ON THE DIVERGENCE OF THE ROGERS-RAMANUJAN CONTINUED FRACTION ON THE UNIT CIRCLE

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Abstract. Let the continued fraction expansion of any irrational number $t \in (0,1)$ be denoted by $[0, a_1(t), a_2(t), \cdots]$ and let the i-th convergent of this continued fraction expansion be denoted by $c_i(t)/d_i(t)$. Let

$$S = \{t \in (0,1) : a_{i+1}(t) \geq \phi^{d_i(t)} \text{ infinitely often}\},$$

where $\phi = (\sqrt{5}+1)/2$. Let $Y_S = \{\exp(2\pi i t) : t \in S\}$. It is shown that if $y \in Y_S$ then the Rogers-Ramanujan continued fraction, $R(y)$, diverges at y. S is an uncountable set of measure zero. Moreover it is shown that there is an uncountable set of points, $G \subset Y_S$, such that if $y \in G$, then $R(y)$ does not converge generally.

It is further shown that $R(y)$ does not converge generally for $|y| > 1$. However we show that $R(y)$ does converge generally if y is a primitive $5m$-th root of unity, $m \in \mathbb{N}$, so that by a theorem of I. Schur, the Rogers-Ramanujan continued fraction converges generally at all roots of unity.

1. Introduction

The Rogers-Ramanujan continued fraction, $R(x)$, is defined as follows:

$$R(x) := \cfrac{x^{\frac{1}{5}}}{1 + \cfrac{x}{1 + \cfrac{x^2}{1 + \cfrac{x^3}{1 + \ddots}}}}.$$

Put $K(x) = x^{\frac{1}{5}}/R(x)$. This continued fraction seems to have been first investigated by L.J. Rogers in 1894 ([8]) and rediscovered by Ramanujan, sometime before 1913.

It is an easy consequence of Worpitsky's theorem (see [5]) that $R(x)$ converges to values in $\hat{\mathbb{C}}$ for any x inside the unit circle. In fact, many explicit evaluations of $R(e^{-\pi\sqrt{n}})$ and $R(-e^{-\pi\sqrt{n}})$ have been given for $n \in \mathbb{Q}^+$ (see,

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for example, [2], [3], [6] and [10]), some of which were asserted by Ramanujan without proof.

It was stated by Ramanujan, and proved in [1], that if $|x| > 1$ then the odd and even convergents tend to different limits.

This leaves the question of convergence on the unit circle. Issai Schur showed in [9] that if x is a primitive m-th root of unity, where $m \equiv 0 \pmod 5$, then $K(x)$ diverges and if x is a primitive m-th root of unity, $m \not\equiv 0 \pmod 5$, then $K(x)$ converges and

$$K(x) = \lambda x^{\frac{1-\lambda\sigma m}{5}} K(\lambda), \tag{1.1}$$

where $\lambda = \left(\frac{m}{5}\right)$ (the Legendre symbol) and σ is the least positive residue of $m \pmod 5$. Note that $K(1) = \phi = (\sqrt{5}+1)/2$, and $K(-1) = 1/\phi$. It follows that $R(x)$ takes only ten possible values at roots of unity. For later use we define

$$R_j = \begin{cases} -\phi \exp\left(2\pi i j/5\right), & 1 \leq j \leq 5, \\ \dfrac{\exp(2\pi i j/5)}{\phi}, & 6 \leq j \leq 10. \end{cases} \tag{1.2}$$

Remark: $\{R_j\}_{j=1}^{10}$ consists of the ten values taken by $R(x)$ at roots of unity.

Since Schur's paper it has been an open problem whether $K(x)$ converges or diverges at any point x on the unit circle which is not a root of unity.

Let the regular continued fraction expansion of any irrational $t \in (0,1)$ be denoted by $[0, a_1(t), a_2(t), \cdots]$. Let the i-th convergent of this continued fraction expansion be denoted by $c_i(t)/d_i(t)$. Occasionally for brevity when there is no danger of confusion we write a_i for $a_i(t)$, d_i for $d_i(t)$ etc. In this paper we prove the following theorem.

Theorem 1. *Let*

$$S = \{t \in (0,1) : a_{i+1}(t) \geq \phi^{d_i(t)} \text{ infinitely often}\}. \tag{1.3}$$

Then S is an uncountable set of measure zero and, if $t \in S$ and $y = \exp(2\pi i t)$, then $K(y)$ diverges.

As an example of a point in S, we give the following corollary to Theorem 1.

Corollary 1. *Let t be the number with continued fraction expansion equal $[0, a_1, a_2, \cdots]$, where a_i is the integer consisting of a tower of i twos with an i an top.*

$$t = [0, 2, 2^{2^2}, 2^{2^{2^3}}, \cdots] =$$
$$0.48484848484848484848484848484848484848484848484848484848484$$
$$8484848484848484848484849277885083112437522992318812011\cdots$$

If $y = \exp(2\pi i t)$ then $K(y)$ diverges.

In [4], Jacobson introduced the concept of general convergence for continued fractions. General convergence is defined in [5] as follows.

Let the n-th convergent of the continued fraction

$$M = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}}$$

be denoted by A_n/B_n and let

$$S_n(w) = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots \cfrac{a_n}{b_n + w}}}} = \frac{A_n + wA_{n-1}}{B_n + wB_{n-1}}.$$

Define

$$d(w, z) = \frac{|z - w|}{\sqrt{1 + |w|^2}\sqrt{1 + |z|^2}}, \tag{1.4}$$

if w and z are both finite, and

$$d(w, \infty) = \frac{1}{\sqrt{1 + |w|^2}}.$$

Definition: M is said to converge generally to $f \in \hat{\mathbb{C}}$ if there exist sequences $\{v_n\}$, $\{w_n\} \subset \hat{\mathbb{C}}$ such that $\liminf d(v_n, w_n) > 0$ and

$$\lim_{n\to\infty} S_n(v_n) = \lim_{n\to\infty} S_n(w_n) = f.$$

If a continued fraction converges generally, then it does, in a certain sense, to the "right" value. More precisely, for $n = 0, 1, 2, \cdots$, let

$$h_n = -S_n^{-1}(\infty).$$

We use the following theorem from [5].

Theorem 2. *The continued fraction $b_0 + K(a_n/b_n)$ converges generally to f if and only if $\lim S_n(u_n) = f$ for every sequence $\{u_n\}$ from $\hat{\mathbb{C}}$ such that*

$$\liminf_{n\to\infty} d(u_n, -h_n) > 0 \qquad \text{if } f \neq \infty,$$
$$\liminf_{n\to\infty} d(u_n, -A_n/A_{n-1}) > 0 \qquad \text{if } f = \infty.$$

This theorem in turn has the following important corollary (also from [5]).

Corollary 2. *Let $b_0 + K(a_n/b_n)$ converge generally to f and to g. Then $f = g$.*

Classical convergence implies general convergence (take $u_n = 0$ and $v_n = \infty$, for all n), but not conversely. Thus general convergence is a natural extension of classical covergence.

As Schur showed in [9], $K(x)$ does not converge in the classical sense when x is an m-th root of unity, where $m \equiv 0 (\text{mod } 5)$. However $K(x)$ can be shown to converge generally in this case. We have the following proposition.

Proposition 1. *If x is an m-th root of unity, where $m \equiv 0(mod\ 5)$, then $K(x)$ converges generally.*

Taking Proposition 1 along with Schur's theorem shows that $K(x)$ converges generally at any root of unity.

This suggests the question of general convergence at points on the unit circle which are not roots of unity. We have the following theorem.

Theorem 3. *Let t be any irrational in $(0,1)$ for which there exist two subsequences of convergents $\{c_{f_n}/d_{f_n}\}$ and $\{c_{g_n}/d_{g_n}\}$ and integers r, $u \in \{0,1,2,3,4\}$, integers s, $v \in \{1,2,3,4\}$ such that*

$$\begin{aligned} c_{f_n} &\equiv r(mod\ 5), & c_{g_n} &\equiv u(mod\ 5), \\ d_{f_n} &\equiv s(mod\ 5), & d_{g_n} &\equiv v(mod\ 5). \end{aligned} \tag{1.5}$$

and

$$a_{h_n+1} > 2\pi(d_{h_n}+1)^2 \phi^{d_{h_n}^2+2d_{h_n}}, \tag{1.6}$$

for all n, where $h_n = f_n$ or g_n.

