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Modified Convergence for q-Continued Fractions Defined by Functional Relations

DOUGLAS BOWMAN

ABSTRACT. There are several definitions of the value of an infinite continued fraction. In this paper we study one which is useful in the context of continued fractions defined by iterating functional relations. Our results are applicable to the study of q-continued fractions, such as some of the continued fractions studied by B. Gordon and K. Alladi. We obtain an explanation of the different limits which have been attached to identical continued fractions by showing that they correspond to basis elements in the solution space of the relevant q-difference equations. A generalization is given for a general q-continued fraction arising from Heine's q-analogue of Gauss's hypergeometric function. Modified and analytic convergence criteria are given for this q-continued fraction.

1 Introduction

The value of an infinite process is usually defined as the limit of the sequence s_n formed by halting the process after n steps. This is the traditional approach to convergence of continued fractions, as well as of infinite series and products. Recently, however, several new notions of convergence for continued fractions have been studied [**4**],[**9**]. A special case of one of these, called *modified convergence,* was applied by Alladi [**2**] to obtain some new limits for certain q-continued fractions. In one example he found that a continued fraction which normally diverged by oscillation now has a limit. In other examples he found that continued fractions which normally had a limit now tended toward new limits. In these examples the new limits are natural in that they agree with those that the continued fraction "should take" in the context of recurrence relations.

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In this note we apply a more general case of modified convergence [**9**] to q-continued fractions derived from three-term recurrences. We first consider arbitrary continued fractions derived from three-term recurrences and define the special case of modified convergence required; then we consider applications to q-continued fractions.

Specifically, suppose we are given a sequence $\{y_m\}$ $(m = 0, 1, 2, \dots)$ which is a solution of the difference equation

$$c_m A_m = b_m A_{m+1} + a_{m+1} A_{m+2}. \tag{1}$$

Putting $A_m = y_m$ and dividing through by y_{m+1} gives

$$\frac{c_m y_m}{y_{m+1}} = b_m + \frac{a_{m+1} c_{m+1}}{\left(\dfrac{c_{m+1} y_{m+1}}{y_{m+2}}\right)}.$$

Hence

$$\frac{c_0 y_0}{y_1} = b_0 + \frac{a_1 c_1}{b_1 +} \frac{a_2 c_2}{b_2 + \dots} \frac{a_m c_m}{\left(\dfrac{c_m y_m}{y_{m+1}}\right)}. \tag{2}$$

We now wish to let $m \to \infty$ to obtain

$$\frac{c_0 y_0}{y_1} = b_0 + \frac{a_1 c_1}{b_1 +} \frac{a_2 c_2}{b_2 +} \frac{a_3 c_3}{b_3 + \dots}. \tag{3}$$

However (3) is not always true if the value of the continued fraction on the right is defined in the standard way. We present one of Alladi's examples where the continued fraction converges to $c_0 y_0 / y_1$ in the modified sense, but not in the ordinary sense.

Throughout this section, assume that y_m, a_m, b_m and c_m are elements of a normed field K with the norm satisfying the product rule $\|xy\| = \|x\| \|y\|$. Convergence is defined with respect to this norm. We say that a sequence converges to infinity if its reciprocal sequence converges to zero. We also follow a somewhat unusual convention by writing our continued fractions in the form

$$b_0 + \frac{a_1 c_1}{b_1 +} \frac{a_2 c_2}{b_2 +} \frac{a_3 c_3}{b_3 + \dots} \tag{4}$$

instead of the more common form

$$b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 + \dots}.$$

There are two reasons for this. First of all, our definition of convergence will depend on how the partial numerators are factored into the sequences $\{a_m\}$ and $\{c_m\}$. Secondly, when the field K is the quotient field of some integral domain, the introduction of the sequence $\{c_m\}$ allows clearing fractions in (1); this is often desirable.

