$

and

$$\mathcal{L}[P_m P_n] = 0 \qquad m \neq n. \tag{2.17}$$

Obviously $\mathcal{L}$ is linear–

$$\mathcal{L}[af + bg] = a\mathcal{L}[f] + b\mathcal{L}[g], \tag{2.18}$$

where f and g are polynomials and $a, b \in \mathbb{R}$.

It is clear that

$$\mathcal{L}\left[\sum c_n x^n\right] = \sum c_n \mu_n.$$

Thus $\mathcal{L}$ takes $x^n \mapsto \mu_n$. In particular if there are some free parameters in $w(x)$, then the μ_n will be functions of these parameters.

Notice that additionally

$$\mathcal{L}[P_n^2] \neq 0$$

since $\int w\,dx > 0$ and $P_n^2 > 0$.

One way to generalize the above is to note that (2.16) and (2.18) suffice to define $\mathcal{L}$ on polynomials without reference to (2.15). Thus in place of (2.15), one may consider an arbitrary sequence of complex numbers μ_n and use (2.16) and (2.18) to define $\mathcal{L}$ on polynomials. Then if there exists a sequence of polynomials satisfying (2.17) and also

$$\mathcal{L}[P_n^2] \neq 0, \tag{2.19}$$

then the sequence of polynomials P is said to be an OPS with respect to $\mathcal{L}$.

For brevity and clarity we now work formally. Suppose that $P_0(x) = 1$ and $\int_a^b w(x)\,dx = 1$. Additionally suppose that some function $f(x)$ defined on (a, b) may be expanded formally in terms of the polynomials $\{P_n\}$ so that

$$f(x) = \sum_{n \geq 0} a_n P_n(x). \tag{2.20}$$

Then formally

$$\int_a^b f(x)w(x)\,dx = \sum_{n\geq 0} a_n \int_a^b P_n(x)w(x)\,dx$$
$$= \sum_{n\geq 0} a_n \int_a^b P_0(x)P_n(x)w(x)\,dx = \sum_{n\geq 0} a_n\delta_{0,n} = a_0.$$

Thus *integration of a function against the weight function gives the constant term in the expansion (2.20).*

With this in mind we return to the polynomials A_n. Recall that for $|u|=1$

$$\frac{1}{P(z)} = \sum_{n\geq 0} \frac{A_n(u)}{(q)_n} z^n.$$

Also since $|u|=1$, let $u=e^{\theta}$ and by abuse of notation put $A_n(\theta)=A_n(u)$. Then it transpires (see [**3**], [**4**], or [**5**]) that A_n is orthogonal on the interval $(0,\pi)$ with respect to the weight function

$$w(\theta) = \frac{(q)_\infty}{2\pi}(e^{2i\theta})_\infty(e^{-2i\theta})_\infty, \tag{2.21}$$

with the orthogonality relation

$$\int_0^\pi A_n(\theta)A_m(\theta)w(\theta)\,d\theta = \frac{\delta_{m,n}}{(q)_n}.$$

In fact (1.25) is an algebraic version of this orthogonality relation for the polynomials $\{A_n\}$.

In particular if we integrate a function

$$f(\theta) = \sum_{n\geq 0} K_n A_n$$

with respect to the weight function $w(\theta)$, then by the above computation

$$\int_0^\pi f(\theta)w(\theta)\,d\theta = K_0.$$

That is, the constant term function defined in Section 1.3 has been identified with an integral operator:

$$C(f) = \int_0^\pi f(\theta)w(\theta)\,d\theta. \tag{2.22}$$

Now consider equation (1.32) which occurs in Rogers's paper. It states that

$$C\left(\frac{1}{P(\lambda_1)P(\lambda_2)P(\lambda_3)P(\lambda_4)}\right)$$
$$= \frac{(\lambda_1\lambda_2\lambda_3\lambda_4,)_\infty}{(\lambda_1\lambda_2,\lambda_1\lambda_3,\lambda_1\lambda_4,\lambda_2\lambda_3,\lambda_2\lambda_4,\lambda_3\lambda_4)_\infty}.$$

By (2.21) and (2.22) if follows that

$$\begin{aligned}&\int_0^\pi \frac{(e^{2i\theta},e^{-2i\theta})_\infty}{(\lambda_1e^{i\theta},\lambda_1e^{-i\theta},\lambda_2e^{i\theta},\lambda_2e^{-i\theta},\lambda_3e^{i\theta},\lambda_3e^{-i\theta},\lambda_4e^{i\theta},\lambda_4e^{-i\theta},)_\infty}d\theta\\ &= \frac{2\pi}{(q)_\infty}\frac{(\lambda_1\lambda_2\lambda_3\lambda_4,)_\infty}{(\lambda_1\lambda_2,\lambda_1\lambda_3,\lambda_1\lambda_4,\lambda_2\lambda_3,\lambda_2\lambda_4,,\lambda_3\lambda_4,)_\infty}.\end{aligned} \tag{2.23}$$

This is the famous Askey-Wilson integral [**12**],[**25**]. It is the key to proving the orthogonality of the Askey and Wilson ${}_4\Phi_3$ polynomials. In fact Ismail, Stanton and Viennot say in the introduction to their paper [**34**],

> *"The Askey-Wilson polynomials are the most general classical orthogonal polynomials that are known. So any new evaluations of the Askey-Wilson integral are desirable."*

The deduction of the orthogonality of the ${}_4\Phi_3$ polynomials from the Askey-Wilson integral can be found on pages 11-13 of [**12**]; we refer the reader there for the proof.

Now using the definition of Φ_4 and Theorem 2.1–

$$\begin{aligned} &C\left(\frac{1}{P(\lambda_1)P(\lambda_2)P(\lambda_3)P(\lambda_4)P(\lambda_5)}\right) \\ &= \frac{(\lambda_1\lambda_2\lambda_3\lambda_4, \lambda_2\lambda_3\lambda_4\lambda_5)_\infty}{(\lambda_1\lambda_2, \lambda_1\lambda_3, \lambda_1\lambda_4, \lambda_2\lambda_3, \lambda_2\lambda_4, \lambda_2\lambda_5, \lambda_3\lambda_4, \lambda_3\lambda_5, \lambda_4\lambda_5)_\infty} \\ &\qquad \times {}_3\Phi_2\left(\begin{matrix} \lambda_2\lambda_3,\ \lambda_2\lambda_4,\ \lambda_3\lambda_4 \\ \lambda_1\lambda_2\lambda_3\lambda_4,\ \lambda_2\lambda_3\lambda_4\lambda_5 \end{matrix} ; \lambda_1\lambda_5\right). \end{aligned}$$

Interchanging λ_4 and λ_5 and applying (2.22) to this gives the following theorem.

THEOREM 2.6.

$$\begin{aligned} &\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta})_\infty}{(\lambda_1 e^{i\theta}, \lambda_1 e^{-i\theta}, \lambda_2 e^{i\theta}, \lambda_2 e^{-i\theta}, \lambda_3 e^{i\theta}, \lambda_3 e^{-i\theta}, \lambda_4 e^{i\theta}, \lambda_4 e^{-i\theta}, \lambda_5 e^{i\theta}, \lambda_5 e^{-i\theta})_\infty} d\theta \\ &= \frac{2\pi}{(q)_\infty} \frac{(\lambda_1\lambda_2\lambda_3\lambda_5, \lambda_2\lambda_3\lambda_4\lambda_5, \lambda_1\lambda_4)_\infty}{\prod_{1\le j<k\le 5}(\lambda_j\lambda_k)_\infty} {}_3\Phi_2\left(\begin{matrix} \lambda_2\lambda_3,\ \lambda_2\lambda_5,\ \lambda_3\lambda_5 \\ \lambda_1\lambda_2\lambda_3\lambda_5,\ \lambda_2\lambda_3\lambda_4\lambda_5 \end{matrix} ; \lambda_1\lambda_4\right). \end{aligned}$$

This theorem originally appeared in [**34**]. Conversely Theorem 2.1 may be derived from Theorem 2.6. Specifically, orthogonality of A_n with knowledge of the weight function (2.21) gives equivalence. The same may be said of the Askey-Wilson integral and (1.32). Letting $\lambda_1 = 0$ in the Theorem 2.6 reduces it to the Askey-Wilson integral above.

Consider next the evaluation of Φ_5 in Theorem 1.14 which we reproduce below for convenience:

$$\begin{aligned} &C\left(\frac{1}{P(\lambda_1)P(\lambda_2)P(\lambda_3)P(\lambda_4)P(\lambda_5)P(\lambda_6)}\right) \\ &= \frac{1}{(\lambda_1\lambda_2, \lambda_1\lambda_3, \lambda_2\lambda_3, \lambda_4\lambda_5, \lambda_4\lambda_6, \lambda_5\lambda_6)_\infty} \\ &\qquad \times \sum_{n\ge 0} \frac{h_n(\lambda_1\lambda_2\lambda_3; \lambda_1, \lambda_2, \lambda_3) h_n(\lambda_4\lambda_5\lambda_6; \lambda_4, \lambda_5, \lambda_6)}{(q)_n}. \end{aligned}$$

Applying (2.22) to this gives the following generalization of the Theorem 2.6.

THEOREM 2.7.

$$\begin{aligned} &\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta})_\infty}{P(\lambda_1)P(\lambda_2)P(\lambda_3)P(\lambda_4)P(\lambda_5)P(\lambda_6)} d\theta \\ &= \frac{2\pi}{(q)_\infty} \frac{1}{(\lambda_1\lambda_2, \lambda_1\lambda_3, \lambda_2\lambda_3, \lambda_4\lambda_5, \lambda_4\lambda_6, \lambda_5\lambda_6)_\infty} \\ &\qquad \times \sum_{n\ge 0} \frac{h_n(\lambda_1\lambda_2\lambda_3; \lambda_1, \lambda_2, \lambda_3) h_n(\lambda_4\lambda_5\lambda_6; \lambda_4, \lambda_5, \lambda_6)}{(q)_n}. \end{aligned} \tag{2.24}$$

This result appears to be new.

2.4. Synthesis: The continuous q-ultraspherical polynomials

In [**42**] Rogers suggested parallel development of the properties of the polynomials B_n defined by:

$$B_n = B_n(u) = h_n(-qu, -qu^{-1}; 0) = q^{\binom{n+1}{2}} \chi_n(u, u^{-1}).$$

In terms of the polynomials A_n and the continuous q-Hermite polynomials H_n,

$$B_n = q^{\binom{n+1}{2}} A_n(u, u^{-1}; q^{-1}) = q^{\binom{n+1}{2}} H_n(x; q^{-1}),$$

where $x = (u + u^{-1})/2$. Rogers developed some interesting formulas involving the B_n's which exactly correspond to formulas for the A_n. Theorem 1.5 is a good example, corresponding to Theorem 1.3. He also attempted to find a formula corresponding to Theorem 1.12, but unfortunately the resulting equation diverged. The problems with the polynomials B_n arise because they are orthogonal with respect to a signed measure which is not unique. Nevertheless certain similarities do exist and this suggests a question: Is there a common generalization of the polynomials A_n and B_n?

Rogers answers this question in his third memoir [**44**]. Perhaps the easiest way to understand his synthesis is from the point of view of generating functions. Recall the generating function of A_n is

$$\frac{1}{P(z)} = \sum_{n\geq 0} \frac{A_n}{(q)_n} z^n \qquad P(z) = (uz, u^{-1}z)_\infty \qquad u = e^{i\theta},$$

and by the q-multinomial theorem the generating function of B_n is

$$P(-zq) = \sum_{n\geq 0} \frac{B_n}{(q)_n} z^n.$$

Rogers's synthesis of A_n and B_n is $L_n = L_n(\theta)$ defined by

$$\frac{P(\lambda z)}{P(z)} = \sum_{n\geq 0} \frac{L_n}{(q)_n} z^n \qquad |z| < 1.$$

After introducing L_n Rogers notes several special cases of his polynomials. For instance, when $\lambda = 0$ it is obvious that $L_n = A_n$, while for B_n it is easy to see that $\lim_{\lambda\to\infty} (-q/\lambda)^n L_n = B_n$. Another interesting special case is $\lambda = q$ where $L_n = \frac{\sin(n+1)\theta}{\sin\theta}(q)_n$. Note that in terms of the symmetric polynomials h_n, $L_n = h_n(\lambda u, \lambda u^{-1}; u, u^{-1})$ with $u = e^{i\theta}$. The standard notation for the continuous q-ultraspherical polynomials is $C_n(x; \lambda|q)$ which equals $L_n/(q)_n$ with $x = \cos\theta$. The C_n are polynomials of degree n in x and thus form a basis for formal power series in x over $\mathbb{C}$.

Now the linearization formula (1.24) for A_n is, after re-indexing the sum,

$$\frac{A_j A_k}{(q)_j (q)_k} = \sum_{0\leq t\leq k} \begin{bmatrix} j+k-2t \\ k-t \end{bmatrix} \frac{A_{j+k-2t}}{(q)_t (q)_{j+k-2t}},$$

where we assume $j \geq k$ without loss of generality. Similarly the linearization formula for B_n is

$$\frac{B_j B_k}{(q)_j(q)_k} = \sum_{0 \leq t \leq k} \begin{bmatrix} j+k-2t \\ k-t \end{bmatrix} \frac{(-1)^t q^{\binom{t+1}{2}-(k-t)(j-t)} B_{j+k-2t}}{(q)_t(q)_{j+k-2t}}. \tag{2.25}$$

This equation can be derived from Theorem 1.5. In [**44**] Rogers gave a result which includes both of these as special cases, namely

$$\frac{L_j L_k}{(q)_j(q)_k} = \sum_{0 \leq t \leq k} \frac{(1-\lambda q^{j+k-2t})(\lambda)_t(\lambda)_{j-t}(\lambda)_{k-t}(\lambda^2)_{j+k-t}}{(\lambda)_{j+k-t+1}(q)_t(q)_{j-t}(q)_{k-t}(\lambda^2)_{j+k-2t}} L_{j+k-2t}.$$

Rogers suggests an inductive proof of this theorem. This proof is rather tedious and other proofs have since been given. Gasper [**24**],[**25**] gives a somewhat easier proof and other proofs can be found in [**17**] and [**39**]. Although a proof will not be given here, we would like to take the opportunity to point out some consequences of this formula which follows from the techniques we have developed. As an application we note that just as with the polynomials A_n the algebra structure of series $\mathbb{C}[[L_n]]$ follows immediately. Here $\mathbb{C}[[L_n]]$ denotes the set whose elements are series of the form

$$\sum a_n L_n \qquad a_n \in \mathbb{C}$$

converging absolutely and uniformly in u by the Weierstrass M-test. We shall tacitly assume that the series below belong to $\mathbb{C}[[L_n]]$. Then from Rogers' linearization formula the following theorem holds.

THEOREM 2.8. *$\mathbb{C}[[L_n]]$ is a $\mathbb{C}$-algebra.*

We next obtain a generalization of Theorem 1.8.

DEFINITION 2.9. If $y = \sum a_n L_n \in \mathbb{C}[[L_n]]$, put $C_L(y) = a_0$.

Note that if y is a function analytic at $x = 0$, then since it has a unique power series expansion at 0, and since L_n form a basis for expansion of formal power series in x, it follows that y has a unique expansion in $\mathbb{C}[[L_n]]$. Thus $C_L(y)$ is well defined for such analytic functions.