Suppose further that

$$R(\exp(2\pi i r/s)) = R_a \neq R_b = R(\exp(2\pi i u/v)), \tag{1.7}$$

for some a, $b \in \{1,2,\cdots,10\}$.

Let $S^{\diamond}$ denote the set of all $t \in (0,1)$ satisfying (1.5), (1.6) and (1.7) and set

$$G = \{\exp(2\pi i t) : t \in S^{\diamond}\}. \tag{1.8}$$

Then G is an uncountable set of measure zero such that if $y \in G$, then $K(y)$ does not converge generally.

Remark: It follows from (1.6) that $S^{\diamond} \subset S$. Once again it is possible to give explicit examples of points y for which $K(y)$ does not converge generally and in Corollary 4 we show that $K(y)$ does not converge generally for the the point y in Corollary 1.

An interesting question is what forms can divergence take. In fact there are uncountably many points y on the unit circle such that $R(y)$ has subsequences of convergents tending to all ten of the R_j's defined by (1.2). We prove the following proposition.

Proposition 2. *There exists an uncountable subset* $G^* \subset Y_S$ *such that if* $y \in G^*$ *then there exist ten sequences of positive integers,* $\left\{n_{i,j}\right\}_{i=1}^{\infty}$, $1 \leq j \leq 10$, *say, such that* $\lim_{i\to\infty} R_{n_{i,j}}(y) = R_j$.

This proposition is not strictly necessary for the proof of Theorem 3 but we find the existence of the set G^* to be of interest. It is possible to give explicit examples of such points in G^*. We have the following corollary to Proposition 2.

Corollary 3. *Let the sequence of integers* $\{a_i\}_{i=1}^{\infty}$ *be defined by*

$$\{a_1, a_2, \cdots\} = \{0, 2, \overline{1, 2, 1, 0, 0, 1, 2, 1, 0, 2, 2, 4}\},$$

where the bar indicates that the terms under it repeat infinitely often. Let t *be the number with continued fraction expansion given by*

$$t = [0, g_1 + a_1, g_2 + a_2, g_3 + a_3, \cdots],$$

where g_i *is the integer consisting of a tower of* i *sixteens with an* i *an top and the* a_i*'s are as above.*

$$t = [0, 16, 16^{16^2} + 2, 16^{16^{16^3}} + 1, \cdots] =$$

$$\begin{aligned}&0.06249999999999999999999999999999999999999999999999999999999\\&9999999999999999999999999999999999999999999999999999999999999\\&9999999999999999999999999999999999999999999999999999999999999\\&9999999999999999999999999999999999999999999999999999999999999\\&9999999999999999999999999999999999999999999999999999999999999\\&9999999782707631005156114932594461198007415603592189407975407\\&\qquad 1725266194446089419127033011861051603999\cdots.\end{aligned}$$

If $y = \exp(2\pi i t)$ *then* $R(y)$ *has subsequences of convergents tending to all ten values taken by the Rogers-Ramanujan continued fraction at roots of unity.*

We also consider the question of general convergence outside the unit circle. It was proved in [1] that if $0 < |x| < 1$ then the odd convergents of $1/K(1/x)$ tend to

$$1 - \frac{x}{1} + \frac{x^2}{1} - \frac{x^3}{1} + \cdots := F_1(x) \tag{1.9}$$

while the even convergents tend to

$$\frac{x}{1} + \frac{x^4}{1} + \frac{x^8}{1} + \frac{x^{12}}{1} + \cdots := F_2(x). \tag{1.10}$$

Worpitsky's theorem gives that each continued fraction converges inside the unit circle to values in $\hat{\mathbb{C}}$. It is not clear that $F_1(x) \neq F_2(x)$ for *all* x inside the unit circle but Worpitsky's theorem again gives that $F_1(x) \neq F_2(x)$ for $|x| < 1/4$ and for such x we have the following proposition which implies that the Rogers-Ramanujan continued fraction does not converge generally at $1/x$.

Proposition 3. *Let $C = b_0 + K_{n=1}^{\infty} a_n/b_n$ be such that the odd and even convergents tend to different limits. Further suppose that there exist positive constants c_1, c_2 and c_3 such that, for $i \geq 1$,*

$$c_1 \leq |b_i| \leq c_2, \tag{1.11}$$

and

$$\left| \frac{a_{2i+1}}{a_{2i}} \right| \leq c_3. \tag{1.12}$$

Then C does not converge generally.

2. Divergence In The Classical Sense

Let

$$K_n(x) := 1 + K_{j=1}^{n} \frac{x^j}{1} = \frac{P_n(x)}{Q_n(x)}$$

denote the n-th convergent of $K(x)$ and let $R_n(x) = x^{\frac{1}{5}}/K_n(x)$. It is elementary that

$$Q_{n+1}(x) = Q_n(x) + x^{n+1} Q_{n-1}(x). \tag{2.1}$$

It can also easily be checked that if $|x| = 1$, then for $n \geq 1$,

$$|P_n(x) Q_{n-1}(x) - Q_n(x) P_{n-1}(x)| = 1. \tag{2.2}$$

It follows easily from the triangle inequality that, for $n \geq 2$,

$$|Q_n(x)| \leq F_{n+1}. \tag{2.3}$$

where $\{F_i\}_{i=1}^{\infty}$ denotes the Fibonacci sequence defined by $F_1 = F_2 = 1$ and $F_{i+1} = F_i + F_{i-1}$.

Suppose $\lim_{n \to \infty} P_n(y)/Q_n(y) = L \in \mathbb{C}$ for some y on the unit circle. Then, by (2.2),

$$\begin{aligned}\frac{1}{|Q_n(y)Q_{n-1}(y)|} &= \left|\frac{P_n(y)}{Q_n(y)} - \frac{P_{n-1}(y)}{Q_{n-1}(y)}\right| \\ &\leq \left|\frac{P_n(y)}{Q_n(y)} - L\right| + \left|\frac{P_{n-1}(y)}{Q_{n-1}(y)} - L\right|.\end{aligned}$$

Thus $\lim_{n\to\infty}|Q_n(y)Q_{n-1}(y)| = \infty$. We will exhibit an uncountable set of points, of measure zero, on the unit circle such that if y is one of these points then $\lim_{n\to\infty}|Q_n(y)Q_{n-1}(y)| \neq \infty$ so that $K(y)$ does not converge.

Lemma 1. *With the notation of Theorem1, for $t \in S$, we have*

$$\left|t - \frac{c_i(t)}{d_i(t)}\right| < \frac{1}{d_i(t)^2\phi^{d_i(t)}}$$

for infinitely many i.

Proof. Let i be one of the infinitely many integers for which $a_{i+1}(t) \geq \phi^{d_i(t)}$ and let $t_{i+1} = [a_{i+1}(t), a_{i+2}(t), \cdots]$ denote the i-th tail of the continued fraction expansion for t. Then

$$\begin{aligned}\left|t - \frac{c_i(t)}{d_i(t)}\right| &= \left|\frac{t_{i+1}c_i(t) + c_{i-1}(t)}{t_{i+1}d_i(t) + d_{i-1}(t)} - \frac{c_i(t)}{d_i(t)}\right| \\ &= \frac{1}{d_i(t)(t_{i+1}d_i(t) + d_{i-1}(t))} \\ &< \frac{1}{d_i(t)(a_{i+1}(t)d_i(t) + d_{i-1}(t))} < \frac{1}{d_i(t)^2\phi^{d_i(t)}}.\end{aligned}$$

□

Lemma 2. *Let x and y be two points on the unit circle. Then, for all integers $n \geq 0$,*

$$|Q_n(x) - Q_n(y)| \leq n^2\phi^n|x-y|. \tag{2.4}$$

and

$$|P_n(x) - P_n(y)| \leq (n+1)^2\phi^{n+1}|x-y|. \tag{2.5}$$

Proof. The assertions of the lemma can easily be checked for $n = 0, 1$.

