ACTA ARITHMETICA
* (200*)

Polynomial continued fractions

by

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1. Introduction. A *polynomial continued fraction* is a continued fraction $\mathrm{K}_{n=1}^{\infty} a_n/b_n$ where a_n and b_n are polynomials in n. Most well known continued fractions are of this type. For example the first continued fractions giving values for π (due to Lord Brouncker, first published in [10]) and e ([3]) are of this type:

$$\frac{4}{\pi} = 1 + \operatorname*{K}_{n=1}^{\infty} \frac{(2n-1)^2}{2}, \tag{1.1}$$

$$e = 2 + \cfrac{1}{1 + \operatorname*{K}_{n=1}^{\infty} \frac{n}{n+1}}. \tag{1.2}$$

Here we use the standard notations

$$\operatorname*{K}_{n=1}^{N} \frac{a_n}{b_n} := \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ldots + \cfrac{a_N}{b_N}}}} = \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+}\cdots\frac{a_N}{b_N}.$$

We write A_N/B_N for the above finite continued fraction written as a rational function of the variables $a_1, \ldots, a_N, b_1, \ldots, b_N$. By $\mathrm{K}_{n=1}^{\infty} a_n/b_n$ we mean the limit of the sequence $\{A_n/B_n\}$ as n goes to infinity, if the limit exists.

The first systematic treatment of this type of continued fraction seems to be in Perron [7] where degrees through two for a_n and degree one for b_n are studied. Lorentzen and Waadeland [6] also study these cases in detail and they evaluate all such continued fractions in terms of hypergeometric series. There is presently no such systematic treatment for cases of higher degree and examples in the literature are accordingly scarcer. Of particular interest are cases where the degree of a_n is less than or equal to the degree of b_n. These cases are interesting from a number theoretic standpoint since the values of the continued fraction can then be approximated exceptionally well

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by rationals and irrationality measures may be given. When the degrees are equal, the value may be rational or irrational and certainly the latter when the first differing coefficient is larger in b_n. (Here we count the first coefficient as the coefficient of the largest degree term.) Irrationality follows from the criterion given by Tietze, extending the famous Theorem of Legendre (see Perron [7, pp. 252–253]):

TIETZE'S CRITERION. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of integers and* $\{b_n\}_{n=1}^{\infty}$ *be a sequence of positive integers, with* $a_n \neq 0$ *for any* n. *If there exists a positive integer* N_0 *such that*

$$\begin{cases} b_n \geq |a_n|, \\ b_n \geq |a_n| + 1 \quad \text{for } a_{n+1} < 0 \end{cases} \tag{1.3}$$

for all $n \geq N_0$ *then* $\mathrm{K}_{n=1}^{\infty} a_n/b_n$ *converges and its limit is irrational.*

It would seem from the literature that finding cases of equal degrees or even close degrees is difficult. If one picks a typical continued fraction from published tables, the degree of the numerator tends to be twice that of the denominator. One easy way in which this can arise is when the continued fraction is equal to a series after using the Euler transformation:

$$\sum_{n\geq 0} a_n = a_0 + \frac{a_1}{1+}\,\frac{-a_2}{a_1+a_2+}\,\frac{-a_1 a_3}{a_2+a_3+}\,\frac{-a_2 a_4}{a_3+a_4+}\,\frac{-a_3 a_5}{a_4+a_5+\dots}. \tag{1.4}$$

If one side of this equality converges, then so does the other as the nth approximants are equal. The Euler transformation is easily proved by induction.

In this formula, if the terms of the series are rational functions of the index of fixed degree, then in the continued fraction after simplification, one will get the degrees of the numerators to be at least twice those of the denominators. The continued fraction for π given by (1.1) is an example of this phenomenon. Another example of this is the series definition of Catalan's constant:

$$C = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2},$$

which, by (1.4), transforms into the continued fraction given by

$$C = \frac{1}{1+}\,\frac{1^4}{8+}\,\frac{3^4}{16+}\,\frac{5^4}{24+\dots}.$$

Here the degree of the numerator is four times that of the denominator. This continued fraction appears to be new.

Taking contractions of continued fractions (see, for example, Jones and Thron [5, pp. 38–43]) also leads to a relative increase in the degree of the numerator over that of the denominator. For example, forming the even part of the continued fraction will cause a continued fraction with equal degrees to be transformed into one with twice the degree in the numerator as the denominator.