The continued fraction (4) is usually said to converge if the sequence of convergents

$$\frac{P_n}{Q_n} = b_0 + \frac{a_1c_1}{b_1+}\,\frac{a_2c_2}{b_2+}\,\frac{a_3c_3}{b_3+\dots}\,\frac{a_nc_n}{b_n}. \tag{5}$$

tends to a limit as $n \to \infty$. Alladi's modified convergents are

$$\frac{P_n^*}{Q_n^*} = b_0 + \frac{a_1c_1}{b_1+}\,\frac{a_2c_2}{b_2+}\,\frac{a_3c_3}{b_3+\dots}\,\frac{a_nc_n}{b_n + a_{n+1}c_{n+1}}$$

and the modified limit of (4) is the limit of the modified convergents (if it exists). The more general modified convergents of [**9**] are given by

$$b_0 + \frac{a_1c_1}{b_1+}\,\frac{a_2c_2}{b_2+}\,\frac{a_3c_3}{b_3+\dots}\,\frac{a_nc_n}{b_n + w_n}, \tag{6}$$

where $\{w_n\}$ is some sequence of elements from K. Thus the case studied in [**2**] is $w_n = a_{n+1}c_{n+1}$. If (6) approaches a limit as $n \to \infty$, it is called the modified limit of (4) with respect to the sequence $\{w_n\}$.

For example, Alladi showed that in his modified sense

$$q + \frac{1}{q^3+}\,\frac{1}{q^5+\dots} = \prod_{n=0}^{\infty} \frac{(1-q^{5n+2})(1-q^{5n+3})}{(1-q^{5n+1})(1-q^{5n+4})} \qquad |q|<1, \tag{7}$$

while in the ordinary sense the continued fraction does not converge.

Gordon [**3**] showed that under ordinary convergence

$$1 + \frac{q^2-q}{1+q+}\,\frac{q^4-q}{1+q+}\,\frac{q^6-q}{1+q+\dots} = \prod_{n=0}^{\infty} \frac{(1+q^{8n+3})(1+q^{8n+5})}{(1+q^{8n+1})(1+q^{8n+7})} \qquad |q|<1, \tag{8}$$

while Alladi showed that in his modified sense

$$1 + \frac{q^2-q}{1+q+}\,\frac{q^4-q}{1+q+}\,\frac{q^6-q}{1+q+\dots} = q\prod_{n=0}^{\infty} \frac{(1+q^{8n+1})(1+q^{8n+7})}{(1+q^{8n+3})(1+q^{8n+5})} \qquad |q|<1. \tag{9}$$

Thus we have a case where the same continued fraction has two different limits. It can also happen that both definitions of convergence lead to the same limit. This leaves matters in a somewhat confusing state. Which should be taken as the 'natural limit'?

The purpose of this paper is to remedy this by looking at a natural modification of the convergence of (4) which yields (7), (8), and (9) and is designed for giving (3) in general. It will be found that in the last two examples, both limits are natural when taken with respect to the proper recurrence solutions. First we define the modification we will use, and then give some useful results which help in our analysis of the above cases. Finally, we examine them as formal identities and see how the theory explains the results.

2 A special modification of convergence

Throughout this section P_n and Q_n are defined by (5).

DEFINITION. *Let the sequence* $\{y_m\}$ $(m = 0, 1, 2, \dots)$ *satisfy* (1). *Assume*

$$\lim_{m\to\infty} y_m/y_{m+1} = \lambda.$$

If

$$\lim_{n\to\infty} \frac{\lambda P_{n-1} + a_n P_{n-2}}{\lambda Q_{n-1} + a_n Q_{n-2}} \quad \left(= \lim_{n\to\infty} b_0 + \frac{a_1 c_1}{b_1 +} \frac{a_2 c_2}{b_2 +} \frac{a_3 c_3}{b_3 + \dots} \frac{a_{n-1} c_{n-1}}{b_{n-1} +} \frac{a_n}{\lambda}\right)$$

exists, it is called the recurrence limit of the continued fraction (3) *with respect to the solution* $\{y_m\}$ *of the recurrence* (1).