Let $\beta_i = |Q_i(x) - Q_i(y)|$ and $\delta_i = (i+1)F_i|x-y|$. Using (2.1) and (2.3) it easily follows that

$$\beta_n \leq \beta_{n-1} + \beta_{n-2} + \delta_{n-1}. \tag{2.6}$$

Iterating this last inequality downwards gives that, for $r = 2, \cdots, n-1$,

$$\beta_n \leq F_r\beta_{n-r+1} + F_{r-1}\beta_{n-r} + \sum_{i=1}^{r-1} F_i\delta_{n-i}. \tag{2.7}$$

The claim is true for $r = 2$ by (2.6). Suppose it is true for $r = 2, \cdots, s$. Then

$$\begin{aligned}\beta_n &\leq F_s \beta_{n-s+1} + F_{s-1}\beta_{n-s} + \sum_{i=1}^{s-1} F_i \delta_{n-i} \\ &\leq F_s(\beta_{n-s} + \beta_{n-s-1} + \delta_{n-s}) + F_{s-1}\beta_{n-s} + \sum_{i=1}^{s-1} F_i \delta_{n-i} \\ &= F_{s+1}\beta_{n-s} + F_s \beta_{n-s-1} + \sum_{i=1}^{s} F_i \delta_{n-i}\end{aligned}$$

and (2.7) is true by induction for $2 \leq r \leq n-1$.

Recall that $\beta_1 = 0$ and $\beta_2 = |(1+x^2) - (1+y^2)| \leq 2|x-y|$. Now in (2.7) let $r = n-1$. This gives

$$\beta_n \leq 2F_{n-1}|x-y| + \sum_{i=1}^{n-2} F_i \delta_{n-i} = \sum_{i=1}^{n-1} F_i (n-i+1) F_{n-i}|x-y|$$
$$\leq \sum_{i=1}^{n-1} \phi^n (n-i+1)|x-y|,$$

using the bound $F_j \leq \phi^j$. This last expression simplifies to

$$\phi^n |x-y| \sum_{i=2}^{n} i < n^2 \phi^n |x-y|.$$

(2.5) follows similarly.

□

To show our set has measure zero, we use the following lemma.

Lemma 3. ([7]) *Let $f(n) > 1$ for $n = 1, 2, \cdots$ and suppose $\sum_{n=1}^{\infty} 1/f(n) < \infty$. Then the set $S^* = \{t \in (0,1) : a_k(t) > f(k)$ infinitely often $\}$ has measure zero.*

Suppose x is a primitive m-th root of unity. In proving the lemma we will use Table 1 which gives values for $P_{m-2}(x)$, $P_{m-1}(x)$, $Q_{m-2}(x)$ and $Q_{m-1}(x)$. These values are from [9].

Proof of Theorem 1: Let $t \in S$ with convergents $\{c_n/d_n\}_{n=0}^{\infty}$. Let $y = \exp(2\pi i t)$ and let $x_n = \exp(2\pi i\, c_n/d_n)$. By Table 1,

$$\max\{\ |Q_{d_n-1}(x_n)|,\ \ |Q_{d_n-2}(x_n)|\ \} \leq 2. \tag{2.8}$$

m	P_{m-2}	P_{m-1}	Q_{m-2}	Q_{m-1}
5μ	0	$-x^{\frac{2m}{5}} - x^{\frac{-2m}{5}}$	$-x^{\frac{2m}{5}} - x^{\frac{-2m}{5}}$	0
$5\mu+1$	$x^{\frac{1-m}{5}}$	1	0	$x^{\frac{-1+m}{5}}$
$5\mu-1$	$x^{\frac{1+m}{5}}$	1	0	$x^{\frac{-1-m}{5}}$
$5\mu+2$	$-x^{\frac{1+2m}{5}}$	0	1	$-x^{\frac{-1-2m}{5}}$
$5\mu-2$	$-x^{\frac{1-2m}{5}}$	0	1	$-x^{\frac{-1+2m}{5}}$

TABLE 1

Using in turn, Lemma 2, the fact that chord length is shorter than arc length, and Lemma 1, it follows that, for infinitely many n,

$$
\begin{aligned}
(2.9) \qquad |Q_{d_n-1}(x_n) - Q_{d_n-1}(y)| &\leq (d_n-1)^2\phi^{d_n-1}|x_n - y| \\
&< (d_n-1)^2\phi^{d_n-1}2\pi\left|\frac{c_n}{d_n} - t\right| \\
&< \left(\frac{d_n-1}{d_n}\right)^2\frac{2\pi}{\phi} < 4.
\end{aligned}
$$

Similarly,

$$
(2.10) \qquad |Q_{d_n-2}(x_n) - Q_{d_n-2}(y)| < \left(\frac{d_n-2}{d_n}\right)^2\frac{2\pi}{\phi^2} < 4.
$$

Applying the triangle inequality to (2.9) and (2.10) and using (2.8) gives $|Q_{d_n-1}(y)| < 6$ and $|Q_{d_n-2}(y)| < 6$. Finally, we have that

$$
|Q_{d_n-1}(y)Q_{d_n-2}(y)| < 36.
$$

Since this holds for infinitely many terms of the sequence $\{d_n\}_{n=0}^{\infty}$ it follows that $\lim_{n\to\infty} Q_n(y)Q_{n-1}(y) \neq \infty$ and thus $K(y)$ does not converge.

We next show that S has measure zero (it is clearly an uncountable set). Let $f(i) = \phi^{F_i}$, $i = 1, 2, \cdots$. Then it follows that $\sum_{i=1}^{\infty} 1/f(i) < \infty$. Let

$$
S^* = \{t \in (0,1) : a_{i+1}(t) \geq \phi^{F_{i+1}} \text{ infinitely often}\}
$$

so by Lemma(3) S^* has measure zero.

Recall $S = \{t \in (0,1) : a_{i+1}(t) \geq \phi^{d_i(t)} \text{ infinitely often}\}$. $S \subset S^*$, since $d_i(t) \geq F_{i+1}$, and thus S, being a subset of a set of measure zero, has measure zero.

□

Proof of Corollary 1: Denote the i-th convergent of the continued fraction expansion of t by c_i/d_i. We will show that, for $i = 1, 2, \cdots,$

$$a_{i+1} \geq 2^{d_i} > \phi^{d_i}. \tag{2.11}$$

Then $K(y)$ will diverge by Theorem 1.

$$2^{d_i} \leq a_{i+1} \Longleftrightarrow d_i \leq \underbrace{2^{\cdot^{\cdot^{\cdot^{2^{i+1}}}}}}_{i \text{ twos}}, \tag{2.12}$$

where the notation indicates that the last integer consists of a tower of i twos with an $i+1$ on top. It can be easily checked that the second inequality holds for $i = 1, 2$. Suppose it holds for for $i = 1, 2, \cdots, r-1$. Then

$$d_r = a_r d_{r-1} + d_{r-2} \leq \underbrace{2^{\cdot^{\cdot^{\cdot^{2^{r}}}}}}_{r \text{ twos}} \times \underbrace{2^{\cdot^{\cdot^{\cdot^{2^{r}}}}}}_{(r-1) \text{ twos}} + \underbrace{2^{\cdot^{\cdot^{\cdot^{2^{r-1}}}}}}_{(r-2) \text{ twos}}$$

$$\leq \underbrace{2^{\cdot^{\cdot^{\cdot^{2^{r+1}}}}}}_{r \text{ twos}}.$$

Thus the first inequality in (2.12) holds for all positive integers i and the result follows.

□

We will in fact show later that $K(y)$ does not converge generally when y has the value stated in the corollary above. Note that the convergents of the continued fraction expansion converge very fast to t – the third convergent agrees with t to over 10^{19700} decimal places!

Remark: It is possible to replace the set S by a similar set, say

$$S_\kappa = \{t \in (0,1) : a_{i+1}(t) \geq \kappa\, \phi^{d_i(t)} \text{ infinitely often}\},$$

where κ is a positive constant and Theorem 1 will still hold for all t in S_κ. However $\bigcup_{\kappa \in \mathbb{R}^+} S_\kappa$ will still have measure zero.

3. Divergence in the Generalized Sense

For ease of notation later, define $Y_S = \{\exp(2\pi i t) : t \in S\}$. We first prove the general convergence of $K(y)$ when y is a primitive $5m$-th root of unity, some $m \in \mathbb{N}$.