The even part of the continued fraction $\mathrm{K}_{n=1}^{\infty} a_n/b_n$ is equal to

$$\frac{a_1 b_2}{a_2 + b_1 b_2 +} \; \frac{-a_2 a_3 b_4}{a_3 b_4 + b_2(a_4 + b_3 b_4) +} \; \frac{-a_4 a_5 b_2 b_6}{a_5 b_6 + b_4(a_6 + b_5 a_6) +} \; \frac{-a_6 a_7 b_4 b_8}{a_7 b_8 + b_6(a_8 + b_7 a_8) + \ldots}.$$

Other work on polynomial continued fractions was done by Ramanujan [1, Chapter 12]. He gave several cases of equal degree in which the sum is rational. For example, Ramanujan gave the following: If x is not a negative integer then

$$\underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{x+n}{x+n-1} = 1. \tag{1.5}$$

Despite the simplicity of this formula, Ramanujan did not give a proof: the first proof seems to have been given by Berndt [1, p. 112].

In this paper we examine a large number of infinite classes of polynomial continued fractions in which the degrees are equal, or close. Our results follow from a theorem of Pincherle and a variant of the Euler transformation discussed above. We obtain generalizations of Ramanujan's results in which the degrees are equal and the values rational as well as cases of equal degree with irrational limits. Many of our theorems give infinite families of continued fractions. While we concentrate on polynomial continued fractions, many of the results hold in more general cases. Here are some special cases of our general results (proofs are given throughout the paper):

$$\underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{n^{\alpha}+1}{n^{\alpha}} = 1 \qquad \text{for } \alpha > 0. \tag{1.6}$$

$$1 - \cfrac{1}{1 + \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{n^2}{n^2+2n}} = J_0(2), \tag{1.7}$$

where $J_0(x)$ is the Bessel function of the first kind of order 0.

$$2 + \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{2n^2+n}{2n^2+5n+2} = \frac{1}{\sqrt{2}\csc(\sqrt{2}) - 1}. \tag{1.8}$$

(Notice that the irrationality criterion mentioned above means that the last two quantities on the right are irrational.)

$$\underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{n^{12}+2n^{11}+n^{10}+4n+5}{n^{12}+4n-4} = 4. \tag{1.9}$$

$$\underset{n=2}{\overset{\infty}{\mathrm{K}}} \frac{6n^7+6n^6+2n^5+3n+2}{6n^7-6n^6+2n^5+3n-5} = \frac{19}{7}. \tag{1.10}$$

$$(1.11)\qquad \operatorname*{K}_{n=1}^{\infty} \frac{3n^6 - 3n^4 + n^3 + 6n^2 + 6n - 1}{3n^6 - 9n^5 + 6n^4 + n^3 + 6n^2 - 12n - 2} = 1.$$

2. Infinite polynomial continued fractions with rational limits. In this section we derive some general results about the convergence of polynomial continued fractions in some infinite families and give some examples of how these results can be used to find the limit of such continued fractions. Many of the results in this paper are consequences of the following theorem of Pincherle [9]:

THEOREM 1 (Pincherle). *Let* $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ *and* $\{G_n\}_{n=-1}^{\infty}$ *be sequences of real or complex numbers satisfying* $a_n \neq 0$ *for* $n \geq 1$ *and for all* $n \geq 1$,

$$(2.1)\qquad G_n = a_n G_{n-2} + b_n G_{n-1}.$$

Let $\{B_n\}_{n=1}^{\infty}$ *denote the denominator convergents of the continued fraction* $\operatorname{K}_{n=1}^{\infty} a_n/b_n$. *If* $\lim_{n\to\infty} G_n/B_n = 0$ *then* $\operatorname{K}_{n=1}^{\infty} a_n/b_n$ *converges and its limit is* $-G_0/G_{-1}$.

Proof. See, for example, Lorentzen and Waadeland [6, p. 202]. ∎

For many sequences it may be difficult to decide whether the condition $\lim_{n\to\infty} G_n/B_n = 0$ is satisfied. Below are some conditions governing the growth of the B_n's which will be useful later.

(i) Let a_n and b_n be non-constant polynomials such that $a_n \geq 1, b_n \geq 1$ for $n \geq 1$, and suppose b_n is a polynomial of degree k. If the leading coefficient of b_n is D, then given $\varepsilon > 0$, there exists a positive constant $C_1 = C_1(\varepsilon)$ such that $B_n \geq C_1(|D|/(1+\varepsilon))^n (n!)^k$.

(ii) If a_n and b_n are positive numbers ≥ 1, then there exists a positive constant C_3 such that $B_n \geq C_3 \phi^n$ for ≥ 1, where ϕ is the golden ratio $(1+\sqrt{5})/2$.