Obviously our definition of the limit depends not only on the continued fraction, but also on the recurrence solution which generates it. It is this dependence which allows us to resolve the ambiguities in the above examples. Clearly our definition is meaningless in the case where λ does not exist. Fortunately, λ exists in a large class of cases. This is guaranteed by the following theorem, special cases of which were first given by Poincaré and Pincherle. The general result over $\mathbb{C}$ is due to Perron [**8**]. See also [**6**]. Here we note that Perron's proof extends to an arbitrary complete normed field.

THEOREM. *For* $1 \le i \le r$ *let* $\{a_i^{(n)}\}$ $(n \ge 0)$ *be a sequence of elements of* K *tending to a limit* $a_i \in K$ *as* $n \to \infty$. *Suppose that* $a_r^{(n)} \neq 0$ *(although* $a_r = 0$ *is allowed.) Consider the difference equation*

$$D_{n+r} + a_1^{(n)} D_{n+r-1} + a_2^{(n)} D_{n+r-2} + \dots + a_r^{(n)} D_n = 0. \tag{10}$$

Let the roots of the characteristic polynomial $x^r + a_1 x^{r-1} + a_2 x^{r-2} + \dots + a_r$ *be* $x_1, \dots, x_r \in K$. *Assume* $\|x_1\| > \|x_2\| > \dots > \|x_r\|$. *Then there exist* r *independent solutions* $D_n^{(i)}$ $1 \le i \le r$ *of* (10) *such that*

$$\lim_{n\to\infty} \frac{D_n^{(i)}}{D_{n+1}^{(i)}} = \frac{1}{x_i}.$$

The proof given in [**8**] applies *mutatis mutandis* to this more general case.

When $r = 2$, this theorem shows that λ exists when the roots of the characteristic polynomial have distinct norms and lie in K.

The next proposition will be used to prove Theorem 1 which shows why the recurrence limit is a natural definition of limit for continued fractions generated by the recurrence (1).

PROPOSITION 1.

$$\frac{\lambda P_{n-1}+a_nP_{n-2}}{\lambda Q_{n-1}+a_nQ_{n-2}}-\frac{c_0y_0}{y_1}=\frac{(-1)^n(\lambda y_{n+1}-y_n)a_1\dots a_n}{y_1(\lambda Q_{n-1}+a_nQ_{n-2})}$$

PROOF. We use the equations

$$c_0c_1\dots c_{n-1}y_0=P_{n-1}y_n+a_nP_{n-2}y_{n+1}$$
$$c_1\dots c_{n-1}y_1=Q_{n-1}y_n+a_nQ_{n-2}y_{n+1}.$$

When all $c_i=1$, these equations are proved in [**7**], vol. 1, §2, III. Our version is proved similarly. We have

$$\begin{aligned}\frac{\lambda P_{n-1}+a_nP_{n-2}}{\lambda Q_{n-1}+a_nQ_{n-2}}-\frac{c_0y_0}{y_1}&=\frac{(\lambda P_{n-1}+a_nP_{n-2})y_1-c_0y_0(\lambda Q_{n-1}+a_nQ_{n-2})}{(\lambda Q_{n-1}+a_nQ_{n-2})y_1}\\&=\frac{(\lambda P_{n-1}+a_nP_{n-2})(Q_{n-1}y_n+a_nQ_{n-2}y_{n+1})}{c_1\dots c_{n-1}(\lambda Q_{n-1}+a_nQ_{n-2})y_1}\\&\quad-\frac{(P_{n-1}y_n+a_nP_{n-2}y_{n+1})(\lambda P_{n-1}+a_nP_{n-2})}{c_1\dots c_{n-1}(\lambda Q_{n-1}+a_nQ_{n-2})y_1}\\&=\frac{(\lambda y_{n+1}-y_n)a_n(P_{n-1}Q_{n-2}-P_{n-2}Q_{n-1})}{c_1\dots c_{n-1}(\lambda Q_{n-1}+a_nQ_{n-2})y_1},\end{aligned}$$

and the determinant formula, see [**5**], gives the proposition. □

THEOREM 1. *Equation* (3) *holds as a recurrence limit if and only if*

$$\lim_{n\to\infty}\frac{(\lambda y_{n+1}-y_n)a_1\dots a_n}{\lambda Q_{n-1}+a_nQ_{n-2}}=0. \tag{11}$$