THEOREM 2.10. *When the sequences a_j and b_k are independent of u,*

$$C_L\left(\left(\sum_{j\geq 0} \frac{(1-\lambda q^j)L_j}{(1-\lambda)(\lambda^2)_j(q)_j} a_j\right)\left(\sum_{k\geq 0} \frac{(1-\lambda q^k)L_k}{(1-\lambda)(\lambda^2)_k(q)_k} b_k\right)\right)$$

$$= \sum_{j\geq 0} \frac{1-\lambda q^j}{(1-\lambda)(\lambda^2)_j(q)_j} a_j b_j.$$

PROOF. It follows immediately from the linearization formula that

$$C_L(L_j L_k) = \begin{cases} 0 & \text{if} \quad j \neq k \\ \frac{1-\lambda}{1-\lambda q^j}(\lambda^2)_j(q)_j & \text{if} \quad j = k. \end{cases} \tag{2.26}$$

The rest of the proof is completed just like the proof of Theorem 1.8. $\square$

The orthogonality for L_n for $|\lambda| < 1$ (see [**12**],[**25**]) is given by

$$\int_0^\pi L_m L_n W(\theta) d\theta = \frac{\delta_{m,n}}{d_n}, \tag{2.27}$$

where

$$W(\theta) = \frac{(e^{2i\theta}, e^{-2i\theta})_\infty}{(\lambda e^{2i\theta}, \lambda e^{-2i\theta})_\infty},$$

and

$$d_n = \frac{(\lambda^2, q)_\infty}{2\pi(\lambda)_\infty^2} \frac{(1-\lambda q^n)}{(\lambda^2)_n (q)_n}.$$

One notices that (2.26) is an algebraic form of the orthogonality relation for the polynomials L_n. We have derived it in exactly the same way in which Rogers derived the special case for the polynomials A_n. It is therefore possible that Rogers was aware of this orthogonality as well. In one sense it is more general than the orthogonality relation (2.27): (2.26) holds even for $|\lambda| > 1$ as long as denominators do not vanish and even implies the constant term orthogonality of the polynomials B_n in the limit.

A corollary of Theorem 2.26 is a simultaneous generalization (synthesis) of Rogers's result Corollary 1.9 as well as the analogous result for the polynomials B_n suggested by Rogers in the last section of [**42**].

COROLLARY 2.11. *When the sequence b_j is independent of u,*

$$C_L\left(\frac{P(\lambda z)}{P(z)} \sum_{j\geq 0} \frac{(1-\lambda q^j)L_j}{(1-\lambda)(\lambda^2)_j (q)_j} b_j\right) = \sum_{j\geq 0} \frac{z^j}{(q)_j} b_j.$$

PROOF. In the left hand side of the assertion of Theorem 2.10, substitute

$$a_j = \frac{(1-\lambda)(\lambda^2)_j}{1-\lambda q^j} z^j$$

and use the fact that

$$\frac{P(\lambda z)}{P(z)} = \sum_{i\geq 0} \frac{L_i}{(q)_i} z^i.$$

□

Just as Rogers's linearization formula for A_n is equivalent to the q-Mehler formula [**33**], there is a generalization of the q-Mehler formula equivalent to the linearization formula for the continuous q-ultraspherical polynomials. This result is essentially formula (2.1) in [**17**]. We simplify this formula further by employing Rogers's symmetric polynomials h_n.

THEOREM 2.12 (Generalization of q-Mehler formula). *For $|r|, |s| < 1$, $\lambda^{-1} \neq q^k$, $k = 0, 1, 2, \ldots$,*

$$\frac{P(r\lambda)P(s\lambda)}{P(r)P(s)} = \sum_{l\geq 0} \frac{h_l(r\lambda, s\lambda; r, s)}{(\lambda)_l (q)_l} {}_2\Phi_1\left(\begin{matrix} \lambda, \lambda^2 q^l \\ \lambda q^{l+1} \end{matrix}; rs\right) L_l.$$

PROOF. First write the linearization formula in the more convenient form

$$\frac{L_j L_k}{(q)_j (q)_k} = \sum_{0\leq t\leq \min(j,k)} \frac{(\lambda)_t (\lambda)_{j-t} (\lambda)_{k-t} (\lambda^2 q^{j+k-2t})_t}{(\lambda)_{j+k-2t} (\lambda q^{j+k-2t+1})_t (q)_t (q)_{j-t} (q)_{k-t}} L_{j+k-2t}. \tag{2.28}$$

Now multiply both sides by $r^j s^k$ and sum over $j, k \geq 0$. Observe that by the generating function for L_n, the left hand side simplifies to

$$\frac{P(r\lambda)P(s\lambda)}{P(r)P(s)}.$$

The right hand side is then

$$\sum_{j,k\geq 0}\left(\sum_{0\leq t\leq \min(j,k)} \frac{(\lambda)_t(\lambda)_{j-t}(\lambda)_{k-t}(\lambda^2 q^{j+k-2t})_t}{(\lambda)_{j+k-2t}(\lambda q^{j+k-2t+1})_t(q)_t(q)_{j-t}(q)_{k-t}} L_{j+k-2t} r^j s^k\right).$$

Putting $m = j - t$ and $n = k - t$ the sum becomes

$$\begin{aligned}&\sum_{m,n,t\geq 0} \frac{(\lambda)_m(\lambda)_n(\lambda)_t(\lambda^2 q^{m+n})_t L_{m+n} r^{m+t} s^{n+t}}{(\lambda)_{m+n}(\lambda q^{m+n+1})_t(q)_m(q)_n(q)_t}\\ &= \sum_{m,n\geq 0} \frac{(\lambda)_m(\lambda)_n r^m s^n}{(\lambda)_{m+n}(q)_m(q)_n} {}_2\Phi_1\left(\begin{matrix}\lambda,\ \lambda^2 q^{m+n}\\ \lambda q^{m+n+1}\end{matrix};rs\right) L_{m+n}.\end{aligned}$$

Finally changing indices to $l = m + n$ and using the definition of h_l completes the proof. □

Note that the restriction on λ may be removed by simply multiplying both sides by $(\lambda)_\infty$. Applying Heine's transformation to the ${}_2\Phi_1$ on the right hand gives the following convenient form of the Theorem 2.12.

$$\frac{P(r\lambda)P(s\lambda)(rs)_\infty}{P(r)P(s)(rs\lambda)_\infty} = \sum_{l\geq 0} \frac{h_l(r\lambda, s\lambda; r, s)}{(\lambda)_l(q)_l}\, {}_2\Phi_1\left(\begin{matrix}q/\lambda,\ rs\\ \lambda rs\end{matrix};\lambda^2 q^l\right) L_l. \tag{2.29}$$

We have the following generalization of (1.32).

THEOREM 2.13. *For* $|s_1|, |s_2|, |s_3|, |s_4| < 1$, $\lambda^{-1} \neq q^k$ $k = 0, 1, 2, \ldots,$

$$\begin{aligned}&C_L\left(\frac{P(s_1\lambda)P(s_2\lambda)P(s_3\lambda)P(s_4\lambda)}{P(s_1)P(s_2)P(s_3)P(s_4)}\right)\\ &= \sum_{j\geq 0} \frac{(\lambda^2)_j}{(q)_j(\lambda)_j(\lambda q)_j} {}_2\Phi_1\left(\begin{matrix}\lambda,\ \lambda^2 q^j\\ \lambda q^{j+1}\end{matrix};s_1 s_2\right) {}_2\Phi_1\left(\begin{matrix}\lambda,\ \lambda^2 q^j\\ \lambda q^{j+1}\end{matrix};s_3 s_4\right)\\ &\qquad\times h_j(\lambda s_1, \lambda s_2; s_1, s_2) h_j(\lambda s_3, \lambda s_4; s_3, s_4).\end{aligned} \tag{2.30}$$

PROOF. Rewrite Theorem 2.10 in the form

$$C_L\left(\left(\sum_{j\geq 0} a_j L_j\right)\left(\sum_{k\geq 0} b_k L_k\right)\right) = \sum_{j\geq 0} \frac{(1-\lambda)(\lambda^2)_j(q)_j}{1-\lambda q^j} a_j b_j. \tag{2.31}$$

Now express the functions

$$\frac{P(s_1\lambda)P(s_2\lambda)}{P(s_1)P(s_2)}$$

and

$$\frac{P(s_3\lambda)P(s_4\lambda)}{P(s_3)P(s_4)}$$

as series in L_n by Theorem 2.12 and apply (2.31) to their product. □

Again it is easy to remove the restriction on λ.

Just as Theorem 2.1 had an integral equivalent (after knowing the weight function), Theorem 2.13 also has an integral equivalent. The following theorem is a generalization of Proposition 3.2 in [**33**] to which it reduces when $|s_1| = |s_2|$ and $|s_3| = |s_4|$, so that the polynomials h_j on the series side reduce to continuous q-ultraspherical polynomials. Our proof seems to be simpler.

THEOREM 2.14. *For* $|\lambda|, |s_i| < 1$ $(1 \le i \le 4)$,

$$\begin{aligned}&\int_0^\pi \frac{P(s_1\lambda)P(s_2\lambda)P(s_3\lambda)P(s_4\lambda)(e^{2i\theta}, e^{-2i\theta})_\infty}{P(s_1)P(s_2)P(s_3)P(s_4)(\lambda e^{2i\theta}, \lambda e^{-2i\theta})_\infty} d\theta \\ &= \frac{2\pi(\lambda)_\infty^2}{(\lambda^2, q)_\infty} \sum_{j\ge 0} \frac{(\lambda^2)_j}{(q)_j(\lambda)_j(\lambda q)_j} {}_2\Phi_1\left(\begin{matrix}\lambda,\ \lambda^2 q^j \\ \lambda q^{j+1}\end{matrix}; s_1 s_2\right) {}_2\Phi_1\left(\begin{matrix}\lambda,\ \lambda^2 q^j \\ \lambda q^{j+1}\end{matrix}; s_3 s_4\right) \\ &\qquad \times h_j(\lambda s_1, \lambda s_2; s_1, s_2) h_j(\lambda s_3, \lambda s_4; s_3, s_4).\end{aligned} \tag{2.32}$$

Putting $\lambda = 0$ in this formula and summing the right hand side by the q-Mehler formula (1.16) again recovers the Askey-Wilson integral (2.23).

PROOF. This follows immediately from Theorem 2.13 and (2.27). $\square$

Theorems 2.12, 2.13 and 2.14 have several special cases which are worthy of mention here. These arise in cases when the ${}_2\Phi_1$ sums by either the q-Gauss or q-Kummer theorems. We begin with two cases of Theorem 2.12. When $s = q/r\lambda^2$ applying the q-Gauss sum gives the following corollary.

COROLLARY 2.15. *For* $|r| < 1$ *and* $|\lambda|^2 > |q/r|$,

$$\begin{aligned}&\frac{(ru\lambda)_\infty(r\lambda/u)_\infty(qu/r\lambda)_\infty(q/r\lambda u)_\infty(\lambda)_\infty(q/\lambda^2)_\infty}{(ru)_\infty(r/u)_\infty(qu/r\lambda^2)_\infty(q/r\lambda^2 u)_\infty(q)_\infty(q/\lambda)_\infty} \\ &\qquad = \sum_{l\ge 0} \frac{(1-\lambda q^l)h_l(r\lambda, q/r\lambda; r, q/r\lambda^2)}{(q)_l^2} L_l.\end{aligned} \tag{2.33}$$

When $s = -q/r\lambda$ the q-Kummer theorem [**25**] may be applied and Theorem 2.12 reduces to

COROLLARY 2.16. *For* $|r| < 1$ *and* $|\lambda| > |q/r|$,

$$\begin{aligned}&\frac{(r\lambda u)_\infty(r\lambda/u)_\infty(-qu/r)_\infty(-q/ru)_\infty(\lambda)_\infty(-q/\lambda)_\infty}{(ru)_\infty(r/u)_\infty(-qu/r\lambda)_\infty(-q/r\lambda u)_\infty(-q)_\infty} \\ &\qquad = \sum_{l\ge 0} \frac{(1-\lambda q^l)h_l(r\lambda, -q/r\,; r, -q/r\lambda)(\lambda^2 q^{l+1}, q^{l+2}; q^2)_\infty}{(q)_l} L_l.\end{aligned} \tag{2.34}$$

Applying (2.31) to Corollary 2.15 using two different r's gives

COROLLARY 2.17. *For* $|r_1|, |r_2| < 1$ *and* $|\lambda|^2 > \max(|q/r_1|, |q/r_2|)$,

$$\begin{aligned}&C_L\left(\frac{P(r_1\lambda)P(q/r_1\lambda)P(r_2\lambda)P(q/r_2\lambda)}{P(r_1)P(q/r_1\lambda^2)P(r_2)P(q/r_2\lambda^2)}\right) = \frac{(q, q/\lambda)_\infty^2}{(\lambda q, q/\lambda^2)_\infty^2} \sum_{j\ge 0} \frac{(\lambda^2)_j(1-\lambda q^j)}{(q)_j^3(1-\lambda)} \\ &\qquad \times h_j(r_1\lambda, q/r_1\lambda; r_1, q/r_1\lambda^2) h_j(r_2\lambda, q/r_2\lambda; r_2, q/r_2\lambda^2).\end{aligned} \tag{2.35}$$

Applying (2.31) to Corollary 2.16 with two different r's gives

COROLLARY 2.18. *For* $|r_1|, |r_2| < 1$ *and* $|\lambda| > \max(|q/r_1|, |q/r_2|)$,

$$\begin{aligned}
&C_L\left(\frac{P(r_1\lambda)P(-q/r_1)P(r_2\lambda)P(-q/r_2)}{P(r_1)P(-q/r_1\lambda)P(r_2)P(-q/r_2\lambda)}\right)\\
&= \frac{(-q)^2_\infty}{(\lambda q, -q/\lambda)^2_\infty} \sum_{j\geq 0} \frac{(1-\lambda q^j)(\lambda^2)_j(\lambda^2 q^{j+1}, q^{j+2}; q^2)^2_\infty}{(q)_j(1-\lambda)}\\
&\quad\times h_j(r_1\lambda, -q/r_1; r_1, -q/r_1\lambda) h_j(r_2\lambda, -q/r_2; r_2, -q/r_2\lambda).
\end{aligned} \tag{2.36}$$

The last possibility is to apply (2.31) to Corollary 2.15 with r_1 for r and to Corollary 2.16 with r_2 for r to obtain the following corollary.

COROLLARY 2.19. *For* $|r_1|, |r_2| < 1$ *and* $|\lambda| > \max(|q/r_1|^{1/2}, |q/r_2|)$,

$$\begin{aligned}
&C_L\left(\frac{P(r_1\lambda)P(q/r_1\lambda)P(r_2\lambda)P(-q/r_2)}{P(r_1)P(q/r_1\lambda^2)P(r_2)P(-q/r_2\lambda)}\right)\\
&= \frac{(q, -q, q/\lambda)_\infty}{(\lambda q)^2_\infty(-q/\lambda, q/\lambda^2)_\infty} \sum_{j\geq 0} \frac{(1-\lambda q^j)(\lambda^2)_j(\lambda^2 q^{j+1}, q^{j+2}; q^2)_\infty}{(q)^2_j(1-\lambda)}\\
&\quad\times h_j(r_1\lambda, q/r_1\lambda; r_1, q/r_1\lambda^2) h_j(r_2\lambda, -q/r_2; r_2, -q/r_2\lambda).
\end{aligned} \tag{2.37}$$

The last three results are all Corollaries of Theorem 2.13. Of course they all have integral versions via (2.27) which we give here. These are special cases of Theorem 2.14. They are recorded in the following corollary.