COROLLARY 1. *If* m *is a positive integer and* b_n *is any polynomial of degree* ≥ 1 *such that* $b_n \geq 1$ *for* $n \geq 1$, *then*

$$\operatorname*{K}_{n=1}^{\infty} \frac{mb_n + m^2}{b_n} = m.$$

Proof. With $a_n = mb_n + m^2$ for $n \geq 1$, and $G_n = (-1)^{n+1} m^{n+1}$ for $n \geq -1$, $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{G_n\}_{n=-1}^{\infty}$ satisfy equation (2.1). By (i) above

$$\lim_{n\to\infty} G_n/B_n = \lim_{n\to\infty} (-1)^{n+1} m^{n+1}/B_n = 0 \;\Rightarrow\; \operatorname*{K}_{n=1}^{\infty} \frac{a_n}{b_n} = -G_0/G_{-1} = m. \;\blacksquare$$

A special case is where $m = 1$, in which case $|G_n| = 1$ for all n, and all that is necessary is that $\lim_{n\to\infty} B_n = \infty$. The following generalization of the result (1.5) of Ramanujan for positive numbers greater than 1 follows easily:

If $\{b_n\}_{n=1}^{\infty}$ is any sequence of positive numbers with $b_n \geq 1$ for $n \geq 1$ then

$$\mathop{\mathrm{K}}_{n=1}^{\infty} \frac{b_n + 1}{b_n} = 1.$$

Letting $b_n = n^{\alpha}$, $\alpha > 0$, gives (1.6) in the introduction ([1]).

Entry 12 from the chapter on continued fractions in Ramanujan's second notebook [1, p. 118] follows as a consequence of the above theorem:

Corollary 2. *If x and a are complex numbers, where $a \neq 0$ and $x \neq -ka$, where k is a positive integer, then*

$$\frac{x+a}{a + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{(x+na)^2 - a^2}{a}} = 1.$$

Proof. Note that

$$\frac{x+a}{a + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{(x+na)^2 - a^2}{a}} = \frac{x/a+1}{1 + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{(x/a+n)^2 - 1}{1}}.$$

Replace x/a by m to simplify notation; the result will follow if it can be shown that

$$m = \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{(m+n)^2 - 1}{1} = \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{(m+n-1)(m+n+1)}{1}.$$

With $G_{-1} = 1$, $G_n = (-1)^{n+1} \prod_{i=0}^{n} (m+i)$ for $n \geq 0$, and $b_n = 1$, $a_n = (m+n-1)(m+n+1)$ for $n \geq 1$, $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{G_n\}_{n=-1}^{\infty}$ satisfy equation (2.1) so that the result will follow from Theorem 1 if it can be shown that $\lim_{n\to\infty} G_n/B_n = 0$, in which case the continued fraction will converge to $-G_0/G_{-1} = m$. However, an easy induction shows that for $k \geq 1$,

$$B_{2k+1} = (k+1) \prod_{i=2}^{2k+1} (m+i) \quad \text{and} \quad B_{2k} = (m+k+1) \prod_{i=2}^{2k} (m+i).$$

Thus $\lim_{n\to\infty} G_n/B_n = 0$, and the result follows. ∎

Corollary 3. *Let m be a positive integer and let b_n be any polynomial of degree ≥ 1 such that $b_n \geq 1$ for $n \geq 1$ and either degree $b_n > 1$ or if degree $b_n = 1$ then its leading coefficient is $D > m$. Then*

$$\mathop{\mathrm{K}}_{n=1}^{\infty} \frac{mnb_n + m^2 n(n+1)}{b_n} = m.$$

([1]) Lorentzen and Waadeland give an exercise [6, p. 234, question 15(d)] which effectively involves a similar result in the case where b_n belongs to a certain family of quadratic polynomials in n over the complex numbers.

Proof. Letting $G_n = (-1)^{n+1} m^{n+1} (n+1)!$ for $n \geq -1$ and $a_n = mnb_n + m^2 n(n+1)$ for $n \geq 1$, one sees that $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}, \{G_n\}_{n=-1}^{\infty}$ satisfy equation (2.1). By (i), $\lim_{n\to\infty} G_n/B_n = 0 \Rightarrow \mathrm{K}_{n=1}^{\infty}\, a_n/b_n = -G_0/G_{-1} = m$. ∎

Theorem 1 does not say directly how to find the value of all polynomial continued fractions $\mathrm{K}_{n=1}^{\infty} a_n/b_n$ as it does not say how the sequence G_n can be found or even if such a sequence can be found. However, Algorithm Hyper (see [8]) can be used to determine if a hypergeometric solution G_n exists to equation (2.1) and, if such a solution exists, the algorithm will output G_n, enabling the limit of the continued fraction to be found, if G_n satisfies $\lim_{n\to\infty} G_n/B_n = 0$.