Proof of Proposition 1: From [9], for $0 \leq r < m$,

$$P_{qm+r} = P_r P_{m-1}^q, \quad Q_{qm+r} = Q_r Q_{m-2}^q. \tag{3.1}$$

From Table 1,

$$P_{m-1} = -x^{\frac{2m}{5}} - x^{-\frac{2m}{5}}, \quad Q_{m-2} = -x^{\frac{m}{5}} - x^{-\frac{m}{5}}.$$

Let $\{u_n\}_{n=1}^{\infty}$ be a sequence in $\hat{\mathbb{C}}$. It is convenient to split $n \in \mathbb{Z}^+$ into residue classes modulo m. We put $n = qm + r$. From (3.1),

(3.2)

$$S_n(u_n) = \frac{P_n + u_n P_{n-1}}{Q_n + u_n Q_{n-1}} = \begin{cases} \left(\frac{P_{m-1}}{Q_{m-2}}\right)^q \frac{P_r + u_n P_{r-1}}{Q_r + u_n Q_{r-1}}, & 1 \le r \le m-1, \\ \left(\frac{P_{m-1}}{Q_{m-2}}\right)^q \frac{P_{m-1} + u_n P_{m-1}}{Q_{m-2} + u_n Q_{m-1}}, & r = m. \end{cases}$$

Suppose that $x^{\frac{m}{5}}$ is in the second or third quadrants. Then

$$|P_{m-1}| = 2\cos\left(\frac{2\pi}{5}\right) < |Q_{m-2}| = 2\cos\left(\frac{\pi}{5}\right).$$

Hence

$$\left|\frac{P_{m-1}}{Q_{m-2}}\right| < 1. \tag{3.3}$$

We now construct two sequences $\{v_n\}$ and $\{w_n\}$ which satisfy the conditions for general convergence at x. Let

$$M = \max_{1 \le r \le m} \left\{ \left|\frac{Q_r}{Q_{r-1}}\right| : Q_{r-1} \neq 0 \right\}.$$

Put $v_n = M + 1$ and $w_n = M + 2$, for $n = 1, 2, \cdots$. Hence

$$\liminf d(v_n, w_n) > 0,$$

and by (3.2) and (3.3),

$$\lim_{n\to\infty} \frac{P_n + v_n P_{n-1}}{Q_n + v_n Q_{n-1}} = \lim_{n\to\infty} \frac{P_n + w_n P_{n-1}}{Q_n + w_n Q_{n-1}} = 0.$$

Thus $K(x)$ converges generally to 0 in this case.

Next suppose that $x^{\frac{m}{5}}$ is in the first or fourth quadrants so that

$$|P_{m-1}| = 2\cos\left(\frac{\pi}{5}\right) > |Q_{m-2}| = 2\cos\left(\frac{2\pi}{5}\right).$$

Then

$$\left|\frac{P_{m-1}}{Q_{m-2}}\right| > 1. \tag{3.4}$$

In this case let

$$M = \max_{1 \le r \le m} \left\{ \left|\frac{P_r}{P_{r-1}}\right| : P_{r-1} \neq 0 \right\}.$$

As before, let $v_n = M + 1$ and $w_n = M + 2$, for $n = 1, 2, \cdots$. Hence

$$\liminf d(v_n, w_n) > 0,$$

and by (3.2) and (3.4),

$$\lim_{n\to\infty}\frac{P_n+v_nP_{n-1}}{Q_n+v_nQ_{n-1}}=\lim_{n\to\infty}\frac{P_n+w_nP_{n-1}}{Q_n+w_nQ_{n-1}}=\infty.$$

Thus $K(x)$ converges generally to ∞ in second case.

□

Proof of Proposition 3 Let the i-th convergent of $C=b_0+K_{n=1}^{\infty}a_n/b_n$ be denoted A_i/B_i. Suppose the odd convergents tend to f_1 and that the even convergents tend to f_2. Further suppose that C converges generally to $f\in\hat{\mathbb{C}}$ and that $\{v_n\}$, $\{w_n\}\subset\hat{\mathbb{C}}$ are two sequences such that

$$\lim_{n\to\infty}\frac{A_n+v_nA_{n-1}}{B_n+v_nB_{n-1}}=\lim_{n\to\infty}\frac{A_n+w_nA_{n-1}}{B_n+w_nB_{n-1}}=f$$

and

$$\liminf_{n\to\infty} d(v_n,w_n)>0.$$

It will be shown that these two conditions lead to a contradiction. Suppose first that $|f|<\infty$ and, without loss of generality, that $f\neq f_1$. (If $f=f_1$ then $f\neq f_2$ and we proceed similarly). We write

$$\frac{A_n+w_nA_{n-1}}{B_n+w_nB_{n-1}}=f+\gamma_n,\qquad \frac{A_n+v_nA_{n-1}}{B_n+v_nB_{n-1}}=f+\gamma_n^{'},$$

where $\gamma_n\to 0$ and $\gamma_n^{'}\to 0$ as $n\to\infty$. By assumption we have, for $n\geq 0$, that $A_{2n}=B_{2n}(f_2+\alpha_{2n})$, $A_{2n+1}=B_{2n+1}(f_1+\alpha_{2n+1})$, where $\alpha_i\to 0$ as $i\to\infty$. Then

$$\begin{aligned}\frac{A_{2n}+w_{2n}A_{2n-1}}{B_{2n}+w_{2n}B_{2n-1}}&=\frac{B_{2n}(f_2+\alpha_{2n})+w_{2n}B_{2n-1}(f_1+\alpha_{2n-1})}{B_{2n}+w_{2n}B_{2n-1}}\\&=f+\gamma_{2n}.\end{aligned}$$

By simple algebra we have

$$w_{2n}=\frac{B_{2\,n}\,\left(-f+f_2+\alpha_{2\,n}-\gamma_{2\,n}\right)}{B_{2\,n-1}\,\left(f-f_1-\alpha_{2\,n-1}+\gamma_{2\,n}\right)}.$$

Similarly,

$$v_{2n} = \frac{B_{2\,n}\left(-f + f_2 + \alpha_{2\,n} - \gamma'_{2\,n}\right)}{B_{2\,n-1}\left(f - f_1 - \alpha_{2\,n-1} + \gamma'_{2\,n}\right)}.$$

If $f \neq f_2$ then

$$\lim_{n\to\infty} d(v_{2n}, w_{2n}) \leq \lim_{n\to\infty} \frac{|v_{2n} - w_{2n}|}{|w_{2n}|} = 0.$$

Hence $f = f_2$,

$$w_{2n} = \frac{B_{2\,n}\,(\alpha_{2\,n} - \gamma_{2\,n})}{B_{2\,n-1}\,(f - f_1 - \alpha_{2\,n-1} + \gamma_{2\,n})}$$

and

$$v_{2n} = \frac{B_{2\,n}\,(\alpha_{2\,n} - \gamma'_{2\,n})}{B_{2\,n-1}\left(f - f_1 - \alpha_{2\,n-1} + \gamma'_{2\,n}\right)}.$$

Now we show that

$$\lim_{n\to\infty} \left| \frac{B_{2n}}{B_{2n-1}} \right| = \infty.$$

For iff not, then there is a sequence $\{n_i\}$ and a positive constant M such that $|B_{2n_i}/B_{2n_i-1}| \leq M$ for all n_i and then

$$\begin{aligned}
&\lim_{i\to\infty} d(v_{2n_i}, w_{2n_i}) \leq \lim_{i\to\infty} |v_{2n_i} - w_{2n_i}| \\
&\leq \lim_{i\to\infty} M \left| \frac{\alpha_{2\,n_i} - \gamma'_{2\,n_i}}{f - f_1 - \alpha_{2\,n_i-1} + \gamma'_{2\,n_i}} - \frac{\alpha_{2\,n_i} - \gamma_{2\,n_i}}{f - f_1 - \alpha_{2\,n_i-1} + \gamma_{2\,n_i}} \right| = 0.
\end{aligned}$$

Similarly,

$$w_{2n+1} = \frac{B_{2n+1}}{B_{2n}} \left(\frac{f_1 - f_2 + \alpha_{2n+1} - \gamma_{2n+1}}{\gamma_{2n+1} - \alpha_{2n}} \right)$$

and

$$v_{2n+1} = \frac{B_{2n+1}}{B_{2n}} \left(\frac{f_1 - f_2 + \alpha_{2n+1} - \gamma'_{2n+1}}{\gamma'_{2n+1} - \alpha_{2n}} \right)$$

We now show that

$$\lim_{n\to\infty}\left|\frac{B_{2n+1}}{B_{2n}}\right| = 0.$$

If not, then there is a sequence $\{n_i\}$ and some $M > 0$ such that $|B_{2n_i+1}/B_{2n_i}| \geq M$ for all n_i. Then $\lim_{i\to\infty} w_{2n_i+1} = \lim_{i\to\infty} v_{2n_i+1} = \infty$ and $\lim_{i\to\infty} d(v_{2n_i+1}, w_{2n_i+1}) = 0$.