COROLLARY 2.20. *For* $|r_1|, |r_2| < 1$ *and* $\max(|q/r_1|, |q/r_2|) < |\lambda|^2 < 1$,

$$\begin{aligned}
&\int_0^\pi \frac{P(r_1\lambda)P(q/r_1\lambda)P(r_2\lambda)P(q/r_2\lambda)(e^{2i\theta}, e^{-2i\theta})_\infty}{P(r_1)P(q/r_1\lambda^2)P(r_2)P(q/r_2\lambda^2)(\lambda e^{2i\theta}, \lambda e^{-2i\theta})_\infty} d\theta\\
&= \frac{2\pi(q)_\infty(q/\lambda)^2_\infty(1-\lambda)}{(\lambda^2)_\infty(q/\lambda^2)^2_\infty} \sum_{j\geq 0} \frac{(\lambda^2)_j(1-\lambda q^j)}{(q)^3_j}\\
&\quad\times h_j(r_1\lambda, q/r_1\lambda; r_1, q/r_1\lambda^2) h_j(r_2\lambda, q/r_2\lambda; r_2, q/r_2\lambda^2).
\end{aligned} \tag{2.38}$$

For $|r_1|, |r_2| < 1$ *and* $\max(|q/r_1|, |q/r_2|) < |\lambda| < 1$,

$$\begin{aligned}
&\int_0^\pi \frac{P(r_1\lambda)P(-q/r_1)P(r_2\lambda)P(-q/r_2)(e^{2i\theta}, e^{-2i\theta})_\infty}{P(r_1)P(-q/r_1\lambda)P(r_2)P(-q/r_2\lambda)(\lambda e^{2i\theta}, \lambda e^{-2i\theta})_\infty} d\theta\\
&= \frac{2\pi(-q)^2_\infty(1-\lambda)}{(\lambda^2)_\infty(-q/\lambda)^2_\infty} \sum_{j\geq 0} \frac{(1-\lambda q^j)(\lambda^2)_j(\lambda^2 q^{j+1}, q^{j+2}; q^2)^2_\infty}{(q)_j}\\
&\quad\times h_j(r_1\lambda, -q/r_1; r_1, -q/r_1\lambda) h_j(r_2\lambda, -q/r_2; r_2, -q/r_2\lambda).
\end{aligned} \tag{2.39}$$

For $|r_1|, |r_2| < 1$ *and* $\max(|q/r_1|^{1/2}, |q/r_2|) < |\lambda| < 1$,

$$\begin{aligned}
&\int_0^\pi \frac{P(r_1\lambda)P(q/r_1\lambda)P(r_2\lambda)P(-q/r_2)(e^{2i\theta}, e^{-2i\theta})_\infty}{P(r_1)P(q/r_1\lambda^2)P(r_2)P(-q/r_2\lambda)(\lambda e^{2i\theta}, \lambda e^{-2i\theta})_\infty} d\theta\\
&= \frac{2\pi(-q, q/\lambda)_\infty(1-\lambda)}{(\lambda^2, -q/\lambda, q/\lambda)_\infty} \sum_{j\geq 0} \frac{(1-\lambda q^j)(\lambda^2)_j(\lambda^2 q^{j+1}, q^{j+2}; q^2)_\infty}{(q)^2_j}\\
&\quad\times h_j(r_1\lambda, q/r_1\lambda; r_1, q/r_1\lambda^2) h_j(r_2\lambda, -q/r_2; r_2, -q/r_2\lambda).
\end{aligned} \tag{2.40}$$

In closing this section we make some remarks on generalizations and problems which remain. First of all, the continuous q-ultraspherical polynomials are not the end of the line for q-orthogonal polynomials. More generally the ${}_4\Phi_3$ polynomials (also known as the Askey-Wilson polynomials) [**12**],[**25**] have a linearization formula [**39**]. However the formula is complicated; its linearization coefficients are sums of terminating very well-poised ${}_{10}\Phi_9$'s. The method we have just used may be applied to these polynomials yielding generalizations of theorems of this section. Nicer generalizations may be obtained with the continuous q-Jacobi polynomials where the coefficients are specialized ${}_{10}\Phi_9$'s; see [**39**]. A refinement of the method used here was employed in [**16**] to give further integral evaluations. Finally, Theorem 2.10 enables us to make a commutative diagram analogous to that of A_n given in Chapter 1; we do not know the identity of the corresponding operator $?_L$ in the present case.

2.5. Convergence criteria for operator theorems

It is appropriate to give some convergence criteria for the operators used here. Rogers did not remark on convergence or divergence of his results, so we tie up this loose end here. We give results on the operators $\sum b_m \eta^m$, $1/(x\delta)_\infty$, and $E(x\sigma)$. As operators on polynomials convergence can be defined either formally or in the complex topology. We extend the action of these operators to infinite series as follows. Given a function

$$f(\lambda) = \sum_{n\geq 0} a_n \lambda^n,$$

where the coefficients a_n are independent of λ, define the action of the operators $g(x\eta) = \sum b_m x^m \eta^m$, $1/(x\delta)_\infty$, and $E(x\sigma)$ by the equations

$$g(x\eta)f(\lambda) = \sum_{n\geq 0} a_n \left(g(x\eta)\lambda^n\right),$$

$$\frac{1}{(x\delta)_\infty} f(\lambda) = \sum_{n\geq 0} a_n \left(\frac{1}{(x\delta)_\infty}\lambda^n\right),$$

and

$$E(x\sigma)f(\lambda) = \sum_{n\geq 0} a_n \left(E(x\sigma)\lambda^n\right).$$

Convergence of the resulting function in the first two cases is given by the following proposition and in the third by the proposition following.

PROPOSITION 2.21. *Suppose $f(\lambda) = \sum_{n\geq 0} a_n \lambda^n$ and $g(z) = \sum_{m\geq 0} b_m z^m$ are analytic in the unit circle, and $|q| < 1$. Then*

(1) *$g(x\eta)f(\lambda)$ is analytic for $|\lambda| < q^{-k}$ and $|x| < q^{-l}$, where g and f have k-fold and l-fold zeros at the origin, respectively.*
(2) *$\dfrac{1}{(x\delta)_\infty} f(\lambda)$ is analytic for $|\lambda|, |x| < 1$.*

PROOF. For (i), we have

$$g(x\eta)f(\lambda) = \sum_{n\geq 0} a_n g(x\eta)\lambda^n = \sum_{n\geq 0} a_n \left(\sum_{m\geq 0} b_m x^m q^{mn} \lambda^n\right)$$

$$= \sum_{n\geq 0} a_n \left(\sum_{m\geq 0} b_m x^m q^{mn}\right)\lambda^n = \sum_{n\geq 0} a_n g(xq^n)\lambda^n.$$

Now

$$\limsup_{n\to\infty} |a_n g(xq^n)|^{1/n} = \limsup_{n\to\infty} |a_n x q^{kn} h(xq^n)|^{1/n} \le |q|^k,$$

where $g(z) = z^k h(z)$. Thus $g(x\eta)f(\lambda)$ is analytic for $|\lambda| < q^{-k}$. Similarly for $|x| < q^{-l}$.

For (ii), we have formally

$$\begin{aligned}\frac{1}{(x\delta)_\infty} f(\lambda) &= \sum_{n\ge 0} a_n h_n(\lambda, x) = \sum_{n\ge 0} a_n \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k}\lambda^k \right) \\ &= \sum_{k\ge 0} \left(\sum_{m\ge 0} \begin{bmatrix} m+k \\ k \end{bmatrix} x^m a_{m+k} \right) \lambda^k.\end{aligned}$$

We analyze this last series. Taking the absolute value of its terms,

$$\begin{aligned}&\sum_{k\ge 0} \left(\sum_{m\ge 0} \left| \begin{bmatrix} m+k \\ k \end{bmatrix} \right| |x|^m |a_{m+k}| \right) |\lambda|^k \le \sum_{k\ge 0} \left(\sum_{m\ge 0} \begin{bmatrix} m+k \\ k \end{bmatrix}_{|q|} |x|^m |a_{m+k}| \right) |\lambda|^k \\ &\le \sum_{k\ge 0} \left(\sum_{m\ge 0} \frac{(|q|)_{m+k}}{(|q|)_m} |x|^m |a_{m+k}| \right) \frac{|\lambda|^k}{(|q|)_k},\end{aligned}$$

where the first inequality follows from the positivity of the coefficients of q-binomial coefficients when $q > 0$, (3.26). Now since $\sum a_n \lambda^n$ has radius of convergence ≥ 1, $\limsup_{n\to\infty} |a_n|^{1/n} \le 1$. From this it follows that $\limsup_{n\to\infty} |(|q|)_n a_n|^{1/n} \le 1$. Hence for any $\varepsilon > 0$ there exists an N such that $n \ge N$ implies $|(|q|)_n a_n|^{1/n} < 1+\varepsilon$. Choose ε small enough that $|(1+\varepsilon)x| < 1$ and $|(1+\varepsilon)\lambda| < 1$. Then $|(|q|)_n a_n| < (1+\varepsilon)^n$ for $n \ge N$. Looking at the terms of the last series for which $m + k \ge N$ we have

$$\begin{aligned}&\sum_{k\ge 0} \left(\sum_{\substack{m\ge N-k \\ m\ge 0}} \frac{(|q|)_{m+k}}{(|q|)_m} |x|^m |a_{m+k}| \right) \frac{|\lambda|^k}{(|q|)_k} \\ &\le \sum_{k\ge 0} \left(\sum_{\substack{m\ge N-k \\ m\ge 0}} \frac{|x|^m |(1+\varepsilon)^{m+k}|}{(|q|)_m} \right) \frac{|\lambda|^k}{(|q|)_k} \\ &= \sum_{k\ge 0} \left(\sum_{\substack{m\ge N-k \\ m\ge 0}} \frac{|x(1+\varepsilon)|^m}{(|q|)_m} \right) \frac{|\lambda(1+\varepsilon)|^k}{(|q|)_k} \le \sum_{k\ge 0} \left(\sum_{m\ge 0} \frac{|x(1+\varepsilon)|^m}{(|q|)_m} \right) \frac{|\lambda(1+\varepsilon)|^k}{(|q|)_k} \\ &= \sum_{m\ge 0} \frac{|x(1+\varepsilon)|^m}{(|q|)_m} \sum_{k\ge 0} \frac{|\lambda(1+\varepsilon)|^k}{(|q|)_k} = \frac{1}{(x(1+\varepsilon);|q|)_\infty} \frac{1}{(\lambda(1+\varepsilon);|q|)_\infty}.\end{aligned}$$

Hence the series

$$\sum_{k\ge 0} \left(\sum_{m\ge 0} \begin{bmatrix} m+k \\ k \end{bmatrix} x^m a_{m+k} \right) \lambda^k$$

converges absolutely for $|x|, |\lambda| < 1$ and its terms may be rearranged. Therefore the series

$$\sum_{n\geq 0} a_n h_n(\lambda, x)$$

also converges absolutely for $|x|, |\lambda| < 1$. $\square$

It follows from Proposition 2.21 that the identities implied by applying (1.12) or next chapter's (3.5) to any analytic function with radius of convergence 1 are valid for $|\lambda|, |\mu\lambda_1| < 1$.

We now give a convergence criteria for Rogers's operator $E(x\sigma)$ defined in the last chapter.

PROPOSITION 2.22. *Suppose $\alpha \geq 1/4$ and $f(\lambda)$ is an entire function with*

$$f(\lambda) = \sum_{n\geq 0} a_n \lambda^n,$$

where $a_n = O(|q|^{\alpha n^2})$. Then $E(x\sigma)f(\lambda)$ is analytic for $|x|, |\lambda| \leq |q|^{4\alpha-1}$, where this last inequality is strict if $\alpha = 1/4$.

PROOF. We have formally

$$\begin{aligned} E(x\delta)f(\lambda) &= \sum_{n\geq 0} a_n \chi_n(\lambda, x) = \sum_{n\geq 0} a_n \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{-k(n-k)} x^{n-k} \lambda^k \right) \\ &= \sum_{k\geq 0} \left(\sum_{m\geq 0} \begin{bmatrix} m+k \\ k \end{bmatrix} q^{-km} x^m a_{m+k} \right) \lambda^k. \end{aligned}$$

We look for absolute convergence. Taking the absolute value of the terms,

$$\begin{aligned} &\sum_{k\geq 0} \left(\sum_{m\geq 0} \left| \begin{bmatrix} m+k \\ k \end{bmatrix} \right| |q|^{-km} |x|^m |a_{m+k}| \right) |\lambda|^k \\ &\leq \sum_{k\geq 0} \left(\sum_{m\geq 0} \begin{bmatrix} m+k \\ k \end{bmatrix}_{|q|} |q|^{-km} |x|^m |a_{m+k}| \right) |\lambda|^k \\ &= \sum_{k\geq 0} \left(\sum_{m\geq 0} \frac{(|q|)_{m+k}}{(|q|)_m (|q|)_k} |q|^{-km} |x|^m |a_{m+k}| \right) |\lambda|^k, \end{aligned}$$

where again the first inequality follows from the positivity of the coefficients of q-binomial coefficients when $q > 0$; see (3.26). Now since

$$\lim_{k,m\to\infty} \frac{(|q|)_{m+k}}{(|q|)_m (|q|)_k}$$

exists for $0 < |q| < 1$ and the denominator never vanishes, it follows that we can put

$$C_1 = \limsup_{k,m\to\infty} \frac{(|q|)_{m+k}}{(|q|)_m (|q|)_k}.$$

Also since $|a_n| = O(|q|^{\alpha n^2})$, we have $|a_n| < C_2|q|^{\alpha n^2}$. Then the last series is less than

$$C_1C_2 \sum_{k,m\geq 0} |q|^{\alpha(k+m)^2-km}|x|^m|\lambda|^k = C_1C_2 \sum_{k,m\geq 0} |q|^{\alpha k^2+(2\alpha-1)km+\alpha m^2}|x|^m|\lambda|^k.$$