Even if for the particular polynomial sequences a_n and b_n it turns out that the sequence G_n found does not satisfy $\lim_{n\to\infty} G_n/B_n = 0$, then these three sequences a_n, b_n and G_n may be used to find the value of infinitely many other continued fractions when G_n is a polynomial or rational function in n.

The following proposition shows how, given any one solution of (2.1), one can find the value of infinitely many other polynomial continued fractions in an easy way.

PROPOSITION 1. *Suppose that there exist complex sequences* $\{G_n\}_{n=-1}^{\infty}$, $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *satisfying*

$$a_n G_{n-2} + b_n G_{n-1} - G_n = 0. \tag{2.2}$$

Let f_n *be any sequence, let* $s_n = f_n G_{n-1} + a_n$ *be such that* $s_n \neq 0$ *for* $n \geq 1$, *and let* $t_n = f_n G_{n-2} - b_n$. *Let* A_n/B_n *denote the convergents to* $\mathrm{K}_{n=1}^{\infty}\, s_n/t_n$. *If* $\lim_{n\to\infty} G_n/B_n = 0$ *then* $\mathrm{K}_{n=1}^{\infty}\, s_n/t_n$ *converges and its limit is* G_0/G_{-1}.

Proof. Let $G_n' = (-1)^{n+1} G_n$. Then

$$s_n G_{n-2}' + t_n G_{n-1}' - G_n' = (-1)^{n-1}(a_n G_{n-2} + b_n G_{n-1} - G_n) = 0.$$

Thus $\{G_n'\}_{n=-1}^{\infty}$, $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ satisfy the conditions of Theorem 1 so $\mathrm{K}_{n=1}^{\infty} s_n/t_n$ converges and its limit is $-G_0'/G_{-1}' = G_0/G_{-1}$. ∎

Entry 9 from the chapter on continued fractions in Ramanujan's second notebook [1, pp. 114–115] follows in case a is real and positive and x is real as a consequence of the above proposition:

COROLLARY 4. *Let* a *be a real positive number and let* x *be a real number such that* $x \neq -ka$ *where* k *is a positive integer. Then*

$$\frac{x+a+1}{x+1} = \operatorname*{K}_{n=1}^{\infty} \frac{x+na}{x+(n-1)a-1}.$$

Proof. It is enough to prove this for $x - 1 > 0$ since for n sufficiently large $x + (n-1)a - 1 > 0$ and then the result will hold for a tail of the continued fraction and the resulting finite continued fraction will collapse from the bottom up to give the result. Let $G_n = (x+(n+1)a+1)/(x+1)$. Put $f_n = x+1, a_n = -1$ and $b_n = 2$ so that $a_nG_{n-2} + b_nG_{n-1} - G_n = 0$, $x + na = f_nG_{n-1} + a_n$ and $x + (n-1)a - 1 = f_nG_{n-2} - b_n$. Since G_n is a degree 1 polynomial in n, and $x + na, x + (n-1)a - 1 > 0$ for $n \geq 1$, it can easily be shown that $\lim_{n\to\infty} G_n/B_n = 0$ and so by Proposition 1 the continued fraction converges to $G_0/G_{-1} = (x+a+1)/(x+1)$. ■

REMARKS. (1) In Proposition 1 any polynomial G_n satisfying (2.2) can always be assumed to have positive leading coefficient (if necessary multiply (2.2) by -1). If f_n is then taken to be a polynomial of sufficiently high degree with leading positive coefficient then both s_n and t_n will be polynomials with positive leading coefficients so that there exists a positive integer N_0 so that for all $n \geq N_0$, $s_n, t_n > 0$. If it happens that for some $m \geq N_0$ both B_m and B_{m+1} are of the same sign then B_n will go to $+\infty$ or $-\infty$ exponentially fast. In these circumstances $\lim_{n\to\infty} G_n/B_n = 0$, since G_n is only of polynomial growth.

In many of the following corollaries f_n will be restricted so as to have N_0 small (typically in the range $1 \leq N_0 \leq 3$), but of course there are f_n for which this is not the case but for which the results claimed in the corollaries hold.