PROOF. This follows immediately from the last proposition. □

It is of interest to find conditions under which the recurrence limit of a continued fraction equals the usual limit (assuming the latter exists). The following proposition gives some sufficient conditions for this to occur. It should be noted that this has already been studied in that criteria have been developed for finding when (3) holds for ordinary convergence. Perron [**7**] gives four criteria for the case where K is $\mathbb{C}$ or $\mathbb{R}$. Here are some criteria for arbitrary K.

PROPOSITION 2. *Suppose* (4) *converges in the ordinary sense. Then* (3) *holds if either*

(1) P_n, Q_n *approach limits not equal to* $0,\infty$, *and* $a_n+\lambda$ *approaches a nonzero limit, or*
(2) $\lim_{n\to\infty}P_{n-1}/a_n=\lim_{n\to\infty}Q_{n-1}/a_n=0$.

PROOF.

(1) follows easily from the familiar rules for manipulating limits.
(2) follows immediately. □

3 Applications to q-continued fractions

In this section we show how the above examples arise naturally in our theory of convergence. A q-continued fraction is one in which a_n, b_n and c_n are polynomials in q^{K_n}, the three sequences of polynomials are eventually periodic and where K_n is the union of a sequence of arithmetic progressions in $\mathbb{Z}$. The previous examples of Alladi and Gordon are q-continued fractions. In these cases all of the periods are one. In general the least common multiple of the periods and the moduli in the arithmetic progressions is called the q-period of the continued fraction. We view q-continued fractions formally, meaning that convergence is understood in the topology of formal Laurent series in q. Specifically, we study series of the following form:

$$x = \sum_{n \geq k} c_n q^n \qquad c_k \neq 0 \quad k \in \mathbb{Z},$$

where the coefficients c_n belongs to a field L. Formal convergence is defined by the norm $\|x\| \equiv 2^{-k}$. We now look at the formal convergence of the q-continued fractions introduced above.

It is clear that for any q-continued fraction with q-period one, the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ tend to limits equal to the constant terms of their respective polynomials.

We use the standard notation

$$(a)_n = (a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1})$$

for a non-negative integers n, we define $(a)_\infty$ to be the formal (or if we choose the analytic) limit

$$\prod_{m=0}^{\infty}(1-aq^m) = \lim_{n\to\infty}(a)_n.$$

We first consider (7). In [**2**] it was shown that

$$u_m = (1-b+aq^{2m+1})u_{m+1} + bu_{m+2}, \tag{12}$$

where $u_m = f(aq^{2m})$ and

$$f(a) = (-bq^2;q^2)_\infty \sum_{n\geq 0} \frac{q^{n^2}}{(-bq^2;q^2)_n (q^2;q^2)_n} a^n.$$

Obviously the coefficients in (12) tend to limits as $n \to \infty$, and the characteristic equation is $bu^2 + (1-b)u - 1 = 0$. This has roots 1 and $-1/b$. To get (7) we put $b = 1$; then the roots have the same norm and Perron's theorem does not apply. However, for the general case it is easy to see from the definition of f that $\lim_{m\to\infty} u_m = (-bq^2;q^2)_\infty$. Thus it is clear that $\lambda = 1$, and our recurrence limit reduces to Alladi's limit.

For the special case $a = b = 1$, Q_m is defined by $Q_0 = 1$, $Q_1 = q^3$, and $Q_m = q^{2m+1}Q_{m-1} + Q_{m-2}$. It is obvious from this recurrence that $\|Q_m\|$ is equal to 1 or 1/8 according as m is even or odd. This shows that Q_m does not tend to a limit. Similarly, $\|P_m\|$ is equal to $1/2$ or 1 according as m is even or odd. This shows that P_m does not tend to a limit. Together these equations imply that $\|P_m/Q_m\|$ is equal to $1/2$ or 8, according as m is even or odd. This shows that (7) does not converge formally in the ordinary sense.