Now by Example 4, p. 87 of [**18**], it follows easily that

$$\sum_{k,m\geq 0} |q|^{\alpha k^2+(2\alpha-1)km+\alpha m^2}|q|^{(4\alpha-1)m}|q|^{(4\alpha-1)k}$$

is convergent for $\alpha > 1/4$. When $\alpha = 1/4$, convergence follows from the writing the penultimate sum as

$$C_1C_2 \sum_{k,m\geq 0} |q|^{(k-m)^2/4} r^m s^k$$

with $r = |x/q^{4\alpha-1}|$ and $s = |\lambda/q^{4\alpha-1}|$ so that $r, s < 1$ so that the series converges by comparison with

$$\sum_{k,m\geq 0} r^m s^k.$$

Hence the proposition thereafter. □

2.6. A fascinating phenomenon

In [**42**] Rogers's evaluated the function

$$\frac{1}{(\lambda\delta_1)_\infty} \prod_{i=2}^{k} \frac{1}{(\lambda_1\lambda_i)_\infty}, \tag{2.41}$$

when $k = 2, 3$. When $k = 2$ the result was the q-Mehler theorem, while when $k = 3$ Rogers obtained his beautiful symmetric function expansion. In the next chapter we will consider this function for arbitrary k as well as other variants. It will transpire that Rogers's method extends without any trouble and it is in this way that we will prove the generalization of Rogers's symmetric function expansion quoted in the first chapter. We therefore expect similar nice results if we carry out a similar evaluation of the function

$$E(\lambda\sigma_1) \prod_{i=2}^{k} E(\lambda_1\lambda_i), \tag{2.42}$$

with $k > 3$, since when $k = 3$ Rogers obtained Theorem 1.5. Here we explore this question. We continue the operator approach used by Rogers. In the case of (2.41) one obtains Theorem 1.1 when $k = 4$, so this raises the question of what kind of analog of the symmetric expansion is obtained when $k = 4$ in (2.42)? We find using Theorems 1.4 and 1.5:

$$
\begin{aligned}
&E(\lambda\sigma_1)E(\lambda_1\lambda_4)E(\lambda_1\lambda_3)E(\lambda_1\lambda_2)\\
&\quad= E(\lambda_1\lambda_4)E(\lambda\lambda_4\eta_1^{-1})\frac{E(\lambda\lambda_3)E(\lambda_1\lambda_3)E(\lambda\lambda_2)E(\lambda_1\lambda_2)}{E(-\lambda\lambda_1\lambda_2\lambda_3)}\\
&\quad= E(\lambda_1\lambda_4)E(\lambda\lambda_3)E(\lambda\lambda_2)\sum_{n\geq 0}\frac{q^{\binom{n+1}{2}}\lambda^n\lambda_4^n E(\lambda_1\lambda_2q^{-n})E(\lambda_1\lambda_3q^{-n})}{(q)_nE(-\lambda\lambda_1\lambda_2\lambda_3q^{-n})}\\
&\quad= \frac{E(\lambda_1\lambda_4)E(\lambda\lambda_3)E(\lambda\lambda_2)E(\lambda_1\lambda_2)E(\lambda_1\lambda_3)}{E(-\lambda\lambda_1\lambda_2\lambda_3)}\\
&\qquad\qquad\times\sum_{n\geq 0}\frac{(-\lambda_1^{-1}\lambda_2^{-1})_n(-\lambda_1^{-1}\lambda_3^{-1})_n}{(q)_n(\lambda^{-1}\lambda_1^{-1}\lambda_2^{-1}\lambda_3^{-1})_n}(-\lambda_1\lambda_4q)^n.
\end{aligned}
$$

On the other hand the q-multinomial theorem and (1.19) give

$$
E(\lambda\sigma_1)E(\lambda_1\lambda_4)E(\lambda_1\lambda_3)E(\lambda_1\lambda_2)=\sum_{n\geq 0}\frac{q^{\binom{n+1}{2}}\chi_n(\lambda_2,\lambda_3,\lambda_4)\chi_n(\lambda,\lambda_1)}{(q)_n}.
$$

Equating these two results, multiplying by the infinite products, and changing the signs of λ_2, λ_3 and λ_4 gives the following "identity".

$$
\begin{aligned}
&{}_2\Phi_1\left(\begin{matrix}\lambda_1^{-1}\lambda_2^{-1},\ \lambda_1^{-1}\lambda_3^{-1}\\ \lambda^{-1}\lambda_1^{-1}\lambda_2^{-1}\lambda_3^{-1}\end{matrix};\lambda_1\lambda_4q\right)\\
&\quad=\frac{E(-\lambda\lambda_1\lambda_2\lambda_3)}{E(-\lambda_1\lambda_4)E(-\lambda\lambda_3)E(-\lambda\lambda_2)E(-\lambda_1\lambda_2)E(-\lambda_1\lambda_3)}\\
&\qquad\times\sum_{n\geq 0}\frac{(-1)^nq^{\binom{n+1}{2}}\chi_n(\lambda_2,\lambda_3,\lambda_4)\chi_n(\lambda,\lambda_1)}{(q)_n}.
\end{aligned}\tag{2.43}
$$

To compare this "symmetric expansion" with Rogers's it is helpful to change variables so that the Φ's are the same. To do this put $\lambda=\lambda_1^{-1}$ in (2.43), and $a=\lambda_1^{-1}\lambda_2^{-1}$, $b=\lambda_1^{-1}\lambda_3^{-1}$, $c=\lambda_2^{-1}\lambda_3^{-1}$ and $d=\lambda_1\lambda_4q$ so that

$$
\lambda_1=\sqrt{\frac{c}{ab}},\quad \lambda_2=\sqrt{\frac{b}{ac}},\quad \lambda_3=\sqrt{\frac{a}{bc}},\quad\text{and}\quad \lambda_4=\frac{d}{q}\sqrt{\frac{ab}{c}}.
$$

This gives using (1.20)

(2.44)

$$
\begin{aligned}
&{}_2\Phi_1\left(\begin{matrix}a,\ b\\ c\end{matrix};d\right)=\frac{(q/c)_\infty}{(bq/c)_\infty(aq/c)_\infty(q/a)_\infty(q/b)_\infty(d)_\infty}\\
&\quad\times\sum_{n\geq 0}\frac{(-1)^nq^{\binom{n+1}{2}}}{(q)_n}h_n\left(\sqrt{\frac{a}{bc}},\sqrt{\frac{b}{ac}},\sqrt{\frac{ab}{c}}\frac{d}{q}:q^{-1}\right)h_n\left(\sqrt{\frac{ab}{c}},\sqrt{\frac{c}{ab}}:q^{-1}\right).
\end{aligned}
$$

A similar change of variables in (1.5) gives

$$
\begin{aligned}
&{}_2\Phi_1\left(\begin{matrix}a,\ b\\ c\end{matrix};d\right)=\frac{(c/b)_\infty(c/a)_\infty(a)_\infty(b)_\infty(abd/c)_\infty}{(c)_\infty}\\
&\qquad\times\sum_{n\geq 0}\frac{1}{(q)_n}h_n\left(\sqrt{\frac{bc}{a}},\sqrt{\frac{ac}{b}},\sqrt{\frac{ab}{c}}d\right)h_n\left(\sqrt{\frac{ab}{c}},\sqrt{\frac{c}{ab}}\right).
\end{aligned}
$$

Comparing the right sides of these equations it appears that we have obtained a new symmetric expansion. It is interesting to note that the sum on the right side of (2.44) is the same as the sum in Rogers's symmetric expansion with the change of variables $q \to q^{-1}$, $a \to a^{-1}$, $b \to b^{-1}$, $c \to c^{-1}$, $d \to abd/cq$. However by Corollary 1.17 the left hand side is fixed under this change of variables. This implies that the infinite products on the right hand sides are equal, a clear impossibility. What's going on here?

The answer is that equation (2.43) is incorrect, the error being made when we applied the operator $E(\lambda\sigma_1)$ to the q-multinomial expansion

$$E(\lambda_1\lambda_4)E(\lambda_1\lambda_3)E(\lambda_1\lambda_2) = \sum_{n\geq 0} \frac{q^{\binom{n+1}{2}}\chi_n(\lambda_2,\lambda_3,\lambda_4)\lambda_1^n}{(q)_n}.$$

Here the coefficient of λ_1^n is only $O(|q|^{n^2/6})$ so that Proposition 2.22 doesn't imply convergence of the function which results from applying the operator $E(\lambda\sigma_1)$. Indeed the sum on the right hand of (2.43) does not converge.

Since the sum is the same as Rogers's after base inversion, it follows that it does converge when $|q| > 1$. In this case, unfortunately, things go bad in another way; the infinite products on the right hand side of (2.43) now fail to converge. This example as well as the last equation given in [**42**], which is incorrect by virtue of divergence, clearly demonstrate the danger of using analogy and that in using operator methods one must give consideration to convergence.

2.7. Operator theorems and q-Heisenberg algebras

Recently researchers in mathematical physics and algebra have rediscovered Rogers's operator identity Theorem 1.2; see [**27**], [**36**], [**38**]. For a nice overview of this work we recommend Koornwinder [**37**]. Here we look more closely at the connection between Rogers's operator identities and q-Heisenberg algebras and we show that there is an identity in $\mathbb{C}_{q\text{Heis}}[[x,y,c]]$ (defined shortly) which translates into Theorems 1.2 or 1.4 depending on which of two different families of representations of $\mathbb{C}_{q\text{Heis}}[[x,y,c]]$ are chosen. As both families of representations are faithful, this result implies an algebraic equivalence between the two theorems. This is quite interesting given their analytic differences.

The q-Heisenberg algebra we shall work with in this section is denoted by $\mathbb{C}_{q\text{Heis}}[[x,y,c]]$ and consists of all formal series

$$\sum_{k,l,m\geq 0} a_{k,l,m}c^m y^l x^k \quad a_{k,l,m} \in \mathbb{C},$$

where x, y, c satisfy the commutation relations

$$xy - qyx = (1-q)c$$
$$xc = cx$$
$$yc = cy.$$

$q \in \mathbb{C}$ is fixed with $|q| < 1$.

Now Koornwinder [**37**] states that Rogers used the representation $\pi : x \mapsto \lambda\delta_1$, $y \mapsto \lambda_1$, $c \mapsto \lambda$. Of course π is a representation of $\mathbb{C}_{q\text{Heis}}[[x,y,c]]$, however the reader will notice that Theorem 1.2 actually lives in the one dimensional family of representations $\pi_\mu : x \mapsto \lambda\delta_1$, $y \mapsto \mu\lambda_1$, $c \mapsto \mu\lambda$, where $\mu \in \mathbb{C}$. This distinction is of some significance as it is the presence of the parameter μ above which generates the symmetry in the Heine transformation. (It should also be pointed out that alternatively we could adjoin μ formally to $\mathbb{C}_{q\mathrm{Heis}}[[x,y,c]]$ and work in a larger algebra.)

In $\mathbb{C}_{q\mathrm{Heis}}[[x,y,c]]$ Theorem 1.2 takes the form

$$e(x)e(y) = e(y)e(-yx+c)e(x). \tag{2.45}$$

To check this one has only to apply π_μ to (2.45) noting that $\pi_\mu(-yx+c) = \mu\lambda\eta_1$. A concise proof of equation (2.45) is given in [**37**]. Here we derive an alternative form of (2.45). The new statement will easily be seen to be equivalent to Theorem 1.4 under a different family of representations of $\mathbb{C}_{q\mathrm{Heis}}[[x,y,c]]$.

THEOREM 2.23. *In $\mathbb{C}_{qHeis}[[x,y,c]]$ the following identity holds*

$$(-yq)_\infty(-xq)_\infty = (-xq)_\infty(cq-xyq)_\infty(-yq)_\infty.$$

PROOF. We check that the identity is equivalent to (2.45). To do this we may start with (2.45) and note that taking $x \mapsto -qx$, $y \mapsto -qy$, and $c \mapsto q^2c$ gives an automorphism of $\mathbb{C}_{q\mathrm{Heis}}[[x,y,c]]$. Now make this change of variables in (2.45) and take the reciprocal and employ the commutation between x and y to get the identity in the theorem. This procedure obviously may be reversed. $\square$

Next we observe that Theorem 1.18 can be obtained from a different family of representations of the algebra $\mathbb{C}_{q\mathrm{Heis}}[[x,y,c]]$. Indeed the family of representations π^*_μ of $\mathbb{C}_{q\mathrm{Heis}}[[x,y,c]]$ given by $\pi^*_\mu : x \mapsto \mu\lambda_1,\ y \mapsto \lambda\sigma_1,\ c \mapsto -\mu\lambda$ transforms Theorem 2.23 into Theorem 1.18. Once one notes that this family of representations is faithful, one can derive the q-Heisenberg identity from Theorem 1.18. This gives another proof of our theorem. (Alternatively one can follow the proof of Theorem 1.18 in the variables x, y and c noting the appropriate q-commutations.)

We conclude this section by mentioning a variant of Theorem 1.2 which does not have q-Heisenberg type generalization. Consider the following theorem.

THEOREM 2.24. *As operators on functions independent of λ,*

$$(\lambda\delta_1)_\infty \frac{1}{(\mu\lambda_1)_\infty} = \frac{1}{(\mu\lambda_1)_\infty}(\mu\lambda\eta\eta_1)_\infty(\lambda\delta_1)_\infty.$$

PROOF.

$$\begin{aligned}
(\lambda\delta_1)_\infty(fg) &= \sum_{n\geq 0} \frac{(-1)^n q^{\binom{n}{2}}}{(q)_n}\lambda^n \delta_1^n(fg) \\
&= \sum_{n\geq 0} \frac{(-1)^n q^{\binom{n}{2}}}{(q)_n}\lambda^n \sum_{i+j=n} \begin{bmatrix} n \\ i \end{bmatrix} (\delta_1^i f)(\eta_1^i \delta_1^j g) \\
&= \sum_{i,j\geq 0} \frac{(-1)^i(-1)^j q^{\binom{i+j}{2}}}{(q)_i (q)_j} \lambda^i\lambda^j (\delta_1^i f)(\eta_1^i\delta_1^j g) \\
&= \sum_{i,j\geq 0} \frac{(-1)^i(-1)^j q^{\binom{i}{2}} q^{\binom{j}{2}} q^{ij}}{(q)_i (q)_j} \lambda^i\lambda^j (\delta_1^i f)(\eta_1^i\delta_1^j g) \\
&= \sum_{i\geq 0} \frac{(-1)^i q^{\binom{i}{2}}}{(q)_i}\lambda^i (\delta_1^i f)\eta_1^i \sum_{j\geq 0} \frac{(-1)^j q^{\binom{j}{2}}}{(q)_j} (\lambda q^i)^j (\delta_1^j g) \\
&= \sum_{i\geq 0} \frac{(-1)^i q^{\binom{i}{2}}}{(q)_i}\lambda^i (\delta_1^i f)\eta_1^i \eta^i \sum_{j\geq 0} \frac{(-1)^j q^{\binom{j}{2}}}{(q)_j} \lambda^j (\delta_1^j g),
\end{aligned}$$

where we use the independence of g on λ in the last equality. Hence as an equation in operators on function independent of λ,

$$(\lambda\delta_1)_\infty f = \sum_{i\geq 0} \frac{(-1)^i q^{\binom{i}{2}}}{(q)_i}\lambda^i (\delta_1^i f)\eta_1^i\eta^i (\lambda\delta_1)_\infty.$$

Putting $f = 1/(\mu\delta_1)_\infty$ for which $\delta_1^i f = \mu^i f$ gives the theorem. □

To illustrate the utility of this theorem we give a corollary and use the theorem to give a second proof of the Proposition 2.2.