(2) One approach is to take the polynomial G_n as given and search for polynomials a_n and b_n satisfying equation (2.2). It can be assumed that degree(a_n), degree(b_n) < degree(G_n). This follows since if a solution exists with degree(a_n) $\geq$ degree(G_n) then the Euclidean algorithm can be used to write $a_n = p_nG_{n-1} + a'_n$, $b_n = q_nG_{n-2} + b'_n$, where p_n, q_n, a'_n and b'_n are polynomials in n. Substituting into (2.2) and comparing degrees gives that (2.2) holds with a_n replaced with a'_n and b_n replaced with b'_n.

(3) In theory it is possible to find polynomials G_n of arbitrarily high degree and polynomials a_n and b_n of lesser degree (with *rational* coefficients) satisfying (2.2), by using (2.2) to define equations expressing the coefficients of a_n and b_n in terms of those of G_n. If G_n has degree k and a_n and b_n both have degree $k-1$, then (2.2) is a polynomial identity of degree $2k-1$, giving $2k$ equations for the $2k$ coefficients of a_n and b_n ([2]).

In practice these equations and the requirement that the coefficients of G_n be integers introduces conditions on the coefficients of G_n. For example, if there exist $G_n = an^2 + bn + c$, $a_n = dn + e$, and $b_n = fn + g$, polynomials

([2]) Starting with a_n and b_n, arbitrary polynomials of a certain degree, it is possible to look for solutions G_n satisfying (2.2) with coefficients defined in terms of those of a_n and b_n using the Hyper Algorithm (see [8]). However there is no certainty that the solutions (if they exist) will be polynomials or that they will have any particular desired degree.

with integral coefficients, satisfying (2.2), then

$$\begin{aligned} d = -f &= \frac{4a^2}{a^2 - b^2 + 4ac}, \\ e &= \frac{-3a^2 + b^2 + 2ab - 4ac}{a^2 - b^2 + 4ac}, \\ g &= \frac{12a^2 - 2ab - 2b^2 + 8ac}{a^2 - b^2 + 4ac}, \end{aligned}$$

giving restrictions on the allowable values of a, b and c.

(Parts (ii)–(ix) of the following corollary correspond, respectively, to the solutions $\{a = b = m,\, c = 1\}$, $\{a = 1,\, b = 4,\, c = 4\}$, $\{a = m^2,\, b = 3m^2 - 2m,\, c = 2m^2 - 2m + 1\}$, $\{a = m^2,\, b = m^2 + 2m,\, c = 2m + 1\}$, $\{a = m,\, b = 3m,\, c = 2m + 1\}$, $\{a = m,\, b = m - 2,\, c = -1\}$, $\{a = m,\, b = 3m + 2,\, c = 2m + 3\}$ and $\{a = 4m,\, b = 16m^2 + 8m + 1,\, c = 16m^3 + 16m^2 + 5m + 1\}$.)

Proposition 1 is too general to easily calculate the limit of particular polynomial continued fractions. The following corollary enables these limits to be calculated explicitly in many particular cases.

Corollary 5. *Let m be a positive integer, k a positive integer greater than m and $\{f_n\}_{n=1}^{\infty}$ a non-constant polynomial sequence such that $f_n \geq 1$ for $n \geq 1$. For each of continued fractions below assume that f_n is such that no numerator partial quotient is equal to zero. (This holds automatically in cases* (i)–(vi).)

(i)
$$\operatorname*{K}_{n=1}^{\infty} \frac{(mn + k - m)f_n - 1}{(mn + k - 2m)f_n - 2} = \frac{k}{k - m}.$$

(ii)
$$\operatorname*{K}_{n=1}^{\infty} \frac{((n^2 - n)m + 1)f_n + nm - 1}{((n^2 - 3n + 2)m + 1)f_n + mn - (2m + 2)} = 1.$$

(iii)
$$\operatorname*{K}_{n=1}^{\infty} \frac{(n + 1)^2 f_n + 4n + 5}{n^2 f_n + 4n - 4} = 4.$$

(iv)
$$\operatorname*{K}_{n=1}^{\infty} \frac{f_n(m^2n^2 + n(m^2 - 2m) + 1) + mn + m - 2}{f_n(m^2n^2 - n(m^2 + 2m) + 2m + 1) + mn - m - 3} = 2m^2 - 2m + 1.$$

(v)
$$\operatorname*{K}_{n=1}^{\infty} \frac{f_n(m^2n^2 + n(2m - m^2) + 1) + mn}{f_n(m^2n^2 - n(3m^2 - 2m) + 2m^2 + 2m + 1) + mn - 2m - 1} = 2m + 1.$$