Finally, we show that it is impossible to have both $\lim_{n\to\infty}|B_{2n+1}/B_{2n}| = 0$ and $\lim_{n\to\infty}|B_{2n}/B_{2n-1}| = \infty$. For ease of notation let B_n/B_{n-1} be denoted by r_n, so that $r_{2n} \to \infty$ and $r_{2n+1} \to 0$, as $n \to \infty$. From the recurrence relations for the B_i's we have

$$r_{2n}(r_{2n+1} - b_{2n+1}) = a_{2n+1}.$$

and

$$r_{2n-1}(r_{2n} - b_{2n}) = a_{2n}.$$

Thus

$$\frac{r_{2n}}{r_{2n} - b_{2n}} = \frac{a_{2n+1}r_{2n-1}}{a_{2n}(r_{2n+1} - b_{2n+1})},$$

and by (1.11) and (1.12) the left side tends to 1 and the right side tends to 0, as $n \to \infty$, giving the required contradiction.

If $f = \infty$ then we write

$$\frac{A_n + w_n A_{n-1}}{B_n + w_n B_{n-1}} = \frac{1}{\gamma_n},$$

where $\lim_{n\to\infty}\gamma_n = 0$. With the α_i's as above we find that

$$w_{2n} = -\frac{B_{2\,n}\,\left(-1 + f_2\,\gamma_{2\,n} + \alpha_{2\,n}\,\gamma_{2\,n}\right)}{B_{2\,n-1}\,\left(-1 + f_1\,\gamma_{2\,n} + \alpha_{2\,n+1}\,\gamma_{2\,n}\right)}$$

and

$$v_{2n} = -\frac{B_{2\,n}\,\left(-1 + f_2\,\gamma'_{2\,n} + \alpha_{2\,n}\,\gamma'_{2\,n}\right)}{B_{2\,n-1}\,\left(-1 + f_1\,\gamma'_{2\,n} + \alpha_{2\,n+1}\,\gamma'_{2\,n}\right)}.$$

In this case it follows easily that $\lim_{n\to\infty} d(w_{2n}, v_{2n}) = 0$.

□

Before proving Theorem 3 and Proposition 2, it is necessary to prove some technical lemmas. In what follows, x is a primitive m-th root of unity, where $m \not\equiv 0 (\text{mod } 5)$. $\bar{\phi} = (-\sqrt{5}+1)/2$, $K_j = K_j(x)$, $P_j = P_j(x)$ and $Q_j = Q_j(x)$, for $j = 0, 1, 2, \cdots$. Frequent use will be made of Binet's formula for F_k.

$$F_k = \frac{\phi^k - \bar{\phi}^k}{\sqrt{5}}.$$

Recall also that $\lim_{k\to\infty} F_{k+1}/F_k = \phi$.

We also use the following facts, which can be found in [9] or deduced from Table 1.

$$P_n = P_{m-1}P_{n-m} + P_{m-2}Q_{n-m}; \tag{3.5}$$

$$Q_n = Q_{m-1}P_{n-m} + Q_{m-2}Q_{n-m}.$$

$$P_{qm+r} = P_{(q-1)m+r} + P_{(q-2)m+r} \tag{3.6}$$

$$Q_{qm+r} = Q_{(q-1)m+r} + Q_{(q-2)m+r}.$$

For $0 \leq r < m$, there exist constants b_r and $b_r^{'}$ such that

$$Q_{qm+r} = b_r\phi^q + b_r^{'}\bar{\phi}^q. \tag{3.7}$$

$$Q_{2m-1} = Q_{m-1}, \qquad P_{2m-1} = P_{m-1} + 1, \tag{3.8}$$

$$P_{2m-2} = P_{m-2} \quad \textit{and} \quad Q_{2m-2} = 1 + Q_{m-2}.$$

Lemma 4. *For $q \geq 2$,*

$$\phi^{q-1} \leq |Q_{qm+m-1}| \leq \phi^q; \tag{3.9}$$

$$m \equiv 1, -1 (mod\ 5) \implies \phi^{q-2} \leq |Q_{qm+m-2}| \leq \phi^{q-1}; \tag{3.10}$$

$$m \equiv 2, -2 (mod\ 5) \implies \phi^{q} \leq |Q_{qm+m-2}| \leq \phi^{q+1}; \tag{3.11}$$

$$\frac{1}{\phi^2} \leq \left|\frac{Q_{qm+m-1}}{Q_{qm+m-2}}\right| \leq \phi^2. \tag{3.12}$$

Proof. Using (3.7) and (3.8) it follows that

$$Q_{qm+m-1} = \frac{Q_{m-1}}{\sqrt{5}}(\phi^{q+1} - \bar{\phi}^{q+1}) = \phi^{q+1}\frac{Q_{m-1}}{\sqrt{5}}\left(1 - \frac{(-1)^{q+1}}{\phi^{2q+2}}\right).$$

From Table 1, $|Q_{m-1}| = 1$ and since $q \geq 2$, it follows that

$$\frac{\phi^{q+1}}{\sqrt{5}}\left(1 - \frac{1}{\phi^6}\right) \leq |Q_{qm+m-1}| \leq \frac{\phi^{q+1}}{\sqrt{5}}\left(1 + \frac{1}{\phi^2}\right),$$

and (3.9) follows easily.

Applying (3.7) with $r = 2m - 2$ and using the values from (3.8) one finds similarly that

$$Q_{qm+m-2} = \begin{cases} \frac{1}{\sqrt{5}}(\phi^q - \bar{\phi}^q), & m \equiv 1, -1 (\text{mod } 5), \\ \frac{1}{\sqrt{5}}(\phi^{q+2} - \bar{\phi}^{q+2}), & m \equiv 2, -2 (\text{mod } 5). \end{cases}$$

If $m \equiv 1, -1(\text{mod } 5)$, then

$$\frac{\phi^q}{\sqrt{5}}\left(1 - \frac{1}{\phi^4}\right) \leq |Q_{qm+m-2}| \leq \frac{\phi^q}{\sqrt{5}}\left(1 + \frac{1}{\phi^4}\right)$$

and (3.10) follows. If $m \equiv 2, -2(\text{mod } 5)$, then

$$\frac{\phi^{q+2}}{\sqrt{5}}\left(1 - \frac{1}{\phi^8}\right) \leq |Q_{qm+m-2}| \leq \frac{\phi^{q+2}}{\sqrt{5}}\left(1 + \frac{1}{\phi^8}\right)$$

and (3.11) follows. (3.12) is an immediate consequence of the preceding inequalities. . □

Lemma 5. *For $q \geq 2$,*

$$\frac{1}{\phi^{2q+1}} \leq |K_{qm+m-1}(x) - K(x)| \leq \frac{1}{\phi^{2q}}; \tag{3.13}$$

$$\frac{1}{\phi^{2q-1}} \leq |K_{qm+m-2}(x) - K(x)| \leq \frac{1}{\phi^{2q-2}}. \tag{3.14}$$

$$\max\{|R_{qm+m-1}(x) - R(x)|, \quad |R_{qm+m-2}(x) - R(x)|\} \leq \frac{1}{\phi^{2q-6}}. \tag{3.15}$$

Proof. (3.6) implies that

$$P_{qm+r} = F_q P_{m+r} + F_{q-1} P_r$$

and

$$Q_{qm+r} = F_q Q_{m+r} + F_{q-1} Q_r.$$

Using (3.8) it follows that

$$K_{qm+m-1} = \frac{P_{qm+m-1}}{Q_{qm+m-1}} = \frac{F_{q+1} P_{m-1} + F_q}{F_{q+1} Q_{m-1}}.$$

Let $q \to \infty$ to get

$$K(x) = \frac{P_{m-1}\phi + 1}{Q_{m-1}\phi}.$$

Since $|Q_{m-1}| = 1$ we have that

$$|K_{qm+m-1} - K(x)| = \left| \frac{F_q}{F_{q+1}} - \frac{1}{\phi} \right| = \frac{\sqrt{5}}{\phi^{2q+2}\left(1 - \frac{(-1)^{q+1}}{\phi^{2q+2}}\right)}.$$

The last equality follows from Binet's formula. Thus for $q \geq 2$,

$$\frac{\sqrt{5}}{\phi^{2q+2}\left(1 + \frac{1}{\phi^6}\right)} \leq |K_{qm+m-1} - K(x)| \leq \frac{\sqrt{5}}{\phi^{2q+2}\left(1 - \frac{1}{\phi^6}\right)}.$$

(3.13) now follows.