COROLLARY 2.25. *As equations between functions,*

$$(\lambda\delta_1)_\infty \frac{1}{(\lambda_1\lambda_2)_\infty} = \frac{(\lambda\lambda_2)_\infty}{(\lambda_1\lambda_2)_\infty}, \tag{2.46}$$

or

$$(\lambda\delta_1)_\infty \lambda_1^n = [\lambda_1, \lambda]_n = h_n(\lambda;\lambda_1). \tag{2.47}$$

PROOF.

$$\begin{aligned}
(\lambda\delta_1)_\infty \frac{1}{(\lambda_1\lambda_2)_\infty} &= (\lambda\delta_1)_\infty \frac{1}{(\lambda_1\lambda_2)_\infty}(1) = \frac{1}{(\lambda_1\lambda_2)_\infty}(\lambda\lambda_2\eta\eta_1)_\infty(\lambda\delta_1)_\infty(1) \\
&= \frac{1}{(\lambda_1\lambda_2)_\infty}(\lambda\lambda_2\eta\eta_1)_\infty(1) = \frac{(\lambda\lambda_2)_\infty}{(\lambda_1\lambda_2)_\infty}.
\end{aligned}$$

The second result follows immediately by equating coefficients of λ_2^n after expanding both sides with the q-binomial theorem. □

PROOF 2 OF PROPOSITION 2.2. By Theorem 2.24 the left hand side in the proposition is equal to

$$\frac{1}{(\lambda_1\lambda_x)_\infty}(\mu\lambda_x\eta_1\eta_\mu)_\infty(\mu\delta_1)_\infty\lambda_1^n = \frac{1}{(\lambda_1\lambda_x)_\infty}(\mu\lambda_x\lambda_1\lambda_\mu)_\infty[\lambda_1,\mu]_n$$
$$= \frac{1}{(\lambda_1\lambda_x)_\infty}\sum_{m\geq 0}\frac{(-1)^m q^{\binom{m}{2}}}{(q)_m}\mu^m\lambda_x^m q^{mn}[\lambda_1,\mu]_n = \frac{1}{(\lambda_1\lambda_x)_\infty}[\lambda_1,\mu]_n(\mu\lambda_x q^n)_\infty$$
$$= \frac{(\mu\lambda_x)_\infty}{(\lambda_1\lambda_x)_\infty}\frac{[\lambda_1,\mu]_n}{(\mu\lambda_x)_n},$$

where the first equality uses (2.5). The second assertion follows by summing the first assertion. □

It is natura to ask if Theorem 2.24 has some kind of algebraic generalization? One is tempted to consider the more general algebra $\mathbb{C}[[x,y,z,c]]$ subject to the relations

$$xy - qyx = (1-q)c$$
$$zx = qxz, \qquad zc = qcz$$
$$xc = cx, \qquad yc = cy, \qquad yz = zy.$$

The family of representations to consider is $\pi'_\mu : x \mapsto \lambda\delta_1,\ y \mapsto \mu\lambda_1,\ z \mapsto \eta,$ $c \mapsto \mu\lambda$ and the apparent "identity" is:

$$(x)_\infty e(y) = e(y)e(yxz - cz)(x)_\infty.$$

However this is *false* in $\mathbb{C}[[x,y,z,c]]$. The reason for this failure is that the operator in Theorem 2.24 is restricted to operating on functions which are independent of λ. The fact that Theorem 2.24 has some utility and interest raises the question of whether it has an algebraic context? Also does the algebra $\mathbb{C}[[x,y,z,c]]$ above have an interpretation as the quantization of some familiar algebraic object?

We conclude by mentioning that Faddeev and Kashaev [**22**] as well as Kirillov [**36**] indicate how formally the $c=0$ case of Theorem 2.23 quantizes Rogers's five-term relation for the dilogarithm. More information on the q-dilogarithm as well as q-algebra are given in Kirillov [**36**].

CHAPTER 3

Vector operator identities and simple applications

Before moving on to vector identities, we give new results closely tied to those in the last chapter and tie up a couple of loose ends. Then we look more systematically at applications of the operator theorems and derive some vector results.

3.1. A new operator theorem

We give an analogue of (1.12) motivated by the obvious question of finding a corresponding relation for the product

$$\frac{1}{(\lambda\delta_1)_\infty}(\mu\lambda_1)_\infty. \tag{3.1}$$

Taking the reciprocal of both sides of (1.12) and multiplying on the left and right by $1/(\lambda\delta_1)_\infty$ yields

$$\frac{1}{(\lambda\delta_1)_\infty}(\mu\lambda_1)_\infty = (\mu\lambda\eta_1)_\infty(\mu\lambda_1)_\infty\frac{1}{(\lambda\delta_1)_\infty}. \tag{3.2}$$

If we can now interchange the first two factors on the right side we will have an analogue of (1.12). To this end we give a more general operator theorem.

PROPOSITION 3.1. *Under suitable convergence conditions as equations in operators:*

$$\left(\sum a_n\eta^n\right)(x\lambda)_\infty = (x\lambda)_\infty\sum\frac{a_n}{(x\lambda)_n}\eta^n, \tag{3.3}$$

and

$$\left(\sum a_n\eta^n\right)\frac{1}{(x\lambda)_\infty} = \frac{1}{(x\lambda)_\infty}\sum a_n(x\lambda)_n\eta^n. \tag{3.4}$$

Notice that we do not require a_n to be independent of λ.

PROOF. Equation (3.3) follows from:

$$\begin{aligned}\left(\sum a_n\eta^n\right)(x\lambda)_\infty f(\lambda) &= \sum a_n(x\lambda q^n)_\infty f(\lambda q^n) = (x\lambda)_\infty\sum\frac{a_n}{(x\lambda)_n}f(\lambda q^n)\\ &= (x\lambda)_\infty\sum\frac{a_n}{(x\lambda)_n}\eta^n f(\lambda),\end{aligned}$$

The proof of (3.4) is similar. □

Now we apply (3.3) to (3.2) using the q-binomial theorem:

$$\begin{aligned}\frac{1}{(\lambda\delta_1)_\infty}(\mu\lambda_1)_\infty &= (\mu\lambda\eta_1)_\infty(\mu\lambda_1)_\infty\frac{1}{(\lambda\delta_1)_\infty}\\ &= (\mu\lambda_1)_\infty\sum_{n\geq 0}\frac{(-1)^n q^{\binom{n}{2}}(\mu\lambda\eta_1)^n}{(\mu\lambda_1)_n(q)_n}\frac{1}{(\lambda\delta_1)_\infty}\end{aligned}$$

Hence we have the following theorem for which convergence conditions are given by Proposition 2.21.

THEOREM 3.2.

$$\frac{1}{(\lambda\delta_1)_\infty}(\mu\lambda_1)_\infty = (\mu\lambda_1)_\infty \sum_{n\geq 0} \frac{(-1)^n q^{\binom{n}{2}}(\mu\lambda\eta_1)^n}{(\mu\lambda_1)_n (q)_n}\frac{1}{(\lambda\delta_1)_\infty}. \tag{3.5}$$

In the next section we apply (1.12) and (3.5) to derive symmetric function expansions. We will obtain several new expansions and transformations which are just the tip of the iceberg as far as results which may be generated by our method. One advantage of this technique is that it helps to expose the structure of the groups generated by iterating two-term functional relations.

We now give further results on the above operators. Some of them are contained in [**1**],[**42**],[**46**], [**47**] and [**48**].

COROLLARY 3.3. *As operators:*

$$ {}_r\varphi_s\begin{pmatrix} a_1\lambda, a_2\lambda, \ldots, a_r\lambda \\ b_1\lambda, b_2\lambda, \ldots, b_s\lambda \end{pmatrix}; z\eta\Bigg) = \frac{(a_1\lambda)_\infty \cdots (a_r\lambda)_\infty}{(b_1\lambda)_\infty \cdots (b_s\lambda)_\infty}\frac{1}{(z\eta)_\infty}\frac{(b_1\lambda)_\infty \cdots (b_s\lambda)_\infty}{(a_1\lambda)_\infty \cdots (a_r\lambda)_\infty}. \tag{3.6}$$

PROOF. This follows immediately from (3.3) and (3.4) with $a_n = z^n/(q)_n$. □

Applying (3.6) to the constant function 1 gives the following expression for the basic hypergeometric function.

COROLLARY 3.4. *As functions for* $|z| < 1$*:*

$$ {}_r\varphi_s\left(\begin{matrix} a_1\lambda, a_2\lambda, \ldots, a_r\lambda \\ b_1\lambda, b_2\lambda, \ldots, b_s\lambda \end{matrix}; z\right) = \frac{(a_1\lambda)_\infty \cdots (a_r\lambda)_\infty}{(b_1\lambda)_\infty \cdots (b_s\lambda)_\infty}\frac{1}{(z\eta)_\infty}\frac{(b_1\lambda)_\infty \cdots (b_s\lambda)_\infty}{(a_1\lambda)_\infty \cdots (a_r\lambda)_\infty}. \tag{3.7}$$

PROPOSITION 3.5.

$$\delta^n\lambda^r = \begin{cases} \lambda^{r-n}\dfrac{(q)_r}{(q)_{r-n}} & \text{if} \quad 0 \leq n \leq r \\ 0 & \text{if} \quad n > r. \end{cases}.$$

PROOF. This follows easily by induction on n. □

The next proposition gives linear operator for finding the constant terms of a power series in λ. It follows immediately from Proposition 2.3 by putting $x = \lambda$.

PROPOSITION 3.6.

$$(\lambda\delta)_\infty\lambda^r = \delta_{r,0} \quad \text{(Kronecker delta)}. \tag{3.8}$$

Iterating (2.13) and (2.14) gives the following proposition:

PROPOSITION 3.7. *If at most one* y_i *or* z_i *depends on* λ_1, *then*

$$\begin{aligned} &\frac{(y_1\delta_1)_\infty \cdots (y_m\delta_1)_\infty}{(z_1\delta_1)_\infty \cdots (z_m\delta_1)_\infty} h_n(x_1, \ldots, x_k; b\lambda_1, \lambda_2, \ldots, \lambda_l) \\ &\qquad = h_n(by_1, \ldots, by_m, x_1, \ldots, x_k; bz_1, \ldots, bz_m, b\lambda_1, \lambda_2, \ldots, \lambda_l) \end{aligned} \tag{3.9}$$

as an equation in polynomials.

The special case $k = 0$, $b = l = 1$ gives a linear operator which shifts the basis λ^n to the more general basis $h_n(y_1, \ldots, y_m; z_1, \ldots, z_m, \lambda)$.

PROPOSITION 3.8. *If at most one y_i or z_i depends on λ, then*

$$\frac{(y_1\delta)_\infty \cdots (y_m\delta)_\infty}{(z_1\delta)_\infty \cdots (z_m\delta)_\infty}\lambda^n = h_n(y_1, \dots, y_m; z_1, \dots, z_m, \lambda)$$

as an equation in polynomials.

Combining Propositions 3.6 and 3.8, we obtain the following:

PROPOSITION 3.9. *With y_i and z_i independent of λ,*

$$(\lambda\delta)_\infty \frac{(y_1\delta)_\infty \cdots (y_m\delta)_\infty}{(z_1\delta)_\infty \cdots (z_m\delta)_\infty}\lambda^n = h_n(y_1, \dots, y_m; z_1, \dots, z_m) \tag{3.10}$$

as an equation in polynomials.

Of course these last three propositions can also be easily proved directly by applying the q-multinomial theorem and Proposition 3.5 to the left side and simplifying with the definition of the polynomials h or Proposition 3.6.

3.2. Applications to q-series

In this section we apply (1.12) and (3.5) to derive symmetric expansions. The technique is to calculate the function

$$\frac{1}{(\lambda\delta_1)_\infty}\prod_{i=2}^{k}(\lambda_1\lambda_i)_\infty^{\varepsilon_i}, \tag{3.11}$$

where $\varepsilon_i = \pm 1$. Throughout this section convergence will hold when $|\lambda| < 1$ and $|\lambda_i| < 1$ if $\varepsilon_i = -1$. The manipulations are also justified by absolute convergence in this domain. We first examine the case when all ε_i are -1. When $k = 3$ (3.11) yields (1.5). The case $k = 4$ gives (1.6) and we go on to derive the result for general k. The result is a vector identity generalizing Rogers's symmetric function expansion. Finally we look at some cases when certain $\varepsilon_i = 1$; these will yield other new symmetric function expansions.

As many of the expansions we will give in this chapter give transformations of sums of q-products multiplied by symmetric polynomials h_j, for completeness we give our generalization [**15**] of Heine's transformation which relates two such sums in general.

THEOREM 3.10 (Generalized Heine Transformation). *For $m \geq 1$, $|b_i| < 1$, and $|d_i| < 1$ we have*

$$\sum_{k\geq 0}\frac{(d_1)_k \cdots (d_m)_k}{(c_1)_k \cdots (c_m)_k}\frac{h_k(\mathbf{a};\mathbf{b})}{(q)_k}$$
$$= \frac{(d_1)_\infty \cdots (d_m)_\infty (a_1)_\infty \cdots (a_m)_\infty}{(c_1)_\infty \cdots (c_m)_\infty (b_1)_\infty \cdots (b_m)_\infty}\sum_{k\geq 0}\frac{(b_1)_k \cdots (b_m)_k}{(a_1)_k \cdots (a_m)_k}\frac{h_k(\mathbf{c};\mathbf{d})}{(q)_k}.$$

We begin with a generalization of Rogers's symmetric function expansion. Put $k=5$ in the function (3.11). Then using Theorem 1.2

$$
\begin{aligned}
&\frac{1}{(\lambda\delta_1)_\infty}\frac{1}{(\lambda_1\lambda_5)_\infty(\lambda_1\lambda_4)_\infty(\lambda_1\lambda_3)_\infty(\lambda_1\lambda_2)_\infty}\\
&=\frac{1}{(\lambda_1\lambda_5)_\infty}\sum_{m\geq 0}\frac{(\lambda\lambda_5)^m\eta_1^m}{(q)_m}\\
&\qquad\frac{(\lambda\lambda_1\lambda_2\lambda_3)_\infty}{(\lambda_1\lambda_4)_\infty(\lambda\lambda_2)_\infty(\lambda\lambda_3)_\infty(\lambda_1\lambda_2)_\infty(\lambda_1\lambda_3)_\infty}\sum_{n\geq 0}\frac{(\lambda_1\lambda_2)_n(\lambda_1\lambda_3)_n}{(\lambda\lambda_1\lambda_2\lambda_3)_n(q)_n}(\lambda\lambda_4)^n\\
&=\frac{1}{(\lambda_1\lambda_5)_\infty(\lambda\lambda_2)_\infty(\lambda\lambda_3)_\infty}\\
&\qquad\times\sum_{m,n\geq 0}\frac{(\lambda\lambda_5)^m(\lambda\lambda_1\lambda_2\lambda_3q^m)_\infty(\lambda_1\lambda_2q^m)_n(\lambda_1\lambda_3q^m)_n(\lambda\lambda_4)^n}{(\lambda_1\lambda_4q^m)_\infty(\lambda_1\lambda_2q^m)_\infty(\lambda_1\lambda_3q^m)_\infty(q)_m(\lambda\lambda_1\lambda_2\lambda_3q^m)_n(q)_n}\\
&=\frac{(\lambda\lambda_1\lambda_2\lambda_3)_\infty}{(\lambda_1\lambda_5)_\infty(\lambda\lambda_2)_\infty(\lambda\lambda_3)_\infty(\lambda_1\lambda_4)_\infty(\lambda_1\lambda_2)_\infty(\lambda_1\lambda_3)_\infty}\\
&\qquad\times\sum_{m,n\geq 0}\frac{(\lambda_1\lambda_4)_m(\lambda_1\lambda_2)_{m+n}(\lambda_1\lambda_3)_{m+n}}{(\lambda\lambda_1\lambda_2\lambda_3)_{m+n}(q)_m(q)_n}\lambda_4^n\lambda_5^m\lambda^{m+n}\\
&=\frac{(\lambda\lambda_1\lambda_2\lambda_3)_\infty}{(\lambda_1\lambda_5)_\infty(\lambda\lambda_2)_\infty(\lambda\lambda_3)_\infty(\lambda_1\lambda_4)_\infty(\lambda_1\lambda_2)_\infty(\lambda_1\lambda_3)_\infty}\\
&\qquad\times\sum_{k\geq 0}\left(\sum_{m+n=k}\begin{bmatrix}k\\m\end{bmatrix}(\lambda_1\lambda_4)_m\lambda_4^n\lambda_5^m\right)\frac{(\lambda_1\lambda_2)_k(\lambda_1\lambda_3)_k}{(\lambda\lambda_1\lambda_2\lambda_3)_k(q)_k}\lambda^k\\
&=\frac{(\lambda\lambda_1\lambda_2\lambda_3)_\infty}{(\lambda_1\lambda_5)_\infty(\lambda\lambda_2)_\infty(\lambda\lambda_3)_\infty(\lambda_1\lambda_4)_\infty(\lambda_1\lambda_2)_\infty(\lambda_1\lambda_3)_\infty}\\
&\qquad\times\sum_{k\geq 0}\frac{(\lambda_1\lambda_2)_k(\lambda_1\lambda_3)_k}{(\lambda\lambda_1\lambda_2\lambda_3)_k(q)_k}h_k(\lambda_1\lambda_4\lambda_5;\lambda_4,\lambda_5)\lambda^k.
\end{aligned}
$$

Now using the q-multinomial theorem followed by Proposition 2.3 we obtain

$$
\begin{aligned}
&\frac{1}{(\lambda\delta_1)_\infty}\frac{1}{(\lambda_1\lambda_5)_\infty(\lambda_1\lambda_4)_\infty(\lambda_1\lambda_3)_\infty(\lambda_1\lambda_2)_\infty}\\
&=\sum_{n\geq 0}\frac{h_n(\lambda_2,\lambda_3,\lambda_4,\lambda_5)h_n(\lambda,\lambda_1)}{(q)_n}.
\end{aligned}
$$

Thus we have the following generalization of Roger's symmetric function expansion of Heine's ${}_2\Phi_1$.