(vi)
$$\operatorname*{K}_{n=1}^{\infty} \frac{f_n(n(n + 1)m + 1) + mn + m - 1}{f_n(n(n - 1)m + 1) + mn - m - 2} = 2m + 1.$$

(vii) *Let A_n/B_n denote the convergents to the continued fraction below and suppose* $\lim_{n\to\infty} n^2/B_n = 0$. *Then*

$$\mathop{\mathrm{K}}_{n=1}^{\infty} \frac{f_n((n-1)^2m+(m-2)(n-1)-1)-m^2n+m-1}{f_n((n-2)^2m+(m-2)(n-2)-1)-m^2(n-2)+m-2} = -1.$$

(viii) $$\mathop{\mathrm{K}}_{n=1}^{\infty} \frac{f_n(n(n+1)m+2n+1)-(m^2(n+1)+m+1)}{f_n(n(n-1)m+2n-1)-(m^2(n-1)+m+2)} = 2m+3.$$

(ix) $$\mathop{\mathrm{K}}_{n=1}^{\infty} \frac{-1-8m-32m^2-128m^3-64m^2n+ +(m+16m^3+(1+16m^2)n+4mn^2)f_n}{-2-8m+96m^2-128m^3-64m^2n+ +(-1+5m-16m^2+16m^3+(1-8m+16m^2)n+4mn^2)f_n} = \frac{1+5m+16m^2+16m^3}{m+16m^3}.$$

Proof. In each case below an easy check shows that with the given choices for $\{G_n\}_{n=-1}^{\infty}$, $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$: equation (2.2) holds; the continued fraction in question corresponds to the continued fraction $\mathrm{K}_{n=1}^{\infty} s_n/t_n$ of Proposition 1; if $\{A_n/B_n\}$ are the convergents to this continued fraction then $\lim_{n\to\infty} G_n/B_n = 0$; and (by fact or assumption) no s_n is 0. Finally, the limit of the continued fraction is G_0/G_{-1}. The fact that some early partial quotients may be negative does not affect any of the results—a tail of the continued fraction will have all terms positive so that $\lim_{n\to\infty} G_n/B_n = 0$ will hold for the tail which will then converge and the continued fraction will then collapse from the bottom up to give the result.

Remark. In some cases the result holds if f_n is a *constant* polynomial such that $f_n \geq 1$ for $n \geq 1$.

(i) Let $G_n = mn+k$, $a_n = -1$ and $b_n = 2$.

(ii) Let $G_n = n(n+1)m+1$, $a_n = mn-1$ and $b_n = -mn+(2m+2)$.

(iii) Let $G_n = (n+2)^2$, $a_n = 4n+5$ and $b_n = -4n+4$.

(iv) Let $G_n = m^2n^2+n(3m^2-2m)+2m^2-2m+1$, $a_n = mn+m-2$ and $b_n = -mn+m+3$.

(v) Let $G_n = n(n+1)m^2+2m(n+1)+1$, $a_n = mn$ and $b_n = -mn+2m+1$.

(vi) Let $G_n = (n+2)(n+1)m+1$, $a_n = mn+m-1$ and $b_n = -mn+m+2$.

(vii) Let $G_n = mn^2+(m-2)n-1$, $a_n = -m^2n+m-1$ and $b_n = m^2n-2m^2-m+2$.

(viii) Let $G_n = (n+2)(n+1)m+2n+3$, $a_n = -(m^2n+m^2+m+1)$ and $b_n = m^2(n-1)+m+2$.

(ix) Let $G_n = 1+5m+16m^2+16m^3+(1+8m+16m^2)n+4mn^2$, $a_n = -1-8m-32m^2-128m^3-64m^2n$ and $b_n = 2+8m-96m^2+128m^3+64m^2n$. ∎

Examples. 1. Letting $m = 5$ and $f_n = 10n^8$ in (ii) above gives

$$\operatorname*{K}_{n=1}^{\infty} \frac{((n^2-n)5+1)10n^8+5n-1}{((n^2-3n+2)5+1)10n^8+5n-12} = 1.$$

2. Also in (ii), letting $f_n = n^8$ and m be an arbitrary positive integer gives

$$\operatorname*{K}_{n=1}^{\infty} \frac{((n^2-n)m+1)n^8+nm-1}{((n^2-3n+2)m+1)n^8+mn-(2m+2)} = 1.$$

3. Letting $f_n = 2n^5$ and $m = 3$ in (vi) above gives (1.10) of the introduction. Similarly, letting $f_n = n^{10}$ gives (1.9) of the introduction.