Now we apply Theorem 1. We have:

$$\begin{aligned}\left\|\frac{(\lambda u_{m+1}-u_m)a_1\dots a_m}{(\lambda Q_{m-1}+a_mQ_{m-2})}\right\| &= \left\|\frac{f(q^{2m+2})-f(q^{2m})}{Q_{m-1}+Q_{m-2}}\right\| \\ &= \frac{\|(-q^2;q^2)_\infty\|\|q^{2m}(-1+\dots)\|}{\|Q_{m-1}+Q_{m-2}\|} = \frac{2^{-2m}}{\|1+q^3+\dots\|} \\ &= \frac{1}{2^{2m}}.\end{aligned}$$

Hence as $m\to\infty$, the recurrence limit with respect to u_m exists. Using transformations for $f(1)$ and $f(q^2)$ as in [**2**], we obtain a formal version of (7).

We now consider (8) and (9). Alladi [**2**] showed that v_m and w_m are solutions of the equation

$$B_m = (1+q)B_{m+1} + (zq^{2m+2}-q)B_{m+2},$$

where $v_m = C(zq^{2m})$, $w_m = q^{-m}D(zq^{2m})$ and

$$\begin{aligned}C(z) &= \sum_{n\geq 0}\frac{q^{2n^2}}{(q)_{2n+1}}z^n,\\ D(z) &= \sum_{n\geq 0}\frac{q^{2n^2-2n}}{(q)_{2n}}z^n.\end{aligned}$$

From the recurrence for $\{B_m\}$ we get the characteristic equation $-qx^2+(1+q)x-1=0$. This equation has zeros 1 and $1/q$. These have different norms, so Perron's theorem applies and there are two solutions of the general q-difference equation with term ratios tending to 1 and q. It follows immediately from the definitions of $\{v_m\}$ and $\{w_m\}$ that

$$\begin{aligned}\lim_{m\to\infty}\frac{v_m}{v_{m+1}} &= 1, \text{and}\\ \lim_{m\to\infty}\frac{w_m}{w_{m+1}} &= q,\end{aligned}$$

so $\{v_m\}$ and $\{w_m\}$ are such solutions.

For the continued fraction in (8) and (9), $Q_m = (1+q)Q_{m-1}+(zq^{2m}-q)Q_{m-2}$; thus $Q_m - Q_{m-1} = qQ_{m-1} + (zq^{2m}-q)Q_{m-2}$. Hence $Q_m - Q_{m-1} = q(Q_{m-1} - Q_{m-2}) + zq^{2m}Q_{m-2}$. Iterating this gives $Q_m - Q_{m-1} = q^k(Q_{m-k}-Q_{m-k-1}) + zq^{2m}Q_{m-2} + zq^{2m-1}Q_{m-3} + \dots + zq^{2m-k+1}Q_{m-k-1}$. Putting $k=m$ gives

$$Q_m - Q_{m-1} = q^m(1+zq^2+\dots). \tag{13}$$

This shows that $\lim_{m\to\infty}Q_m$ exists. Similarly, $\lim_{m\to\infty}P_m$ exists. We write these limits as Q_∞ and P_∞ respectively.

For $\{u_m\}$, $\lambda = 1$, and so as $n \to \infty$, the denominator in (11) tends to $(1-q)Q_\infty$. The norm of this is easily seen to be 1. It is also clear that the numerator in (11) tends to 0 as $n \to \infty$, so by Theorem 1, the recurrence limit holds with respect to $\{u_m\}$. By Proposition 2(1) this limit is equal to the ordinary limit. Now using transformations for $C(1)$, and $C(q^2)$, a formal version of Gordon's theorem follows.