THEOREM 3.11.

$$
\begin{aligned}
&\frac{(\lambda\lambda_1\lambda_2\lambda_3)_\infty}{(\lambda_1\lambda_5)_\infty(\lambda\lambda_2)_\infty(\lambda\lambda_3)_\infty(\lambda_1\lambda_4)_\infty(\lambda_1\lambda_2)_\infty(\lambda_1\lambda_3)_\infty}\\
&\qquad\times\sum_{k\geq 0}\frac{(\lambda_1\lambda_2)_k(\lambda_1\lambda_3)_k}{(\lambda\lambda_1\lambda_2\lambda_3)_k(q)_k}h_k(\lambda_1\lambda_4\lambda_5;\lambda_4,\lambda_5)\lambda^k\\
&=\sum_{n\geq 0}\frac{h_n(\lambda_2,\lambda_3,\lambda_4,\lambda_5)h_n(\lambda,\lambda_1)}{(q)_n}.
\end{aligned}
\tag{3.12}
$$

This is the example quoted in the introduction (1.6). A further generalization is given by the following theorem.

THEOREM 3.12. *For $m \geq 1$ let the polynomials $J_k^{(m)}$ be defined by*

$$\begin{aligned} J_k^{(m)} &= J_k^{(m)}(\lambda_1; \lambda_4, \lambda_5, \lambda_6, \dots, \lambda_{m+3}) \\ &= \sum_{n_1+\cdots+n_m=k} \begin{bmatrix} k \\ n_1, \dots, n_m \end{bmatrix} (\lambda_1\lambda_5)_{n_1} (\lambda_1\lambda_6)_{n_1+n_2} \\ &\qquad \cdots (\lambda_1\lambda_{m+3})_{n_1+\cdots+n_{m-1}} \lambda_4^{n_1} \cdots \lambda_{m+3}^{n_m}. \end{aligned}$$

Then

(3.13)

$$\begin{aligned} \frac{(\lambda\lambda_1\lambda_2\lambda_3)_\infty}{(\lambda\lambda_2)_\infty(\lambda\lambda_3)_\infty} \prod_{i=2}^{m+3} \frac{1}{(\lambda_1\lambda_i)_\infty} \sum_{k\geq 0} \frac{(\lambda_1\lambda_2)_k(\lambda_1\lambda_3)_k}{(\lambda\lambda_1\lambda_2\lambda_3)_k(q)_k} J_k^{(m)}(\lambda_1; \lambda_4, \lambda_5, \dots, \lambda_{m+3})\lambda^k \\ = \sum_{k\geq 0} \frac{h_k(\lambda, \lambda_1) h_k(\lambda_2, \lambda_3, \dots, \lambda_{m+3})}{(q)_k}. \end{aligned}$$

PROOF. This is merely an iteration of the technique used to prove the previous theorems. The function on the right side of (3.13) is just

$$\frac{1}{(\lambda\delta_1)_\infty} \prod_{i=2}^{m+3} \frac{1}{(\lambda_1\lambda_i)_\infty}.$$

Now apply (1.12) inductively and use the symmetry of the right hand side in λ_4, $\dots$, λ_{m+3} to reverse the order of these variables in the sum. □

This theorem shows that the function on the left side of (3.13) has a symmetry group $S_2 \times S_{m+2}$. In general, because of the nature of the polynomials J, it does not appear to be a special case of our generalized Heine transformation Theorem 3.10.

We now turn our attention to cases of

$$\frac{1}{(\lambda\delta_1)_\infty} \prod_{i=2}^{k} (\lambda_1\lambda_i)_\infty^{\varepsilon_i}$$

with some $\varepsilon_i = 1$. We first study the simplest case $k = 1$. As a function, we have

$$\begin{aligned} \frac{1}{(\lambda\delta_1)_\infty}(\lambda_1\lambda_2)_\infty &= \frac{1}{(\lambda\delta_1)_\infty}(\lambda_1\lambda_2)_\infty(1) \\ &= (\lambda_1\lambda_2)_\infty \sum_{n\geq 0} \frac{(-1)^n q^{\binom{n}{2}} (\lambda\lambda_2\eta_1)^n}{(\lambda_1\lambda_2)_n(q)_n} \frac{1}{(\lambda\delta_1)_\infty}(1) \\ &= (\lambda_1\lambda_2)_\infty \sum_{n\geq 0} \frac{(-1)^n q^{\binom{n}{2}} (\lambda\lambda_2)^n}{(\lambda_1\lambda_2)_n(q)_n}. \end{aligned}$$

On the other hand from the q-multinomial theorem and Proposition 2.3:

$$\frac{1}{(\lambda\delta_1)_\infty}(\lambda_1\lambda_2)_\infty = \sum_{n\geq 0} \frac{q^{\binom{n}{2}}(-\lambda_2)^n}{(q)_n} h_n(\lambda, \lambda_1).$$

Thus

$$(\lambda_1\lambda_2)_\infty \sum_{n\geq 0} \frac{(-1)^n q^{\binom{n}{2}} (\lambda\lambda_2)^n}{(\lambda_1\lambda_2)_n (q)_n} = \sum_{n\geq 0} \frac{q^{\binom{n}{2}}(-\lambda_2)^n}{(q)_n} h_n(\lambda, \lambda_1). \tag{3.14}$$

The transformation of the left side implied by this symmetric expansion is equation (20.24) of Fine [**23**]. Of course the transformation also follows from the Heine transformation.

We continue by taking $k = 2$ and $\varepsilon_2 = \varepsilon_3 = 1$.

$$\begin{aligned}
\frac{1}{(\lambda\delta_1)_\infty}(\lambda_1\lambda_3)_\infty(\lambda_1\lambda_2)_\infty &= (\lambda_1\lambda_3)_\infty \sum_{n\geq 0} \frac{(-1)^n q^{\binom{n}{2}} (\lambda\lambda_3\eta_1)^n}{(\lambda_1\lambda_3)_n (q)_n} \frac{1}{(\lambda\delta_1)_\infty}(\lambda_1\lambda_2)_\infty \\
&= (\lambda_1\lambda_3)_\infty \sum_{n\geq 0} \frac{(-1)^n q^{\binom{n}{2}} (\lambda\lambda_3\eta_1)^n}{(\lambda_1\lambda_3)_n (q)_n} (\lambda_1\lambda_2)_\infty \sum_{m\geq 0} \frac{(-1)^m q^{\binom{m}{2}} (\lambda\lambda_2)^m}{(\lambda_1\lambda_2)_m (q)_m} \\
&= (\lambda_1\lambda_3)_\infty \sum_{n\geq 0} \frac{(-1)^n q^{\binom{n}{2}} (\lambda\lambda_3)^n}{(\lambda_1\lambda_3)_n (q)_n} (\lambda_1\lambda_2 q^n)_\infty \sum_{m\geq 0} \frac{(-1)^m q^{\binom{m}{2}} (\lambda\lambda_2)^m}{(\lambda_1\lambda_2 q^n)_m (q)_m} \\
&= (\lambda_1\lambda_2)_\infty (\lambda_1\lambda_3)_\infty \sum_{m,n\geq 0} \frac{(-1)^{m+n} q^{\binom{m}{2}+\binom{n}{2}} \lambda_2^m \lambda_3^n}{(q)_m (q)_n (\lambda_1\lambda_2)_{m+n} (\lambda_1\lambda_3)_n} \lambda^{m+n} \\
&= (\lambda_1\lambda_2)_\infty (\lambda_1\lambda_3)_\infty \sum_{k\geq 0} \left(\sum_{m+n=k} \begin{bmatrix} k \\ m \end{bmatrix} \frac{q^{\binom{m}{2}+\binom{n}{2}} \lambda_2^m \lambda_3^n}{(\lambda_1\lambda_3)_n} \right) \frac{(-\lambda)^k}{(\lambda_1\lambda_2)_k (q)_k}.
\end{aligned}$$

Hence we have the theorem:

THEOREM 3.13.

$$\begin{aligned}
(\lambda_1\lambda_2)_\infty (\lambda_1\lambda_3)_\infty \sum_{k\geq 0} &\left(\sum_{m+n=k} \begin{bmatrix} k \\ m \end{bmatrix} \frac{q^{\binom{m}{2}+\binom{n}{2}} \lambda_2^m \lambda_3^n}{(\lambda_1\lambda_3)_n} \right) \frac{(-\lambda)^k}{(\lambda_1\lambda_2)_k (q)_k} \\
&= \sum_{k\geq 0} \frac{h_k(\lambda_2, \lambda_3; 0) h_k(\lambda, \lambda_1)}{(q)_k}.
\end{aligned}$$

By putting $a = \lambda_1\lambda_2$, $b = \lambda_1\lambda_3$ and $c = \lambda/\lambda_1$, this theorem implies that the function

$$f(a,b,c) = (a)_\infty (b)_\infty \sum_{k\geq 0} \left(\sum_{m+n=k} \begin{bmatrix} k \\ m \end{bmatrix} \frac{q^{\binom{m}{2}+\binom{n}{2}} a^m b^n}{(b)_n} \right) \frac{(-c)^k}{(a)_k (q)_k}$$

satisfies the transformations

$$f(a,b,c) = \begin{cases} f(ac, bc, c^{-1}) \\ f(b,a,c) \end{cases}.$$

It is clear that these transformations generate a group isomorphic to S_2^2.

Of course this procedure can be iterated and the general theorem in which all $\varepsilon_i = 1$ is the following.

THEOREM 3.14. *For $m \geq 1$ let the rational functions N_k be defined by*

$$\begin{aligned} N_k &= N_k(\lambda_1; \lambda_2, \ldots, \lambda_{m+1}) \\ &= \sum_{n_1+\cdots+n_m=k} \begin{bmatrix} k \\ n_1, \ldots, n_m \end{bmatrix} \frac{q^{\binom{n_1}{2}+\cdots+\binom{n_m}{2}} \lambda_2^{n_1} \cdots \lambda_{m+1}^{n_m}}{(\lambda_1\lambda_2)_{n_1}(\lambda_1\lambda_3)_{n_1+n_2} \cdots (\lambda_1\lambda_m)_{n_1+\cdots+n_{m-1}}}. \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{(\lambda\delta_1)_\infty} \prod_{i=2}^{m+1} (\lambda_1\lambda_i)_\infty &= \prod_{i=2}^{m+1} (\lambda_1\lambda_i)_\infty \sum_{k\geq 0} \frac{(-\lambda)^k N_k}{(q)_k(\lambda_1\lambda_{m+1})_k} \\ &= \sum_{k\geq 0} \frac{h_k(\lambda, \lambda_1) h_k(\lambda_2, \lambda_3, \ldots, \lambda_{m+1}; 0)}{(q)_k}. \end{aligned} \tag{3.15}$$

Notice that here the induced group of transformations is $S_2 \times S_m$.

We now consider the case $k = 4$, $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = -1$, and $\varepsilon_5 = 1$. Using (3.5) and (1.17) gives

$$\begin{aligned} &\frac{1}{(\lambda\delta_1)_\infty} \frac{(\lambda_1\lambda_5)_\infty}{(\lambda_1\lambda_2)_\infty(\lambda_1\lambda_3)_\infty(\lambda_1\lambda_4)_\infty} \\ &= (\lambda_1\lambda_5)_\infty \sum_{n\geq 0} \frac{(-1)^n q^{\binom{n}{2}} (\lambda\lambda_5)^n}{(\lambda_1\lambda_5)_n(q)_n} \eta_1^n \frac{1}{(\lambda\delta_1)_\infty} \frac{1}{(\lambda_1\lambda_2)_\infty(\lambda_1\lambda_3)_\infty(\lambda_1\lambda_4)_\infty} \\ &= \frac{(\lambda_1\lambda_5)_\infty}{(\lambda\lambda_2)_\infty(\lambda\lambda_3)_\infty} \sum_{n\geq 0} \frac{(-1)^n q^{\binom{n}{2}} (\lambda\lambda_5)^n}{(\lambda_1\lambda_5)_n(q)_n} \eta_1^n \\ &\qquad \frac{1}{(\lambda_1\lambda_4)_\infty} \sum_{m\geq 0} \frac{(\lambda\lambda_1\lambda_2\lambda_3 q^m)_\infty (\lambda\lambda_4)^m}{(q)_m(\lambda_1\lambda_2 q^m)_\infty(\lambda_1\lambda_3 q^m)_\infty} \\ &= \frac{(\lambda_1\lambda_5)_\infty}{(\lambda\lambda_2)_\infty(\lambda\lambda_3)_\infty} \sum_{m,n\geq 0} \frac{(-1)^n q^{\binom{n}{2}} (\lambda\lambda_5)^n (\lambda\lambda_1\lambda_2\lambda_3 q^{m+n})_\infty (\lambda\lambda_4)^m}{(\lambda_1\lambda_5)_n(q)_n(\lambda_1\lambda_4 q^n)_\infty(q)_m(\lambda_1\lambda_2 q^{m+n})_\infty(\lambda_1\lambda_3 q^{m+n})_\infty} \\ &= \frac{(\lambda_1\lambda_5)_\infty(\lambda\lambda_1\lambda_2\lambda_3)_\infty}{(\lambda\lambda_2)_\infty(\lambda\lambda_3)_\infty(\lambda_1\lambda_2)_\infty(\lambda_1\lambda_3)_\infty(\lambda_1\lambda_4)_\infty} \\ &\qquad \times \sum_{k\geq 0} \left[\sum_{m+n=k} \begin{bmatrix} k \\ m \end{bmatrix} \frac{(-1)^n q^{\binom{n}{2}} (\lambda_1\lambda_4)_n \lambda_4^m \lambda_5^n}{(\lambda_1\lambda_5)_n} \right] \frac{(\lambda_1\lambda_2)_k(\lambda_1\lambda_3)_k}{(\lambda\lambda_1\lambda_2\lambda_3)_k(q)_k} \lambda^k. \end{aligned}$$

We now obtain another expansion for this function. First apply (1.12), then the q-multinomial theorem and finally Proposition 2.4.

$$\begin{aligned} &\frac{1}{(\lambda\delta_1)_\infty} \frac{(\lambda_1\lambda_5)_\infty}{(\lambda_1\lambda_2)_\infty(\lambda_1\lambda_3)_\infty(\lambda_1\lambda_4)_\infty} \\ &= \frac{1}{(\lambda_1\lambda_4)_\infty} \frac{1}{(\lambda\lambda_4\eta_1)_\infty} \frac{1}{(\lambda\delta_1)_\infty} \frac{(\lambda_1\lambda_5)_\infty}{(\lambda_1\lambda_2)_\infty(\lambda_1\lambda_3)_\infty} \\ &= \frac{1}{(\lambda_1\lambda_4)_\infty} \frac{1}{(\lambda\lambda_4\eta_1)_\infty} \frac{1}{(\lambda\delta_1)_\infty} \sum_{n\geq 0} \frac{h_n(\lambda_5; \lambda_2, \lambda_3)}{(q)_n} \lambda_1^n \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(\lambda_1\lambda_4)_\infty}\frac{1}{(\lambda\lambda_4\eta_1)_\infty}\sum_{n\geq 0}\frac{h_n(\lambda_5;\lambda_2,\lambda_3)h_n(\lambda,\lambda_1)}{(q)_n}\\
&= \frac{1}{(\lambda_1\lambda_4)_\infty}\sum_{n\geq 0}\frac{h_n(\lambda_5;\lambda_2,\lambda_3)h_n(\lambda\lambda_1\lambda_4;\lambda,\lambda_1)}{(q)_n}.
\end{aligned} \tag{3.16}$$

Hence we have the following theorem.