4. In (vii) above a general class of examples may be obtained by choosing $m > 1$ and $f_n > nm^2$ for $n \geq 1$. With the notation of the proposition it can easily be seen that $s_n, t_n \geq 1$ for $n \geq 3$. If f_n is such that B_2 and B_3 are negative, then B_n will be negative for all $n \geq 2$ and by a similar argument to the reasoning behind condition (ii), it will follow that $\lim_{n\to\infty} n^2/B_n = 0$ and the conditions of the corollary will be satisfied. For example, letting $f_n = 16n$ and $m = 3$ gives

$$\operatorname*{K}_{n=1}^{\infty} \frac{48n^3-80n^2+7n+2}{48n^3-176n^2+135n+19} = -1.$$

All the examples in the last corollary were derived from solutions to equation (2.2) where G_n had degree 2. Table 1 below gives several families of solutions to equation (2.2), where G_n is of degree 3 in n.

Table 1. Some infinite families of solutions to (2.2) for G_n of degree 3

G_n	a_n	b_n
$(n^2-1)mn+1$	$2mn(n-1)-1$	$-2m(n^2-4n+3)+2$
$(2n^2+3n+1)mn+1$	$-2n^2m(m-2)+m^2n$ $+m-1$	$-n^2(4m-2m^2)$ $-n(7m^2-12m)$ $+6m^2-7m+2$
$(n^2+3n+2)mn+1$	$2n(n+1)m-1$	$-2n(n-2)m+2$
$(n^3+6n^2+11n+6)m+1$	$2mn^2+6mn+4m-1$	$-2mn^2+2m+2$
mn^3+3mn^2 $+n(2m-3)-2$	$-m^2n^2-n(m^2+m)$ $+m-1$	$m^2n^2-n(2m^2-m)$ $-4m+2$

Considering the third and fourth row of entries in the Table 1, for example, there is the following corollary to Proposition 1:

Corollary 6. *Let f_n be a polynomial in n such that $f_n \geq 1$ for $n \geq 1$ and let m be a positive integer.*

(i) *If* $f(2) > 2$ *then*

$$\mathop{\mathrm{K}}_{n=1}^{\infty} \frac{f_n((n^2-1)nm+1)+2mn(n+1)-1}{f_n((n^2-3n+2)mn+1)+2mn(n-2)-2} = 1.$$

(ii) *We have*

$$\mathop{\mathrm{K}}_{n=1}^{\infty} \frac{f_n((n^2+3n+2)nm+1)+2mn^2+6mn+4m-1}{f_n((n^2-1)nm+1)+2(n^2-1)m-2} = 6m+1.$$

Proof. (i) In the light of the fact that G_n, a_n and b_n satisfy (2.2) simply note that the numerator of the continued fraction is $f_n G_{n-1} + a_n$ and that the denominator is $f_n G_{n-2} - b_n$. It is easily seen that $a_n \geq 1$ for all $n \geq 1$ and that $b_n \geq 1$ for all $n \geq 2$. It can also be shown that B_2 and B_3 are positive for all m and f_n satisfying the conditions of the corollary. In the light of what was said in an earlier remark this is sufficient to ensure the result.

(ii) The proof of this follows the same lines as that of (i) above. ■

Taking f_n to be n^3 and $m = 3$ in part (i) gives (1.11) of the introduction.

One could continue to prove similar results by finding other solutions to equation (2.2) for degrees 2 or 3 or by going to higher degrees, but these corollaries should be sufficient to illustrate the principle at work.

3. Infinite polynomial continued fractions with irrational limits. In this section we use a continued fraction-to-series transformation equivalent to Euler's transformation to sum some polynomial continued fractions with irrational limits.

THEOREM 2. *For* $N \geq 1$,

$$b_0 + \mathop{\mathrm{K}}_{n=1}^{N} \frac{b_{n-1}x}{b_n - x} = \frac{1}{\displaystyle\sum_{n=0}^{N} \frac{(-1)^n x^n}{\prod_{i=0}^{n} b_i}}. \tag{3.1}$$

Thus, when $N \to \infty$, *the continued fraction converges if and only if the series converges.*

Proof. See, for example, Chrystal [2, p. 516, equation (14)]. ■

REMARK. The irrationality criterion mentioned in the introduction means that if $\{b_n\}_{n=0}^{\infty}$ is a sequence of integers, then the number $\sum_{n=0}^{\infty}(-1)^n x^n/(b_0 b_1 \ldots b_n)$ is not rational for $x = 1/m$, m being a non-zero integer, provided $|mb_n - 1| \geq |mb_{n-1}| + 1$ for all n sufficiently large.