We apply Theorem 1 to $\{v_m\}$, where now $\lambda = q$. The norm of the denominator in (11) is $\|(\lambda Q_{m-1} + a_m Q_{m-2})\| = \|qQ_{m-1} + (q^{2m} - q)Q_{m-2}\|$, which by (13) is equal to $1/2^m$. It is obvious that the norm of the numerator of (11) is $1/2^{3m}$. Thus we easily get convergence for the recurrence limit in this case too. Here $\lim_{n\to\infty}(a_n + \lambda) = 0$, so Proposition 2 does not apply. In fact (8) and (9) show that the recurrence limit is not equal to the ordinary limit. Now the fraction in (8) and (9) is not quite the continued fraction used by Alladi in [**2**], his being an equivalent continued fraction. After the equivalence transformation, our recurrence limit becomes Alladi's modified limit. Transformations for $D(1)$ and $D(q^2)$ now give (9) formally; see [**2**].

We now discuss a q-difference equation which generalizes all of the examples considered so far. The fact that its general solution can be given in terms of basic hypergeometric series means that the preceding results are special cases of continued fraction theorems with additional parameters. We present these theorems here. First some notation is required. The fractions are expansions of quotients of the basic hypergeometric series ${}_2\Phi_1$, defined by

$${}_2\Phi_1\left(\begin{matrix} a,\, b \\ c \end{matrix}; x\right) \equiv \sum_{n\geq 0} \frac{(a)_n (b)_n}{(c)_n (q)_n} x^n.$$

We consider the general recurrence

$$(1 - xq^n)C_n = [1 + c - (a+b)xq^n]C_{n+1} + (abxq^n - c)C_{n+2}. \tag{14}$$

As $n \to \infty$ this recurrence has the characteristic equation $cz^2 - (1+c)z + 1 = 0$ with roots 1 and $1/c$. The fundamental solutions of (14) are given by $w_n = F_1(xq^n)$ and $x_n = F_2(xq^n)$, where

$$F_1(x) = {}_2\Phi_1\left(\begin{matrix} a,\, b \\ cq \end{matrix}; x\right) \quad \text{and} \quad F_2(x) = x^{-\gamma}{}_2\Phi_1\left(\begin{matrix} a/c,\, b/c \\ q/c \end{matrix}; x\right),$$

and $c = q^\gamma$, $\gamma \notin \mathbb{Z}$ [**1**]. The next two theorems deal with the continued fraction

$$1 + c - (a+b)x + \frac{(abx - c)(1 - xq)}{1 + c - (a+b)xq +}\, \frac{(abxq - c)(1 - xq^2)}{1 + c - (a+b)xq^2 + \ldots} \tag{15}$$

obtained by iterating (14). The first theorem extends all the previous continued fractions. We do not give the general convergence conditions for it; in the special cases considered above, Theorem 1 gives convergence without too much difficulty.

THEOREM 2. *When* (11) *holds for* (15), *the recurrence limits of* (15) *with respect to* $\{w_n\}$ *and* $\{x_n\}$ *are respectively:*

(1) *When* $\lambda = 1$, $c \neq q^m$, *for any negative integers* m,

$$(1-x)\frac{{}_2\Phi_1\left(\begin{matrix} a,\, b \\ cq \end{matrix}; x\right)}{{}_2\Phi_1\left(\begin{matrix} a,\, b \\ cq \end{matrix}; xq\right)} = 1 + c - (a+b)x + \frac{(abx-c)(1-xq)}{1+c-(a+b)xq+}\,\frac{(abxq-c)(1-xq^2)}{1+c-(a+b)xq^2+\ldots}.$$

(2) *When* $\lambda = c \neq 0, q^n$, *for any non-negative integers* n,

$$(1-x)c\frac{{}_2\Phi_1\left(\begin{matrix} a/c,\, b/c \\ q/c \end{matrix}; x\right)}{{}_2\Phi_1\left(\begin{matrix} a/c,\, b/c \\ q/c \end{matrix}; xq\right)} = 1 + c - (a+b)x + \frac{(abx-c)(1-xq)}{1+c-(a+b)xq+}\,\frac{(abxq-c)(1-xq^2)}{1+c-(a+b)xq^2+\ldots}.$$