Similarly,

$$\frac{P_{qm+m-2}}{Q_{qm+m-2}} = \frac{P_{m-2}F_{q+1}}{Q_{m-2}F_{q+1} + F_q} \Longrightarrow K(x) = \frac{P_{m-2}\phi}{Q_{m-2}\phi + 1}.$$

We consider the cases $m \equiv 1, -1 (\text{mod } 5)$ and $m \equiv 2, -2 (\text{mod } 5)$ separately. In the first case it can be seen from Table 1 that $Q_{m-2} = 0$ and $|P_{m-2}| = 1$. In this case

$$|K_{qm+m-2} - K(x)| = \left| \frac{F_{q+1}}{F_q} - \phi \right| = \frac{\sqrt{5}}{\phi^{2q}\left(1 - \frac{(-1)^q}{\phi^{2q}}\right)}.$$

(3.14) follows. For the second case it can be seen from Table 1 that $Q_{m-2} = 1$ and again $|P_{m-2}| = 1$. In this case

$$|K_{qm+m-2} - K(x)| = \left|\frac{F_{q+1}}{F_{q+2}} - \frac{1}{\phi}\right| = \frac{\sqrt{5}}{\phi^{2q+4}\left(1 - \frac{(-1)^{q+2}}{\phi^{2q+4}}\right)}.$$

and (3.14) again follows. (3.15) follows from (3.13) and (3.14). □

Lemma 6. *Let $q \geq 2$ and let $n = qm+m-1$ or $qm+m-2$. Let y be another point on the unit circle. Suppose $P_n(y) = P_n(x) + \epsilon_1$, $Q_n(y) = Q_n(x) + \epsilon_2$, with $\epsilon = \max\{|\epsilon_1|, |\epsilon_2|\} < 1/2$. Then*

$$|K_n(y) - K_n(x)| \leq \frac{10\epsilon}{\phi^{q-2}}. \tag{3.16}$$

If $q \geq 3$ and the angle between x and y (measured from the origin) is less than $5\pi/3$ and $\epsilon \leq 1/(20\phi^2)$, then

$$|R_n(y) - R_n(x)| < 3\phi|x-y| + \frac{60\epsilon}{\phi^{q-4}}, \tag{3.17}$$

and

$$|R_n(y) - R(x)| \leq 3\phi|x-y| + \frac{60\epsilon}{\phi^{q-4}} + \frac{1}{\phi^{2q-3}}. \tag{3.18}$$

Proof.

$$|K_n(y) - K_n(x)| = \left|\frac{P_n(y)}{Q_n(y)} - \frac{P_n(x)}{Q_n(x)}\right| = \left|\frac{\epsilon_1 Q_n(x) - \epsilon_2 P_n(x)}{Q_n(x)(Q_n(x)+\epsilon_2)}\right|$$

$$\leq \frac{|\epsilon_1 - \epsilon_2|}{|Q_n(x)+\epsilon_2|} + \frac{|\epsilon_2||P_n(x) - Q_n(x)|}{|Q_n(x)||Q_n(x)+\epsilon_2|}$$

$$= \frac{|\epsilon_1 - \epsilon_2|}{|Q_n(x)+\epsilon_2|} + \frac{|\epsilon_2||K_n(x) - 1|}{|Q_n(x)+\epsilon_2|}$$

$$\leq \frac{2\epsilon}{||Q_n(x)| - \epsilon|} + \frac{\epsilon\,\big||K(x)| + 1/\phi^{2q-2} + 1\big|}{||Q_n(x)| - \epsilon|}.$$

Here we have used (3.13), (3.14) and the bounds on ϵ_1 and ϵ_2. Since $|K(x)| \leq \phi$ and $\epsilon < 1/2$, it follows that

$$|K_n(y) - K_n(x)| \leq \frac{2\epsilon}{||Q_n(x)| - 1/2|} + \frac{3\epsilon}{||Q_n(x)| - 1/2|}$$

$$= \frac{5\epsilon}{||Q_n(x)| - 1/2|}$$

$$\leq \frac{10\epsilon}{\phi^{q-2}}.$$

The last inequality follows from (3.9), (3.10), (3.11). Similarily,

$$|R_n(y) - R_n(x)| = \left| \frac{y^{1/5}}{K_n(y)} - \frac{x^{1/5}}{K_n(x)} \right|$$

$$= \left| \frac{K_n(x)(y^{1/5} - x^{1/5}) + x^{1/5}(K_n(x) - K_n(y))}{K_n(x)K_n(y)} \right|$$

$$\leq \frac{|x-y|}{|K_n(y)|} + \frac{|K_n(x) - K_n(y)|}{|K_n(x)||K_n(y)|}$$

$$\leq \frac{|x-y|}{\big||K_n(x)| - 10\epsilon/\phi^{q-2}\big|} + \frac{10\epsilon/\phi^{q-2}}{|K_n(x)|\,\big||K_n(x)| - 10\epsilon/\phi^{q-2}\big|}.$$

Here we have used (3.16) and the fact that the bound on the angle between x and y implies that $|y^{1/5} - x^{1/5}| \leq |x-y|$. Using (3.13), (3.14) and the bound on ϵ it follows that

$$|R_n(y) - R_n(x)| \leq \frac{|x-y|}{\big||K(x)| - 1/\phi^{2q-2} - 1/(2\phi^q)\big|} + \frac{10\epsilon}{\phi^{q-2}\,\big||K(x)| - 1/\phi^{2q-2}\big|\,\big||K(x)| - 1/\phi^{2q} - 1/2\phi^q\big|}.$$

Since $|K(x)| = \phi$ or $1/\phi$ it follows that

$$|R_n(y) - R_n(x)| \leq \frac{|x-y|\phi}{1 - 1/\phi^{2q-3} - 1/(2\phi^{q-1})} + \frac{10\epsilon}{\phi^{q-4}\left(1 - 1/\phi^{2q-3}\right)\left(1 - 1/\phi^{2q-1} - 1/(2\phi^{q-1})\right)}$$

$$\leq \frac{|x-y|\phi}{1 - 1/\phi^3 - 1/(2\phi^2)}$$

$$+\frac{10\epsilon}{\phi^{q-4}\left(1-1/\phi^{3}\right)\left(1-1/\phi^{3}-1/(2\phi^{2})\right)}$$

$$\leq 3\phi|x-y|+\frac{60\epsilon}{\phi^{q-4}}.$$

Finally, (3.18) follows from (3.15) and (3.17). □

Lemma 7. *There exists an uncountable set of points on the unit circle such that if y is one of these points, then there exists two increasing sequences of integers, $\{n_i\}_{i=1}^{\infty}$ and $\{m_i\}_{i=1}^{\infty}$ say, such that,*

$$\lim_{i\to\infty} R_{n_i}(y)=\lim_{i\to\infty} R_{n_i-1}(y)=R_a,$$
$$\lim_{i\to\infty} R_{m_i}(y)=\lim_{i\to\infty} R_{m_i-1}(y)=R_b,$$

for some a, $b\in\{1,2,\cdots,10\}$, where $a\neq b$.