Corollary 7. *For all non-zero integers m (and indeed for all non-zero real numbers m),*

$$6m^2 - 1 + \operatorname*{K}_{n=1}^{\infty} \frac{m^2(4n^2+2n)}{m^2(4n^2+10n+6)-1} = \frac{1}{(1/m)\csc(1/m)-1}.$$

Remarks. (1) Glaisher [4] states continued fraction expansions essentially equivalent to this one and the one in the next corollary.

(2) The irrationality criterion shows that $\sin(1/m)$ is irrational for m either a non-zero integer or the square root of a positive integer.

Proof of Corollary 7. In Theorem 2 let $b_n = (2n+2)(2n+3)m^2$ and $x = 1$. ∎

Corollary 8. *For all non-zero integers m (and indeed for all non-zero real numbers m),*

$$2m^2 + \operatorname*{K}_{n=1}^{\infty} \frac{m^2(4n^2-2n)}{m^2(4n^2+6n+2)-1} = \frac{1}{1-\cos(1/m)}.$$

Note that the irrationality criterion shows that $\cos(1/m)$ is irrational for m either a non-zero integer or the square root of a positive integer.

Proof of Corollary 8. In Theorem 2 let $b_n = (2n+1)(2n+2)m^2$ and $x = 1$. ∎

Corollary 9. *For all positive integers ν and all non-zero integers m (and indeed for all non-zero real numbers m),*

$$(\nu+1)4m^2 + \operatorname*{K}_{n=1}^{\infty} \frac{4m^2 n(n+\nu)}{4m^2(n+1)(n+\nu+1)-1} = \frac{1}{1-(\nu)!(2m)^{\nu}J_{\nu}(1/m)},$$

where $J_{\nu}(x)$ is the Bessel function of the first kind of order ν.

The Tietze irrationality criterion shows that if ν is a non-negative integer and m is a non-zero integer or the square root of a positive integer then $J_{\nu}(\pm 1/m)$ is irrational.

Proof of Corollary 9. In Theorem 2, letting $b_n = 4(n+1)(\nu+n+1)m^2$ and $x = 1$ gives

$$4(\nu+1)m^2 + \operatorname*{K}_{n=1}^{\infty} \frac{n(\nu+n)4m^2}{(n+1)(n+\nu+1)4m^2-1}$$
$$= \frac{1}{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(1/m)^{2n+2}}{\prod_{i=0}^{n} 4(i+1)(\nu+i+1)}} = \frac{1}{1-(\nu)!(2m)^{\nu}J_{\nu}(1/m)},$$

from the power series expansion for $J_{\nu}(x)$. ∎

Taking ν to be 0 and $m = 2$ gives (1.7) of the introduction. (1.8) of the introduction follows by letting $m = 1/\sqrt{2}$ in Corollary 7.

COROLLARY 10. *For all non-zero integers m (and indeed for all non-zero real numbers m),*

$$1 + \cfrac{1}{6m^3 - 1 + \underset{n=2}{\overset{\infty}{\mathrm{K}}} \dfrac{(3n-5)(3n-4)(3n-3)m^3}{(3n-2)(3n-1)(3n)m^3 - 1}} = \left(\frac{1}{3}\exp(-1/m) + \frac{2}{3}\exp(1/(2m))\cos(\sqrt{3}/(2m))\right)^{-1}.$$

Proof. In Theorem 2 let $b_0 = 1$, $b_n = (3n-2)(3n-1)(3n)$ for $n \geq 1$ and $x = 1/m^3$. Then

$$1 + \cfrac{1/m^3}{6 - 1/m^3 + \underset{n=2}{\overset{\infty}{\mathrm{K}}} \dfrac{(3n-5)(3n-4)(3n-3)(1/m^3)}{(3n-2)(3n-1)(3n) - 1/m^3}} = \frac{1}{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(1/m^{3n})}{(3n)!}}.$$

Simplifying the continued fraction gives the left side and finally the right side equals $\left(\frac{1}{3}\exp(-1/m) + \frac{2}{3}\exp(1/(2m))\cos(\sqrt{3}/(2m))\right)^{-1}$. ∎

By the Tietze criterion the irrationality of this last function follows when m is a non-zero integer or the real cube root of a non-zero integer.

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