PROOF. This follows immediately from Theorem 1. Also notice that (2) follows from (1) by the substitutions $c \to 1/c$, $a \to a/c$ and $b \to b/c$. □

Notice that on substituting $b \to b/x$, $x = 0$, then $a \to a/b$, $b = 0$ and finally putting $c = q^{1/2}$, $a = q$ and $q \to q^2$ in Theorem 2, parts (1) and (2) reduce to (8) and (9) respectively.

The following theorem solves the problem of evaluating (15) for ordinary convergence in the complex plane.

THEOREM 3. *We have*

$$1 + c - (a+b)x + \frac{(abx-c)(1-xq)}{1+c-(a+b)xq+}\,\frac{(abxq-c)(1-xq^2)}{1+c-(a+b)xq^2+\ldots} = \begin{cases} (1-x)\dfrac{{}_2\Phi_1\left(\begin{matrix} a,\, b \\ cq \end{matrix}; x\right)}{{}_2\Phi_1\left(\begin{matrix} a,\, b \\ cq \end{matrix}; xq\right)}, & \text{for } |c| < 1 \\[2ex] (1-x)c\dfrac{{}_2\Phi_1\left(\begin{matrix} a/c,\, b/c \\ q/c \end{matrix}; x\right)}{{}_2\Phi_1\left(\begin{matrix} a/c,\, b/c \\ q/c \end{matrix}; xq\right)}, & \text{for } |c| > 1 \end{cases},$$

where $|q| < 1$, $|x| < 1$, *and* $\log_q(c) \notin \mathbb{Z}$.

PROOF. Since

$$\lim_{n\to\infty} \frac{F_1(xq^n)}{F_2(xq^n)} = \lim_{n\to\infty} q^{n\gamma}x^{\gamma} = \lim_{n\to\infty} c^n x^{\gamma} = 0, \text{ if } |c| < 1, \text{ while}$$

$$\lim_{n\to\infty} \frac{F_2(xq^n)}{F_1(xq^n)} = 0, \text{ if } |c| > 1,$$

Theorem 2.46(C) of [**7**] gives our theorem. Alternatively the $|c| > 1$ case follows from the $|c| < 1$ case by putting $c \to 1/c$, $a \to a/c$, $b \to b/c$ and using the standard equivalence transformation for continued fractions. □

4 Conclusion

Note that the two solutions of (14) are obtained from the same function by a simple change of variables. This phenomenon persists for even more general q-difference equations, so proving ordinary convergence for q-continued fractions of q-period one is easy. For q-period greater than one, this technique can be extended by considering various contractions to continued fractions with q-period one. If these have the same limit, one expects this to be the limit of the original fraction. Thus we expect to obtain convergence proofs in these cases as well. This theory has applications for instance to proving many of the continued fraction theorems left behind by Ramanujan without proof.

Our method shows a natural relation between the two values of the fraction in (8) and (9); they correspond to the basis elements in the solution space of the q-difference equation which generates the fraction. In the extension (15) this relationship persists. Notice that the two values of (15) as an analytic fraction are precisely the different formal limits in Theorem 2. Since the phenomenon of different solutions of q-difference equations being obtained from each other by a simple change of variables holds for even more general equations, it follows that the relationship between the formal limits and the analytic limits holds even more generally.

We believe that our method is sufficiently general to be of use not only for q-continued fractions, but also for limit periodic continued fractions in general. Taking $K = \mathbb{C}$ and using the ordinary absolute value may allow our theory to be applied to the study of analytic continued fractions. Our formal version of Alladi's first example shows that even when Perron's theorem fails, λ may exist. Thus it is an interesting problem to determine more general conditions under which λ exists.

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Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802

E-mail address: bowman@math.psu.edu