Proof. With the notation of Theorem 3, let $t\in S^{\diamond}$ and set $y=\exp(2\pi i t)$. Let c_{f_n}/d_{f_n} be one of the infinitely many convergents satisfying (1.5) and (1.6) and set $x_n=\exp(2\pi i c_{f_n}/d_{f_n})$. Then $R(x_n)=R_a$ and

$$|x_n-y|<\frac{1}{d_{f_n}^2(d_{f_n}+1)^2\phi^{d_{f_n}^2+2d_{f_n}}}. \tag{3.19}$$

For the last inequality we have used the condition on the a_{h_n+1}'s in (1.6) in the same way that the condition on the $a_{i+1}(t)$'s in (1.3) was used in Lemma 1 and the fact that chord length is shorter than arc length. Let $k=d_{f_n}^2+d_{f_n}-1$ or $d_{f_n}^2+d_{f_n}-2$. By (2.4), (2.5) and (3.19) it follows that

$$|P_k(x)-P_k(y)|\leq\frac{1}{\phi^{d_{f_n}}}$$

and

$$|Q_k(x)-Q_k(y)|\leq\frac{1}{\phi^{d_{f_n}}}. \tag{3.20}$$

By (3.18), with k as above, $q=m=d_{f_n}$ and $\epsilon=1/\phi^{d_{f_n}}$, it follows that

$$\begin{aligned}|R_k(y)-R_a|&=|R_k(y)-R(x_n)|\\&\leq\frac{3\phi}{d_{f_n}^2(d_{f_n}+1)^2\phi^{d_{f_n}^2+2d_{f_n}}}+\frac{60}{\phi^{2d_{f_n}-4}}+\frac{1}{\phi^{2d_{f_n}-3}}\\&\leq\frac{500}{\phi^{2d_{f_n}}}.\end{aligned} \tag{3.21}$$

Thus

$$\lim_{n\to\infty} R_{d_{f_n}^2+d_{f_n}-1}(y)=\lim_{n\to\infty} R_{d_{f_n}^2+d_{f_n}-2}(y)=R_a. \tag{3.22}$$

Similarily,

$$\lim_{n\to\infty} R_{d_{gn}^2+d_{gn}-1}(y) = \lim_{n\to\infty} R_{d_{gn}^2+d_{gn}-2}(y) = R_b. \tag{3.23}$$

It is not difficult to show that $S^\diamond$ is an uncountable set and from the remark following Theorem 3, it follows that it has measure zero. Thus $G = \{\exp(2\pi it) : t \in S^\diamond\}$ is an uncountable set of measure zero. □

Proof of Proposition 2: The proof is similar to that of Lemma 7. Let

$$W = \{W_i\}_{i=1}^{12} = \{R_6, R_7, R_8, R_9, R_{10}, R_2, R_3, R_4, R_5, R_1, R_8, R_7\}.$$

Note that W contains all ten of the values taken by the Rogers-Ramanujan continued fraction at roots of unity. Consider the following continued fraction:

$$\alpha = [0, 1, 3, \overline{2, 3, 2, 1, 1, 2, 3, 2, 1, 3, 3, 5}] := [0, a_1, a_2, \cdots]. \tag{3.24}$$

Modulo 5, the convergents are

$$\left\{\frac{0}{1}, \overline{\frac{1}{1}, \frac{3}{4}, \frac{2}{4}, \frac{4}{1}, \frac{0}{1}, \frac{4}{2}, \frac{4}{3}, \frac{2}{3}, \frac{0}{2}, \frac{2}{2}, \frac{2}{4}, \frac{3}{4}}\right\}, \tag{3.25}$$

where the bar indicates that, modulo 5, the convergents repeat in this order.

Let t be any irrational in $(0, 1)$ such that, for $i \geq 1$, the i-th partial quotient, b_i, and the i-th convergent, c_i/d_i, in its continued fraction expansion, $[0, b_1, b_2, \cdots]$, satisfy the following conditions.

$$\begin{aligned} &(i)\ b_i \equiv a_i (\text{mod } 5), \\ &(ii)\ \left| t - \frac{c_i}{d_i} \right| < \frac{1}{2\pi d_i^2 (d_i+1)^2 \phi^{d_i^2+2d_i}}, \end{aligned} \tag{3.26}$$

where the a_i's are as in equation (3.24).

Set $y = \exp(2\pi it)$ and let $x_n = \exp(2\pi i c_n/d_n)$, so that

$$|x_n - y| < \frac{1}{d_n^2 (d_n+1)^2 \phi^{d_n^2+2d_n}}. \tag{3.27}$$

Here we once again have used the fact that chord length is less than arc length. Set $r = n(\text{mod} 12)$, for $n > 0$. Then it can be easily checked, using

(1.1) and (3.25), that

$$R(x_n) = \begin{cases} W_r, & r \neq 0 \\ W_{12} & r = 0 \end{cases} \tag{3.28}$$

Let $k = d_n^2 + d_n - 1$ or $d_n^2 + d_n - 2$. By (2.4), (2.5) and (3.27) it follows that

$$|P_k(x_n) - P_k(y)| \leq \frac{1}{\phi^{d_n}}$$

and

$$|Q_k(x_n) - Q_k(y)| \leq \frac{1}{\phi^{d_n}}. \tag{3.29}$$

By (3.18), with k as above, $q = m = d_n$ and $\epsilon = 1/\phi^{d_n}$, it follows that

$$|R_k(y) - R(x_n)| \leq \frac{3\phi}{d_n^2(d_n+1)^2\phi^{d_n^2+2d_n}} + \frac{60}{\phi^{2d_n-4}} + \frac{1}{\phi^{2d_n-3}} \tag{3.30}$$

$$\leq \frac{500}{\phi^{2d_n}}.$$

Next, for each $j \in \{1, 2, \cdots, 12\}$, define a sequence of integers $\{s_{i,j}\}_{i=1}^{\infty}$, by setting $s_{i,j} = d_{12(i-1)+j}^2 + d_{12(i-1)+j}$. By (3.28), $R(x_{12(i-1)+j}) = W_j$ and so, from (3.30),

$$|R_{(s_{i,j}-1)}(y) - W_j| \leq \frac{500}{\phi^{2d_{12(i-1)+j}}};$$

$$|R_{(s_{i,j}-2)}(y) - W_j| \leq \frac{500}{\phi^{2d_{12(i-1)+j}}}.$$

It follows that

$$\lim_{i\to\infty} R_{(s_{i,j}-1)}(y) = \lim_{i\to\infty} R_{(s_{i,j}-2)}(y) = W_j.$$

Both results hold for $1 \leq j \leq 12$. Since the set W contains all ten of the R_j's the result is proved for this particular t.

Let S' denote the set of all such $t \in (0,1)$ and set $G^* = \{\exp(2\pi i t) : t \in S'\}$. Clearly $G^* \subset Y_S$ and is also uncountable.

□

Proof of Theorem 3: Let y be any point in G, where G is as defined in the proof of Lemma 7, and let t be the irrational in $(0,1)$ for which $y = \exp(2\pi i t)$.

Suppose $R(y)$ converges generally to $f \in \hat{\mathbb{C}}$ and that $\{v_n\}$, $\{w_n\}$ are two sequences such that

$$\lim_{n\to\infty} \frac{P_n + v_n P_{n-1}}{Q_n + v_n Q_{n-1}} = \lim_{n\to\infty} \frac{P_n + w_n P_{n-1}}{Q_n + w_n Q_{n-1}} = \frac{y^{\frac{1}{5}}}{f} := g.$$

Suppose first that $|g| < \infty$. By construction there exists two infinite strictly increasing sequences of positive integers $\{n_i\}_{i=1}^{\infty}$, $\{m_i\}_{i=1}^{\infty} \subset \mathbb{N}$ such that

$$L_a := \frac{y^{\frac{1}{5}}}{R_a} = \lim_{i\to\infty} \frac{P_{n_i}(y)}{Q_{n_i}(y)} = \lim_{i\to\infty} \frac{P_{n_i-1}(y)}{Q_{n_i-1}(y)}$$

and

$$L_b := \frac{y^{\frac{1}{5}}}{R_b} = \lim_{i\to\infty} \frac{P_{m_i}(y)}{Q_{m_i}(y)} = \lim_{i\to\infty} \frac{P_{m_i-1}(y)}{Q_{m_i-1}(y)},$$

for some $a \neq b$, $a, b \in \{1, 2, \cdots, 10\}$. Also by construction each n_i has the form $d_{k_i}^2 + d_{k_i} - 1$, where d_{k_i} is some denominator convergent in the continued fraction expansion of t, and likewise for each m_i. It can be further assumed that $L_a \neq g$, since $L_a \neq L_b$. For ease of notation write

$$\begin{aligned} P_{n_i}(y) &= P_{n_i}, & Q_{n_i}(y) &= Q_{n_i}, \\ P_{n_i-1}(y) &= P_{n_i-1}, & Q_{n_i-1}(y) &= Q_{n_i-1}. \end{aligned}$$