THEOREM 3.15. *Let the rational functions $L_k(c,d)$ be defined by*

$$L_k(c,d) = \sum_{m+n=k}\begin{bmatrix}k\\m\end{bmatrix}\frac{(-1)^n q^{\binom{n}{2}}(c)_n c^m d^n}{(d)_n}.$$

Then

$$\begin{aligned}
&\frac{(\lambda_1\lambda_5)_\infty(\lambda\lambda_1\lambda_2\lambda_3)_\infty}{(\lambda\lambda_2)_\infty(\lambda\lambda_3)_\infty(\lambda_1\lambda_2)_\infty(\lambda_1\lambda_3)_\infty(\lambda_1\lambda_4)_\infty}\\
&\qquad\times\sum_{k\geq 0}\frac{(\lambda_1\lambda_2)_k(\lambda_1\lambda_3)_k}{(\lambda\lambda_1\lambda_2\lambda_3)_k(q)_k}L_k(\lambda_1\lambda_4,\lambda_1\lambda_5)\left(\frac{\lambda}{\lambda_1}\right)^k\\
&= \sum_{n\geq 0}\frac{h_n(\lambda,\lambda_1)h_n(\lambda_5;\lambda_2,\lambda_3,\lambda_4)}{(q)_n}\\
&= \frac{1}{(\lambda_1\lambda_4)_\infty}\sum_{n\geq 0}\frac{h_n(\lambda_5;\lambda_2,\lambda_3)h_n(\lambda\lambda_1\lambda_4;\lambda,\lambda_1)}{(q)_n}.
\end{aligned} \tag{3.17}$$

Again it is obvious that when $\lambda_5 = 0$ this theorem reduces to (1.5). An immediate consequence is:

COROLLARY 3.16. *The function*

$$f(a,b,c,d,e) = \frac{(d)_\infty(abe)_\infty}{(a)_\infty(b)_\infty(c)_\infty(ae)_\infty(be)_\infty}\sum_{k\geq 0}\frac{(a)_k(b)_k}{(abe)_k(q)_k}L_k(c,d)e^k$$

satisfies the following relations:

$$f(a,b,c,d,e) = \begin{cases} f(b,a,c,d,e)\\ f(c,b,a,d,e)\\ f(ae,be,ce,de,e^{-1})\end{cases}. \tag{3.18}$$

These relations generate a transformation group isomorphic to $S_2\times S_3$.

An interesting special case of Theorem 3.15 is obtained by putting $\lambda_4 = 0$. In this case it is convenient to relabel λ_5 as λ_4. Then the theorem becomes

COROLLARY 3.17.

$$\begin{aligned}
&\frac{1}{(\lambda\delta_1)_\infty}\frac{(\lambda_1\lambda_4)_\infty}{(\lambda_1\lambda_2)_\infty(\lambda_1\lambda_3)_\infty}\\
&= \frac{(\lambda_1\lambda_4)_\infty(\lambda\lambda_1\lambda_2\lambda_3)_\infty}{(\lambda\lambda_2)_\infty(\lambda\lambda_3)_\infty(\lambda_1\lambda_2)_\infty(\lambda_1\lambda_3)_\infty}\,{}_2\Phi_2\left(\begin{matrix}\lambda_1\lambda_2,\ \lambda_1\lambda_3\\ \lambda_1\lambda_4,\ \lambda\lambda_1\lambda_2\lambda_3\end{matrix};\lambda\lambda_4\right)\\
&= \sum_{n\geq 0}\frac{h_n(\lambda,\lambda_1)h_n(\lambda_4;\lambda_2,\lambda_3)}{(q)_n}.
\end{aligned} \tag{3.19}$$

Changing variables we find that the function

$$g(a,b,c,d) = \frac{(c)_\infty (d)_\infty}{(a)_\infty (d/a)_\infty (b)_\infty (d/b)_\infty} {}_2\Phi_2\left(\begin{matrix} a,\ b \\ c,\ d \end{matrix}; \frac{cd}{ab}\right) \tag{3.20}$$

satisfies the following relations:

$$g(a,b,c,d) = \begin{cases} g(b,a,c,d) \\ g(d/b, d/a, cd/ab, d). \end{cases} \tag{3.21}$$

The ${}_2\Phi_2$ function in (3.20) is also symmetric in c and d and this symmetry is not generated by the symmetric expansion. This is because g is not invariant under this symmetry. For g the correct relation is:

$$g(a,b,c,d) = \frac{(c/a)_\infty (c/b)_\infty}{(d/a)_\infty (d/b)_\infty} g(a,b,d,c).$$

This relation together with (3.21) generates a group of order 12 isomorphic to $S_2 \times S_3$. This of course leads to the problem of finding a symmetric expansion for the ${}_2\Phi_2$ which gives the full transformation group. We solve this problem at once.

Consider the function

$$\begin{aligned} &\frac{1}{(\lambda_4\delta_1)_\infty} \frac{(\lambda\lambda_1\lambda_2\lambda_3)_\infty}{(\lambda\lambda_2)_\infty(\lambda\lambda_3)_\infty(\lambda_1\lambda_2)_\infty(\lambda_1\lambda_3)_\infty} \\ &= \frac{1}{(\lambda\lambda_2)_\infty(\lambda\lambda_3)_\infty} \frac{1}{(\lambda_4\delta_1)_\infty} \frac{(\lambda\lambda_1\lambda_2\lambda_3)_\infty}{(\lambda_1\lambda_2)_\infty(\lambda_1\lambda_3)_\infty}. \end{aligned}$$

By (3.19) this is equal to

$$\frac{(\lambda\lambda_1\lambda_2\lambda_3)_\infty(\lambda_1\lambda_2\lambda_3\lambda_4)_\infty}{(\lambda\lambda_2)_\infty(\lambda\lambda_3)_\infty(\lambda_1\lambda_2)_\infty(\lambda_1\lambda_3)_\infty(\lambda_2\lambda_4)_\infty(\lambda_3\lambda_4)_\infty} \times {}_2\Phi_2\left(\begin{matrix} \lambda_1\lambda_2,\ \lambda_1\lambda_3 \\ \lambda\lambda_1\lambda_2\lambda_3,\ \lambda_1\lambda_2\lambda_3\lambda_4 \end{matrix}; \lambda\lambda_2\lambda_3\lambda_4\right).$$

On the other hand, from (1.16) and (2.13),

$$\begin{aligned} \frac{1}{(\lambda_4\delta_1)_\infty} \frac{(\lambda\lambda_1\lambda_2\lambda_3)_\infty}{(\lambda\lambda_2)_\infty(\lambda\lambda_3)_\infty(\lambda_1\lambda_2)_\infty(\lambda_1\lambda_3)_\infty} &= \frac{1}{(\lambda_4\delta_1)_\infty} \sum_{n\ge 0} \frac{h_n(\lambda_2,\lambda_3)h_n(\lambda,\lambda_1)}{(q)_n} \\ &= \sum_{n\ge 0} \frac{h_n(\lambda_2,\lambda_3)h_n(\lambda,\lambda_1,\lambda_4)}{(q)_n}. \end{aligned}$$

Combining these two results gives

$$\begin{aligned} &\frac{(\lambda\lambda_1\lambda_2\lambda_3)_\infty(\lambda_1\lambda_2\lambda_3\lambda_4)_\infty}{(\lambda\lambda_2)_\infty(\lambda\lambda_3)_\infty(\lambda_1\lambda_2)_\infty(\lambda_1\lambda_3)_\infty(\lambda_2\lambda_4)_\infty(\lambda_3\lambda_4)_\infty} \\ &\qquad \times {}_2\Phi_2\left(\begin{matrix} \lambda_1\lambda_2,\ \lambda_1\lambda_3 \\ \lambda\lambda_1\lambda_2\lambda_3,\ \lambda_1\lambda_2\lambda_3\lambda_4 \end{matrix}; \lambda\lambda_2\lambda_3\lambda_4\right) \\ &= \sum_{n\ge 0} \frac{h_n(\lambda_2,\lambda_3)h_n(\lambda,\lambda_1,\lambda_4)}{(q)_n}. \end{aligned}$$

This symmetric expansion exhibits the full symmetry group of the ${}_2\Phi_2$. Notice also that the symmetric expansion is the same as that for the Heine transformation.

Equating these two expansions thus relates the general ${}_2\Phi_1$ to a specialized ${}_2\Phi_2$. We have thus found an underlying explanation of Jackson's identity [**35**],

$$ {}_2\Phi_1\left(\begin{matrix} a,\, b \\ c \end{matrix}\,;z\right) = \frac{(az)_\infty}{(z)_\infty}\,{}_2\Phi_2\left(\begin{matrix} a,\, c/b \\ c,\, az \end{matrix}\,;bz\right). $$

We draw attention to one last special case of Theorem 3.15. Putting $\lambda_3 = 0$ (and then relabeling λ_4 as λ_3) in Corollary 3.17 gives:

COROLLARY 3.18.

$$ \frac{(\lambda_1\lambda_3)_\infty}{(\lambda\lambda_2)_\infty(\lambda_1\lambda_2)_\infty}\,{}_1\Phi_1\left(\begin{matrix} \lambda_1\lambda_2 \\ \lambda_1\lambda_3 \end{matrix}\,;\lambda\lambda_3\right) = \sum_{n\geq 0}\frac{[\lambda_2,\lambda_3]_n h_n(\lambda,\lambda_1)}{(q)_n} $$

Finally, putting

$$ f(a,b,c) = \frac{(b)_\infty}{(a)_\infty(ac/b)_\infty}\,{}_1\Phi_1\left(\begin{matrix} a \\ b \end{matrix}\,;c\right), $$

we see that f is invariant under the involution $f(a,b,c) = f(ac/b,c,b)$.

It is easy to see that we have explored only the first few cases of (3.11). A more systematic study of expansions of this function is left for another occasion.

3.3. A vector q-Leibniz rule

One goal of this section is to construct generalizations of the operator δ for which extensions of the q-Leibniz rule (1.13) hold. This is a key step in finding analogs of (1.12) and (3.5). We also present some new operator theorems and consider simple applications.

Recall the standard notation for vector variables. For $\mathbf{k},\mathbf{l},\mathbf{m} \in \mathbb{Q}^n$,

$$ \mathbf{k}\cdot\mathbf{l} = k_1 l_1 + \cdots + k_n l_n, $$

$$ \lambda^{\mathbf{m}} = \lambda_1^{m_1}\cdots\lambda_n^{m_n}, $$

$$ (\eta^{\mathbf{k}} f(\lambda)) = f(\lambda_1 q^{k_1},\ldots,\lambda_n q^{k_n}). $$

Our extension of (1.13) is:

PROPOSITION 3.19. *Fix* $\mathbf{k},\mathbf{l},\mathbf{m} \in \mathbb{Q}^n$. *Suppose*

$$ \sigma = \frac{\eta^{\mathbf{k}} - \eta^{\mathbf{l}}}{\lambda^{\mathbf{m}}}. $$

Then

$$ \sigma^n(fg) = \sum_{j=o}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_{q^{(\mathbf{k}-\mathbf{l})\cdot\mathbf{m}}} q^{\mathbf{m}\cdot\mathbf{l}j(n-j)}(\eta^{\mathbf{l}(n-j)}\sigma^j f)(\eta^{j\mathbf{k}}\sigma^{n-j} g). \tag{3.22} $$

This proposition will follow quickly from the following three lemmas.

LEMMA 3.20. *Let* τ *and* ε *be ring homomorphisms of a ring* R *into a subring of* R. *Define* $\sigma = h(\tau - \varepsilon)$, *where* $hx = xh$ *for all* $x \in \varepsilon(R)$. *(Here* $h \in R$, $\varepsilon(R)$ *is the image of* ε, *and concatenation denotes the ring multiplication.) Then*

$$ \sigma(fg) = (\sigma f)(\tau g) + (\varepsilon f)(\sigma g). $$

PROOF. The right hand side is equal to

$$(\sigma f)(\tau g)+(\varepsilon f)(\sigma g)=h(\tau f-\varepsilon f)(\tau g)+(\varepsilon f)h(\tau g-\varepsilon g)$$

$$=h(\tau f)(\tau g)-h(\varepsilon f)(\tau g)+(\varepsilon f)h(\tau g)-(\varepsilon f)h(\varepsilon g)=h((\tau f)(\tau g)-(\varepsilon f)(\varepsilon g))$$

$$=h(\tau(fg)-\varepsilon(fg))=\sigma(fg).$$

□

LEMMA 3.21. *Let σ, τ and ε be operators on a ring R which satisfy*

$$x\in\mathbb{Z}[q_1,q_2]\subset R \implies \sigma(xf)=x(\sigma f) \qquad f\in R,$$

$\sigma(f+g)=\sigma f+\sigma g$, $\sigma\tau=q_1\tau\sigma$, $\sigma\varepsilon=q_2\varepsilon\sigma$, *and*

$$\sigma(fg)=(\sigma f)(\tau g)+(\varepsilon f)(\sigma g). \tag{3.23}$$

Then

$$\sigma^n(fg)=\sum_{j=0}^{n}\begin{bmatrix}n\\ j\end{bmatrix}_{1,2}(\varepsilon^{n-j}\sigma^j f)(\tau^j\sigma^{n-j}g), \tag{3.24}$$

where the brackets $\left[\begin{smallmatrix}n\\ j\end{smallmatrix}\right]_{1,2}$ *are defined by*

$$\begin{bmatrix}n\\ j\end{bmatrix}_{1,2}=q_1^j\begin{bmatrix}n-1\\ j\end{bmatrix}_{1,2}+q_2^{n-j}\begin{bmatrix}n-1\\ j-1\end{bmatrix}_{1,2}, \tag{3.25}$$

with

$$\begin{bmatrix}n\\ 0\end{bmatrix}_{1,2}=\begin{bmatrix}n\\ n\end{bmatrix}_{1,2}=1.$$