Write $P_{n_i} = Q_{n_i}(L_a + \epsilon_{n_i})$ and $P_{n_i-1} = Q_{n_i-1}(L_a + \delta_{n_i})$, where $\epsilon_{n_i} \to 0$ and $\delta_{n_i} \to 0$ as $i \to \infty$. Thus

$$\frac{Q_{n_i}(L_a + \epsilon_{n_i}) + w_{n_i} Q_{n_i-1}(L_a + \delta_{n_i})}{Q_{n_i} + w_{n_i} Q_{n_i-1}} = g + \gamma_{n_i},$$

where $\gamma_{n_i} \to 0$ as $i \to \infty$. This last equation implies that

$$w_{n_i} + \frac{Q_{n_i}}{Q_{n_i-1}} = \frac{Q_{n_i}}{Q_{n_i-1}} \left(\frac{\epsilon_{n_i} - \delta_{n_i}}{g - L_a + \gamma_{n_i} - \delta_{n_i}} \right).$$

Because of (3.12), the fact that each n_i has the form $d_{k_i}^2 + d_{k_i} - 1$, where d_{k_i} is some denominator convergent in the continued fraction expansion of t and (3.20), it follows that Q_{n_i}/Q_{n_i-1} is absolutely bounded. Therefore the right hand side of the last equality tends to 0 as $i \to \infty$ and thus

$$w_{n_i} + Q_{n_i}/Q_{n_i-1} \to 0 \text{ as } n_i \to \infty. \tag{3.31}$$

Note that $|w_{n_i}| < \infty$ for all i sufficiently large, since $|Q_{n_i}/Q_{n_i-1}| < \infty$. Similarily,

$$v_{n_i} + Q_{n_i}/Q_{n_i-1} \to 0 \text{ as } n_i \to \infty. \tag{3.32}$$

By the (3.31), (3.32) and the triangle inequality

$$\lim_{i\to\infty} |v_{n_i} - w_{n_i}| = 0.$$

Thus

$$\liminf d(v_n, w_n) = 0.$$

Therefore $R(y)$ does not converge generally. A similar argument holds in the case where g is infinite.

Since G is uncountable and of measure zero, this proves the theorem.

□

Corollary 4. *Let y be as in Corollary 1. Then $K(y)$ does not converge generally.*

Proof. Let $t \in (0,1)$ be such that $y = \exp(2\pi i t)$. Recall that $t = [0, a_1, a_2, \cdots]$, where a_i is the integer consisting of a tower of i twos with an i an top. Modulo 5, the convergents in the continued fraction expansion of t are

$$\left\{ \overline{\frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{0}, \frac{3}{3}, \frac{0}{3}, \frac{3}{1}, \frac{3}{4}, \frac{1}{0}, \frac{4}{4}, \frac{0}{4}, \frac{4}{3}, \frac{4}{2}, \frac{3}{0}, \frac{2}{2}, \frac{0}{2}, \frac{2}{4}, \frac{2}{1}, \frac{4}{0}, \frac{1}{1}} \right\}, \tag{3.33}$$

where once again the bar indicates that the convergents repeat modulo 5 in this order. In particular, there are two fractions, r/s and u/v say, such that (1.5) and (1.7) holds. Thus it is sufficient to show that

$$\left| t - \frac{c_i}{d_i} \right| < \frac{1}{2\pi d_i^2 (d_i+1)^2 \phi^{d_i^2 + 2d_i}}, \tag{3.34}$$

for all $i \geq 3$, where c_i/d_i is the i-th convergent in the continued fraction expansion of t. In particular (3.19) will hold and likewise a similar inequality when f_n is replaced by g_n, where $\{c_{f_n}/d_{f_n}\}$ and $\{c_{g_n}/d_{g_n}\}$, are the two sequences of convergents corresponding to r/s and u/v. This in turn will ensure that $y \in G$ so that $K(y)$ will not converge generally by Theorem 3. We will show that, for $i \geq 3$,

$$a_{i+1} > 16^{d_i^2} \tag{3.35}$$

This will be sufficient to prove the result. Indeed, let $t_{i+1} = [a_{i+1}, a_{i+2}, \cdots]$ denote the i-th tail of the continued fraction expansion for t. Then

$$a_{i+1} \geq 16^{d_i^2} = 4^{2 \cdot d_i^2} > 4^{(d_i+1)^2}$$

$$= 2^{(d_i+1)^2} 2^{(d_i+1)^2}$$

$$> 2\pi(d_i+1)^2\phi^{d_i^2+2d_i} \Longrightarrow$$

$$\begin{aligned}
\left|t-\frac{c_i}{d_i}\right| &= \left|\frac{t_{i+1}c_i+c_{i-1}}{t_{i+1}d_i+d_{i-1}}-\frac{c_i}{d_i}\right| \\
&= \frac{1}{d_i(t_{i+1}d_i+d_{i-1})} \\
&< \frac{1}{d_i(a_{i+1}d_i+d_{i-1})} \\
&< \frac{1}{d_i^2 a_{i+1}} \\
&< \frac{1}{2\pi d_i^2(d_i+1)^2\phi^{d_i^2+2d_i}}.
\end{aligned}$$

Thus all that remains is to prove (3.35). The proof of this inequality is similar to that of (2.11).

$$16^{d_i^2} \le a_{i+1} \Longleftrightarrow 4\,d_i^2 \le \underbrace{2^{\cdot^{\cdot^{\cdot^{2^{i}+1}}}}}_{i\text{ twos}}, \tag{3.36}$$

where the notation indicates that the last integer consists of a tower of i twos with an $i+1$ on top. It can be easily checked that the second inequality holds for $i=3,4$. Suppose it holds for for $i=3,4,\cdots,r-1$. Then

$$\begin{aligned}
4\,d_r^2 &= 4\,(a_r d_{r-1}+d_{r-2})^2 \le 4\,(4\,a_r d_{r-1}^2+4\,d_{r-2}^2)^2 \\
&\le 4\left(4\times\underbrace{2^{\cdot^{\cdot^{\cdot^{2^{r}}}}}}_{r\times 2\text{'s}}\times\underbrace{2^{\cdot^{\cdot^{\cdot^{2^{r}}}}}}_{(r-1)\text{ twos}}+4\times\underbrace{2^{\cdot^{\cdot^{\cdot^{2^{r}-1}}}}}_{(r-2)\text{ twos}}\right)^2 \\
&\le \underbrace{2^{\cdot^{\cdot^{\cdot^{2^{r}+1}}}}}_{r\text{ twos}}.
\end{aligned}$$

Thus the first inequality in (3.36) holds for all positive integers $i \ge 3$ and the result follows. $\square$

Proof of Corollary 3: Showing that the i-th partial quotient, b_i, and the i-th convergent, c_i/d_i, of the continued fraction expansion of t satisfy the conditions in (3.26), for $i = 1, 2, \cdots$, will ensure that $y \in G^*$, where G^* is as defined in Proposition 2.

The b_i's satisfy the first of these conditions by construction and so all that remains is to prove the second. By the same reasoning as used in the proof of Corollary 4, it is sufficient to show that

$$g_{i+1} \geq 16^{d_i^2}$$

since $b_{i+1} \geq g_{i+1}$. The details are omitted since the proof is almost identical, the only real difference being that

$$16^{d_i^2} \leq g_{i+1} \Longleftrightarrow d_i^2 \leq \underbrace{16^{\cdot^{\cdot^{16^{i+1}}}}}_{i \times 16\text{'s}},$$

□

4. Concluding Remarks

The set of points on the unit circle for which we have shown that the Rogers-Ramanujan continued fraction diverges has measure zero. On the other hand, the only points for which we know that the continued fraction converges generally are the roots of unity. This leaves open the question of convergence at the remaining points. At present, the authors do not see how to use the methods of the paper to tackle this question; we conjecture that the set of points on the unit circle for which the continued fraction converges has measure zero. Additionally, we do not know how to prove convergence at any points other than the roots of unity, although it seems reasonable to believe that there are such points.

In a later paper we will examine the question of convergence of other q-continued fractions on the unit circle, such as the Göllnitz-Gordon continued fraction and some other q-continued fractions of Ramanujan.

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