PROOF. When $n=1$, (3.24) is just (3.23) in view of the initial conditions defining the brackets. Assume the result holds for $n-1$. Then

$$\sigma^{n-1}(fg)=\sum_{j=0}^{n-1}\begin{bmatrix}n-1\\ j\end{bmatrix}_{1,2}(\varepsilon^{n-j-1}\sigma^j f)(\tau^j\sigma^{n-j-1}g).$$

Applying σ to both sides yields

$$\begin{aligned}\sigma^n(fg)&=\sum_{j=0}^{n-1}\begin{bmatrix}n-1\\ j\end{bmatrix}_{1,2}\sigma[(\varepsilon^{n-j-1}\sigma^j f)(\tau^j\sigma^{n-j-1}g)]\\ &=\sum_{j=0}^{n-1}\begin{bmatrix}n-1\\ j\end{bmatrix}_{1,2}[(\sigma\varepsilon^{n-j-1}\sigma^j f)(\tau\tau^j\sigma^{n-j-1}g)+(\varepsilon\varepsilon^{n-j-1}\sigma^j f)(\sigma\tau^j\sigma^{n-j-1}g)]\\ &=\sum_{j=1}^{n}\begin{bmatrix}n-1\\ j-1\end{bmatrix}_{1,2}q_2^{n-j}(\varepsilon^{n-j}\sigma^j f)(\tau^j\sigma^{n-j}g)]\\ &\qquad+\sum_{j=0}^{n-1}\begin{bmatrix}n-1\\ j\end{bmatrix}_{1,2}q_1^j(\varepsilon^{n-j}\sigma^j f)(\tau^j\sigma^{n-j}g)\end{aligned}$$

$$
\begin{aligned}
&= \begin{bmatrix} n-1 \\ n-1 \end{bmatrix}_{1,2} (\varepsilon^0 \sigma^n f)(\tau^n \sigma^0 g) \\
&\qquad + \sum_{j=1}^{n-1} \left(\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_{1,2} q_2^{n-j} + \begin{bmatrix} n-1 \\ j \end{bmatrix}_{1,2} q_1^j \right) (\varepsilon^{n-j} \sigma^j f)(\tau^j \sigma^{n-j} g) \\
&\qquad + \begin{bmatrix} n-1 \\ 0 \end{bmatrix}_{1,2} q_1^0 (\varepsilon^n \sigma^0 f)(\tau^0 \sigma^n g) \\
&= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{1,2} (\varepsilon^{n-j} \sigma^j f)(\tau^j \sigma^{n-j} g).
\end{aligned}
$$

The following lemma shows that $\left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right]_{1,2}$ can be expressed in terms of q-binomial coefficients.

Lemma 3.22. *Put $q = q_1/q_2$. Then*

$$
\begin{bmatrix} n \\ j \end{bmatrix}_{1,2} = \begin{bmatrix} n \\ j \end{bmatrix} q_2^{j(n-j)}.
$$

Proof. This follows immediately from (3.25) and the fact that ordinary q-binomial coefficients can be defined inductively by the formulas

$$
\begin{bmatrix} n \\ j \end{bmatrix} = \begin{bmatrix} n-1 \\ j \end{bmatrix} + q^{n-j} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}, \tag{3.26}
$$

with

$$
\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1. \qquad \square
$$

Proof of Proposition 3.19. In Lemma 3.20 put $h = \lambda^{-\mathbf{m}}$, $\tau = \eta^{\mathbf{k}}$, $\varepsilon = \eta^{\mathbf{l}}$, and thus $\sigma = (\tau - \varepsilon)/\lambda^{\mathbf{m}}$. (We are assuming R is a commutative ring of functions of $\lambda_1, \ldots, \lambda_n$.) Then the conditions of Lemma 3.20 are fulfilled so

$$
\sigma(fg) = (\sigma f)(\tau g) + (\varepsilon f)(\sigma g).
$$

It is now easy to check that $\sigma\tau = q^{\mathbf{k}\cdot\mathbf{m}} \tau\sigma$ and $\sigma\varepsilon = q^{\mathbf{l}\cdot\mathbf{m}} \varepsilon\sigma$. Then σ, τ and ε satisfy the condition of Lemma 3.21 with $q_1 = q^{\mathbf{k}\cdot\mathbf{m}}$ and $q_2 = q^{\mathbf{l}\cdot\mathbf{m}}$. Then the last two lemmas gives the proposition. $\square$

Equation (1.13) follows from (3.22) by putting $k = 0$, $l = m = 1$. Putting $k = 0$, $l = -1$ and $m = 1$ gives

$$
\sigma^n(fg) = \sum_{i+j=n} \begin{bmatrix} n \\ i \end{bmatrix} q^{-ij} (-1)^n (\sigma^i f)(\eta^{-i} \sigma^j g).
$$

Using this equation a proof of Theorem 1.18 may now be given similar to our proof of Theorem 1.2. We take up the application of other similar operator theorems on another occasion.

3.4. Further operator theorems and simple applications

We next obtain a dual of our result (3.5) which can be viewed as a companion to (1.18). For this we need the following dual of Proposition 3.1.

PROPOSITION 3.23. *As operators under suitable convergence conditions:*

$$\left(\sum a_n \eta^{-n}\right) \frac{1}{E(x\lambda)} = \frac{1}{E(x\lambda)} \left(\sum a_n \frac{(-1)^n q^{\binom{n}{2}}}{[-x\lambda]_n} \eta^{-n}\right) \tag{3.27}$$

$$\left(\sum a_n \eta^{-n}\right) E(x\lambda) = E(x\lambda) \left(\sum a_n \frac{(-1)^n [-x\lambda]_n}{q^{\binom{n}{2}}} \eta^{-n}\right). \tag{3.28}$$

PROOF. The proof is similar to that of Proposition 3.1. □

Putting $a_n = q^{\binom{n}{2}}(-1)^n z^n/(q)_n$ and applying (3.27) and (3.28) iteratively gives the following result dual to Corollary 3.3:

COROLLARY 3.24. *As operators:*

$$\begin{aligned} &{}_r\Phi_s \left(\begin{matrix} a_1^{-1}\lambda^{-1}, \dots, a_r^{-1}\lambda^{-1} \\ b_1^{-1}\lambda^{-1}, \dots, b_s^{-1}\lambda^{-1} \end{matrix} ; \frac{a_1 \cdots a_r}{b_1 \cdots b_s} \lambda^{r-s} z \eta^{-1} \right) \\ &\qquad = \frac{(b_1\lambda q)_\infty \cdots (b_s\lambda q)_\infty}{(a_1\lambda q)_\infty \cdots (a_r\lambda q)_\infty} (z\eta^{-1})_\infty \frac{(a_1\lambda q)_\infty \cdots (a_r\lambda q)_\infty}{(b_1\lambda q)_\infty \cdots (b_s\lambda q)_\infty}. \end{aligned} \tag{3.29}$$

Applying (3.29) to the constant function 1, we obtain an analogue of Corollary (3.4):

COROLLARY 3.25. *As functions:*

$$\begin{aligned} &{}_r\Phi_s \left(\begin{matrix} a_1\lambda, \dots, a_r\lambda \\ b_1\lambda, \dots, b_s\lambda \end{matrix} ; z \right) \\ &= \frac{(b_1^{-1}\lambda^{-1}q)_\infty \cdots (b_s^{-1}\lambda^{-1}q)_\infty}{(a_1^{-1}\lambda^{-1}q)_\infty \cdots (a_r^{-1}\lambda^{-1}q)_\infty} \\ &\qquad \times \left(\frac{a_1 \cdots a_r}{b_1 \cdots b_s} \lambda^{r-s} z \eta^{-1} \right)_\infty \frac{(a_1^{-1}\lambda^{-1}q)_\infty \cdots (a_r^{-1}\lambda^{-1}q)_\infty}{(b_1^{-1}\lambda^{-1}q)_\infty \cdots (b_s^{-1}\lambda^{-1}q)_\infty}. \end{aligned} \tag{3.30}$$

Of course when $r = s + 1$ (3.7) and (3.30) have the same left side, so we have obtained an identity for their right sides.

We now give another example showing how the convergence conditions may not be ignored in working with these operators. From (1.18) it easily follows that as operators

$$E(\lambda\sigma_1) \frac{1}{E(\mu\lambda_1)} = \frac{1}{(-\mu\lambda\eta_1^{-1}q)_\infty} \frac{1}{E(\mu\lambda_1)} E(\lambda\sigma_1).$$

Applying (3.27) to the first two factors on the right side gives the following dual to (3.5):

THEOREM 3.26. *Under suitable convergence conditions:*

$$E(\lambda\sigma_1) \frac{1}{E(\mu\lambda_1)} = \frac{1}{E(\mu\lambda_1)} \sum_{n \geq 0} \frac{q^{\binom{n+1}{2}} (-\lambda/\lambda_1)^n}{(q)_n (-\mu^{-1}\lambda_1^{-1})_n} \eta_1^{-n} E(\lambda\sigma_1). \tag{3.31}$$

Now the phrase "suitable convergence conditions" is quite important. Indeed the poles at $\lambda_1 = \mu^{-1}q^{-n}$ must be cut out by the function being hit by this operator, otherwise there is no hope of obtaining convergence by Proposition 2.22. Thus for example one cannot apply this theorem and Theorem 1.5 to evaluate

$$E(\lambda\sigma_1) \frac{E(\lambda_1\lambda_2) E(\lambda_1\lambda_3)}{E(\lambda_1\lambda_4)}$$

and obtain the "identity"

$$\begin{aligned}&\frac{E(\lambda\lambda_2)E(\lambda\lambda_3)E(\lambda_1\lambda_2)E(\lambda\lambda_2)}{E(\lambda_1\lambda_4)E(-\lambda\lambda_1\lambda_2\lambda_3)}\\ &\qquad\times\sum_{n\geq 0}\frac{(-\lambda_1^{-1}\lambda_2^{-1})_n(-\lambda_1^{-1}\lambda_3^{-1})_n}{(-\lambda_1^{-1}\lambda_4^{-1})_n(\lambda\lambda_1^{-1}\lambda_2^{-1}\lambda_3^{-1})_n(q)_n}q^n\\ &=\sum_{n\geq 0}\frac{h_n(\lambda_2,\lambda_3;\lambda_4)\chi(\lambda,\lambda_1)}{(q)_n}.\end{aligned}$$

This is false; the right hand side does not even converge.

We now give a general operator theorem along with its simplest applications as an indication of future directions. We expect to take up a more complete treatment elsewhere. The proof is just like our proof of Theorem 1.2; we omit it for brevity.

THEOREM 3.27. *As operators on functions independent of the variables x_i and y_j, $1\leq i\leq r+1$, $1\leq j\leq r$,*

$$ {}_{r+1}\Phi_r\left(\begin{matrix}x_1,\ldots,x_{r+1}\\ y_1,\ldots,y_r\end{matrix};z\delta_1\right)\frac{1}{(\mu\lambda_1)_\infty} \tag{3.32}$$
$$=\frac{1}{(\mu\lambda_1)_\infty}{}_{r+1}\Phi_r\left(\begin{matrix}x_1,\ldots,x_{r+1}\\ y_1,\ldots,y_r\end{matrix};\frac{\eta_{1,x_1,\ldots,x_{r+1}}}{\eta_{y_1,\ldots,y_r}}\mu z\right){}_{r+1}\Phi_r\left(\begin{matrix}x_1,\ldots,x_{r+1}\\ y_1,\ldots,y_r\end{matrix};z\delta_1\right).$$

Convergence here is characterized by Proposition 2.21.

COROLLARY 3.28. *As rational functions*

$$ {}_{r+1}\Phi_r\left(\begin{matrix}x_1,\ldots,x_{r+1}\\ y_1,\ldots,y_r\end{matrix};z\delta_1\right)\lambda_1^n=\sum_{i=0}^{n}\begin{bmatrix}n\\ i\end{bmatrix}\frac{(x_1)_{n-i}\cdots(x_{r+1})_{n-i}}{(y_1)_{n-i}\cdots(y_r)_{n-i}}z^{n-i}\lambda_1^i. \tag{3.33}$$

Notice that the right hand side may be recast in terms of basic hypergeometric functions giving:

$$ {}_{r+1}\Phi_r\left(\begin{matrix}x_1,\ldots,x_{r+1}\\ y_1,\ldots,y_r\end{matrix};z\delta_1\right)\lambda_1^n=\lambda_1^n\,{}_{r+2}\Phi_r\left(\begin{matrix}x_1,\ldots,x_{r+1},q^{-n}\\ y_1,\ldots,y_r\end{matrix};\frac{zq^n}{\lambda_1}\right). \tag{3.34}$$

3.5. Conclusion

The next step in the development we have begun here is probably an explanation in terms of symmetric functions of the Sears [**50**] transformation

$$ {}_4\Phi_3\left(\begin{matrix}q^{-n},a,b,c\\ d,e,f\end{matrix};q\right)=\left(\frac{bc}{d}\right)^n\frac{(de/bc,df/bc)_n}{(e,f)_n}{}_4\Phi_3\left(\begin{matrix}q^{-n},a,d/b,d/c\\ d,de/bc,df/bc\end{matrix};q\right), \tag{3.35}$$

where $abc=defq^{n-1}$.

With a little work an implicit explanation of this symmetry follows from general properties of orthogonal polynomials and what we have done here. Make the change of variables $a=\lambda_1\lambda_2\lambda_3\lambda_4q^{n-1}$, $b=\lambda_1e^{i\theta}$, $c=\lambda_1e^{-i\theta}$, $d=\lambda_1\lambda_2$, $e=\lambda_1\lambda_3$, $f=\lambda_1\lambda_4$, and put

$$p_n(x;\lambda_1,\lambda_2,\lambda_3,\lambda_4):=\lambda_1^{-n}(d,e,f)_n\,{}_4\Phi_3\left(\begin{matrix}q^{-n},a,b,c\\ d,e,f\end{matrix};q\right),$$

where $x=\cos\theta$. Then (3.35) merely asserts that $p_n(x;\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ is symmetric in λ_1, λ_2, λ_3, and λ_4. But this follows from the Askey-Wilson integral (2.23). For

(2.23) implies (see [**12**]) that for $\max\{|\lambda_1|, |\lambda_2|, |\lambda_3|, |\lambda_4|\} < 1$, the polynomials p_n are orthogonal on $(0, \pi)$ with respect to the weight function

$$w(\cos\theta; \lambda_1, \lambda_2, \lambda_3, \lambda_4) := \frac{(e^{2i\theta}, e^{-2i\theta})_\infty}{(\lambda_1 e^{i\theta}, \lambda_1 e^{-i\theta}, \lambda_2 e^{i\theta}, \lambda_2 e^{-i\theta}, \lambda_3 e^{i\theta}, \lambda_3 e^{-i\theta}, \lambda_4 e^{i\theta}, \lambda_4 e^{-i\theta})_\infty}.$$

Now this weight function is obviously symmetric in λ_1, λ_2, λ_3, and λ_4. It is positive when $\max\{|\lambda_1|, |\lambda_2|, |\lambda_3|, |\lambda_4|\} < 1$, and the uniqueness of polynomials orthogonal with respect to a positive measure shows that the Askey-Wilson polynomials are symmetric in the four parameters λ_1, λ_2, λ_3, and λ_4. This argument is given in [**12**]. Thus the symmetry in the Askey-Wilson polynomials is in some sense inherited from the function Φ_3. The question of whether a more explicit form of this symmetry can be given in terms of symmetric polynomials or constant terms remains an interesting question.

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